Università degli Studi di Milano
Dipartimento di Informatica

Dottorato di Ricerca in Informatica

Ph.D. Thesis

Strongly Constructive Formal Systems

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Last Revision: July 21, 1997

Acknowledgments

I would like to thank Pierangelo Miglioli who constantly helped me since my graduation, following and stimulating my research activity. Moreover, Mario Ornaghi deserves a special thank for many stimulating discussions.

But, more importantly, I would like to dedicate this thesis to my parents: they gave me the opportunity to make such a wonderful experience.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

Since the early researches in modern Logic and the Foundations of Mathematics there has been a good deal to do with constructive formal systems, which has given rise to several techniques and results, both on the semantical and the syntactical ground. In the semantical framework one may find typically model theoretic tools such as the Kripke Models (see e.g. [Smorynski, 1973, Gabbay, 1981]), and "operational" interpretations such as the ones based on Recursive Realizability (see e.g. [Kleene, 1945, Kleene, 1952, Troelstra, 1973c, Troelstra and van Dalen, 1988b]): the former allow to characterize various "constructive" logics and theories according to a style which is close, under many aspects, to the one involved in the classical model theoretic treatments; the latter spring from intuitionistic tradition and relate the meaning of the logical connectives and quantifiers to the theory of Recursive Functions according to many variants which allow to interpret a family of number theoretic systems, including Heyting Arithmetic. As for the syntactical framework, it includes various proof theoretic results involving intuitionistic Natural and Sequent Calculi, as well as Normalization and Cut-elimination Theorems and their relations to the explicit definability and the disjunction property, more generally, to the subformula property (the reader is referred to [Prawitz, 1965, Prawitz, 1971, Troelstra, 1973b, Girard et al., 1989] for a discussion about Natural Calculi and Normalization, and to [Takeuti, 1975, Girard, 1987, Girard et al., 1989] for a discussion about Sequent Calculi and Cut-elimination).

However, several links can be found between the two areas, which give a "proof theoretic flavour" to many approaches to constructive semantics and a "semantical flavour" to constructive proof theory. In this sense, the above quoted Recursive

Realizability interpretations entangle algorithmic (operational) aspects recalling the typical syntactical manipulations (according to Kreisel, the General Theory of Syntax is Recursiveness Theory, while the General Theory of Semantics is Set Theory). But, more than this, other constructive interpretations of logical formulas have been proposed in the very spirit of the founders of Intuitionism, such as the so called Intended Interpretation of Brouwer and Heyting, or the BHK interpretation described in [Troelstra, 1977], which is a more formalized (even if still lacking formalization) version of the former: such interpretations are based on a semantical (or even epistemic) notion of proof, i.e., the so called "intuitive proofs", which are seen as constructions providing immediate evidence of the validity of logical formulas. Also, attempts of singling out an operational interpretation of the logical constants involved in the introduction-elimination mechanism of Natural Calculi have been made by Prawitz on the basis of proof theoretic notions such as the Inversion Principle and the Strong Validity of proofs; in this context, the *syntactical* notion of *normalized proof* has been proposed as the *formal explanation* of a *semantical* notion of *intuitive proof* such as the one involved in the Intended Interpretation and in the BHK interpretation (see [Prawitz, 1977, Prawitz, 1978]). Further approaches with impressive interconnections between proof theoretic and semantical aspects, which in the recent years have given rise to important applications in Computer Science, are the so called Formulae as Types Paradigm, based on the Curry-Howard isomorphism (see e.g. [Howard, 1980]) and (on a more ambitious ground involving also the Foundations of Mathematics) the Theory of Types of Martin Löf (see e.g. [Martin-Löf, 1984]). Finally, in a more philosophically oriented attitude, Dummett's Theory of Meaning provides an interpretation of intuitionistic logical constants which (like the previously discussed approach of Prawitz) is concerned with an interpretation of a *semantical* notion such as the one of *intuitive proof* in terms of a *syntactical* notion of *formal proof* ([Dummett, 1977]).

The above approaches (apart perhaps from the Kripke Models, which have a different style of interpretation and are applied also to intermediate and modal logics which cannot be considered constructive) correspond to (more or less wide) "oscillations" around an intuitionistic paradigm, and can be included in some "extended intuitionistic tradition". As such, they have given rise to a contraposition of two notions of constructive formal system. On the one hand a notion of "good" constructive formal system is involved, intended as a formal system not only satisfying the disjunction and the explicit definability properties, but, much more importantly, fulfilling some further (syntactical and/or semantical) requests *in agreement with acknowledged paradigms*; on the other hand, the existence of a "naive" notion of constructive formal system is stressed, according to which a formal system can be considered constructive if it satisfies the above quoted properties of explicit definability and disjunction, without any further qualification (for propositional systems only the disjunction property is required). In this framework, any "constructivist with a good paradigm" agrees that the two above properties involving the disjunction connective and the existential quantifier are necessary in order that a formal system be constructive; but he is aware that such properties alone do not uniquely determine (even in presence of strong assumptions) any formal system. In other words, everyone who is engaged in a well founded notion of constructivism considers naive constructivism as a source of too many and too under-characterized formal systems, which, as such, are scarcely interesting.

Now, we believe that naive constructivism should deserve more attention; we look at it as a possible point for a fresh start. After all, the disjunction property and the explicit definability property are important. For instance, suppose that the addition of some non classical logic **L** (intended as a deductive apparatus) to a first order theory **T** provides a constructive (in the naive sense) and recursively axiomatizable formal system $\mathbf{T}+\mathbf{L}$; suppose that **T** is classically consistent and that the set of theorems of $\mathbf{T}+\mathbf{CL}$ includes the one of $\mathbf{T}+\mathbf{L}$, where $\mathbf{T}+\mathbf{CL}$ denotes the formal system obtained by adding to **T** classical logic **CL** (i.e., using **CL** as the deductive apparatus of the system, whose mathematical axioms are provided by the formulas of **T**); suppose also that **T** has a (uniquely determined up to isomorphisms) *isoinitial model* $\mathcal{M}$ (in the classical sense), that is, a model $\mathcal{M}$ which can be isomorphically embedded in a unique way in every model of **T** (for a detailed discussion about the notion of isoinitial model and its relation with the problem of Abstract Data Types in Computer Science see e.g. [Bertoni et al., 1983, Bertoni et al., 1984, Miglioli et al., 1994a]); suppose that the isoinitial model of **T** is reachable, that is every element of its domain is denoted by a closed term of the language; finally suppose that every closed term t can be reduced to a normal form term t^N (that is $t = t^N$ holds in every model of the theory), that any normal term t_1^N cannot be reduced to any normal term t_2^N different from t_1^N, and that the set of normal terms is recursive. Now, let a formula such as

$$\forall x_1 \ldots \forall x_n \exists! y A(x_1, \ldots, x_n, y)$$

be provable in the system $\mathbf{T}+\mathbf{L}$; then, from the constructivity of $\mathbf{T}+\mathbf{L}$ (even if in the naive sense), its axiomatizability, and the properties of the closed terms in normal form, we get the following fact:

- Let $f(x_1, \ldots, x_n)$ be the function whose domain is the set of all the n-tuples of terms in normal form of the system, and such that, for every n-tuple $\langle t_1^N, \ldots, t_n^N \rangle$ of such terms, $f(t_1^N, \ldots, t_n^N)$ is the unique normal term t^N such that $A(t_1^N, \ldots, t_n^N, t^N)$ is provable in $\mathbf{T}+\mathbf{L}$. Then $f(x_1, \ldots, x_n)$ is a computable function; moreover, an algorithm to compute $f(x_1, \ldots, x_n)$ can be extracted from the formal system $\mathbf{T}+\mathbf{L}$.

Situations of this kind are rather complex from the semantical point of view, since, presumably, the function $f(x_1, \ldots, x_n)$ is interesting as a function in the framework of a classical model of **T** (it is a function of the kind $\mathbf{D}^n \rightarrow \mathbf{D}$, where **D** is the domain of the isoinitial model $\mathcal{M}$) turning out to be recursive and requiring, to be computed, a constructive subsystem $\mathbf{T}+\mathbf{L}$ of the classical system $\mathbf{T}+\mathbf{CL}$.

Hardly, we believe, these kinds of examples, crossing the contraposition of classical (non constructive) systems versus constructive systems, can be dealt within the paradigms of the "extended intuitionistic tradition" quoted above (in the previous example we have a simultaneous presence of a classical semantics and a "constructive", even if naive, proof theory). Nevertheless, these situations are extremely interesting for people working in areas such as Program Synthesis, or Abstract Data Type Specification, or Program Synthesis together with Abstract Data Type Specification (for a better analysis of examples such as the above, we refer the reader to [Miglioli et al., 1988, Miglioli et al., 1994b], where the classical notion of isoinitial model, considered as the intended model of an Abstract Data Type Specification, is combined with constructive proof theoretic notions).

Examples such as the above introduce a different kind of paradigm, which is, so to say, more "formalist" than the previously discussed ones concerning the extended intuitionistic tradition; indeed, such a paradigm directly involves the contraposition *informal* versus *formal*. From this point of view, presumably, the main property to be assured by a computer scientist using logical formulas to specify programs, i.e. functions to be computed, is the computability itself of the involved functions, and this is simply guaranteed by the good quality (i.e. *effectiveness*, which corresponds to *recursive axiomatizability*) of the formalization of the logical framework in hand, together with the fulfillment of two *purely formal requirements* such as the disjunction property and the explicit definability property (which only entangle the very notion of calculus, intended as an algorithm to enumerate formulas).

Thus (as in any formalist attitude), the formalist constructive paradigm we are discussing is largely independent of semantical notions (even if the latter are more or less linked with syntactical, proof theoretic concepts). Of course, it does not exclude the possibility of being used in connection with semantical treatments; but it does not necessarily require the presence of some "orthodox" constructive semantics (in our previous example involving the isoinitial models, the underlying semantics is classical). More than this, in the most constructive approaches to program synthesis, this paradigm fits very well with acknowledged paradigms of constructive semantics, to the point that one might look at it as a particular form of the latter (in this perspective, according to us, one can explain the great success in Computer Science of the Formulae as Types Paradigm and of the Theory of Martin-Löf); however, we believe that the formalist constructive paradigm deserves to be analyzed on its own right, and that the possibility of any necessary connection with some well defined semantical framework is to be justified by this analysis, taking into account the needs of massive practices such as the ones involved in Computer Science rather than some *a priori* fixed point of view.

Now, the present Thesis is an attempt of giving some contribution to the analysis of the formalist constructive paradigm. What is involved here is the "algorithmic content" of recursively axiomatizable formal systems with the disjunction property and the explicit definability property, and the possibility of reasonably distinguishing the "global" algorithmic content from the "local" one.

To be more precise, let us come back to the previous example in the framework of isoinitial models. Even if the function $f(x_1, \ldots, x_n)$ there described is computable, a key point is: *which parts of the involved formal system are necessary in order to compute, for every input value $\langle t_1^N, \ldots, t_n^N \rangle$, the required term in normal form $t^N = f(t_1^N, \ldots, t_n^N)$?* At first, one can see that *the whole formal system* can be used as an algorithm to make such computations: indeed, one can define a recursive enumeration of all the theorems of the system, so that, given any input value $\langle t_1^N, \ldots, t_n^N \rangle$ for the function, one can start with the enumeration and successively generate provable formulas, until a formula of the form $A(t_1^N, \ldots, t_n^N, t^N)$ is reached; the term t^N is the desired result of the computation. However, such a *global algorithm* generally involves an "horrendously complex" enumeration of formulas and cannot be considered satisfactory at all. Moreover, the algorithm does not use the information contained in any proof of the formula $\forall x_1 \ldots \forall x_n \exists ! y A(x_1, \ldots, x_n, y)$. In this framework, from an intuitive point of view, one naturally looks at those formal systems which are equipped by calculi where any proof of a formula such as

$$\forall x_1 \ldots \forall x_n \exists ! y A(x_1, \ldots, x_n, y)$$

contains sufficient information (and this information can be extracted from the proof) *to compute the associated function for every input value*; likewise, one is interested in extracting, from any proof of a formula such as

$$\forall x_1 \ldots \forall x_n (B(x_1, \ldots, x_n) \vee \neg B(x_1, \ldots, x_n)) ,$$

the whole information needed to decide, for every n-tuple $\langle t_1^N, \ldots, t_n^N \rangle$ of closed terms in normal form, whether $B(t_1^N, \ldots, t_n^N)$ holds or $\neg B(t_1^N, \ldots, t_n^N)$ holds. Systems satisfying these intuitive requirements can be viewed as systems with *a good local algorithmic content*.

The aim of our Thesis is to formally define the intuitive notion of "formal system with a good local algorithmic content", and to precisely explain how one can extract the related "local information"; we will call *strongly constructive* such systems. In our treatment a strongly constructive formal system will be nothing but a *formal system with local properties of disjunction and explicit definability*, which can be intuitively so explained: there is a presentation (calculus) for the formal system such that every proof in the calculus of a (closed) formula such as $A \vee B$ contains sufficient information to build up (without any "essential" reference to other parts of the calculus) a proof of A or a proof of B (belonging to the calculus); and the like for a (closed) formula $\exists x A(x)$.

Our notion of strongly constructive formal system intends also to give a contribution to formally explain the meaning of expressions such as "proofs-as-programs" in the following sense: any recursively axiomatizable formal system with global (but not necessarily local) properties of disjunction and explicit definability can be looked at as a universal algorithm allowing to compute all the functions provably definable in it, but its proofs cannot be considered (in absence of stronger properties) algorithms (programs) to compute such functions; on the contrary, there are appropriate strongly constructive presentations (calculi), where *proofs* (of suitable formulas) *directly work as programs*. Only calculi conforming to such a paradigm of the proofs-as-programs are reasonable candidates, we believe, to be used for program synthesis.

One of the main results about the "computational content of proofs" is, without any doubt, Prawitz's Normalization Theorem [Prawitz, 1965, Prawitz, 1971]. In the framework of Intuitionistic Arithmetic this result has a clear computational meaning and can be considered one of the most interesting "implementations" of the proof-as-programs paradigm. To briefly discuss this role of Normalization, we suppose to have a proof π_t in the Natural Deduction calculus for Intuitionistic Arithmetic of a closed formula of the kind $\exists z A(t,z)$ (where t is a closed term of the language $\mathcal{L}_{\mathbf{A}}$ of Intuitionistic Arithmetic). We represent this proof as

$$\begin{array}{c}\pi_t\\ \exists z A(t,z)\end{array}\ ,$$

that is, π_t is the proof-tree whose top-formula is $\exists z A(x,z)$. In general, this proof does not contain a subproof of some formula $A(t,t')$, with t' a closed term of $\mathcal{L}_{\mathbf{A}}$. But, as a consequence of the Normalization Theorem [Prawitz, 1965, Troelstra, 1973b], the proof π_t can be normalized in a finite number of steps, giving rise to a proof π_t^* of the kind

$$\pi_t^* \equiv \cfrac{\begin{array}{c}\pi_t'\\ A(t,t')\end{array}}{\exists x A(t,z)}\,\text{I}\exists\ ,$$

where t' is a closed term of $\mathcal{L}_{\mathbf{A}}$ and π_t' is a normalized proof of the closed formula $A(t,t')$. Similarly, if τ_t is a proof in the Natural Deduction calculus for Intuitionistic Arithmetic of the closed formula $B(t) \vee \neg B(t)$, that is τ_t is

$$\begin{array}{c}\tau_t\\ B(t) \vee \neg B(t)\end{array}\ ,$$

we have that this proof, in general, does not contain a subproof of the formula $B(t)$ or a subproof of the formula $\neg B(t)$; but normalizing it, we obtain, in a finite number of steps, a proof τ_t^* which has one of the following two forms:

$$\tau_t^{*1} \equiv \cfrac{\begin{array}{c}\tau_t'\\ B(t)\end{array}}{B(t) \vee \neg B(t)}\,\text{I}\vee \qquad\qquad \tau_t^{*2} \equiv \cfrac{\begin{array}{c}\tau_t''\\ \neg B(t)\end{array}}{B(t) \vee \neg B(t)}\,\text{I}\vee\ ,$$

where τ_t' is a normalized proof of the closed formula $B(t)$ and τ_t'' is a normalized proof of the closed formula $\neg B(t)$.

According to these examples, we can consider an *open proof* $\pi(x)$ of the *open* formula $\exists z A(x, z)$, where x is the only free variable, as a *program* to compute the recursive function $f_\pi(x)$, from closed terms of $\mathcal{L}_\mathbf{A}$ into closed terms of $\mathcal{L}_\mathbf{A}$, defined as follows:

- For any closed term t of $\mathcal{L}_\mathbf{A}$, $f_\pi(t)$ is the normal form of the closed term t' of $\mathcal{L}_\mathbf{A}$ such that the normalized proof π^*_t contains the proof π'_t of $A(t, t')$ as an immediate subproof.

Similarly, we can consider an *open proof* $\tau(x)$ of the *open* formula $B(x) \vee \neg B(x)$, where x is the only free variable, as a *program* to decide the predicate $p_\tau(x)$ on closed terms of $\mathcal{L}_\mathbf{A}$, defined as follows:

- For any closed term t of $\mathcal{L}_\mathbf{A}$, $p_\tau(t)$ is true if the normalized proof τ^*_t contains the proof τ'_t of $B(t)$ as an immediate subproof (that is τ^*_t is τ^{*1}_t), and $p_\tau(t)$ is false if the normalized proof τ^*_t contains the proof τ''_t of $\neg B(t)$ as an immediate subproof (that is τ^*_t is τ^{*2}_t).

This should explain how Normalization interprets proofs as programs. A deeper discussion on the advantages and the limits of Normalization as a mechanism to extract computationally relevant information from proofs is out of the scope of this Thesis (for a more extensive discussion, see [Miglioli and Ornaghi, 1981]). The aspect we want to point out here is that the framework where the Normalization Theorems hold is too narrow, essentially coinciding (disregarding the non-constructive classical systems) with a family of purely intuitionistic calculi. In this sense, taking the normalizable calculi with the disjunction property and the explicit definability property as the only strongly constructive ones, the assertion that there are formal systems which are not strongly constructive, yet satisfying the disjunction property and the explicit definability property, becomes quite trivial.

On the contrary, there is a great number of formal systems which are, according to us, quite reasonable candidates to be included in the strongly constructive ones, even if no presentation (calculus) for them can be reasonably seen as normalizable. Systems of this kind contain interesting mathematical principles which hardly can be handled in an attitude oriented to normalization, even if they have, without any doubt, a clear algorithmic content, giving rise to proofs interpretable as programs which cannot be simulated by proofs interpreted as programs by Normalization in the framework of intuitionistic systems. Also, a lot of interesting axiom-schemes of intermediate predicate logics with the disjunction property and the explicit definability property should be excluded from the realm of strong constructiveness, even if their algorithmic content is quite analyzable in local terms (the use of extra-intuitionistic logical principles in constructive program synthesis is rather restricted nowadays, but a more extensive field of applications might put into evidence their relevance; after all, we are at the very beginning of a serious development of program synthesis from constructive proofs).

Thus one of our goals is to widely extend the field of applications of techniques such as normalization or cut-elimination, yet providing a good paradigm of proofs-as-programs. Our approach is directly inspired by a notion of extraction of information (differently from the normalization approach, where the claimed goal is to delete the detours, i.e. to transform the proof into a very regular proof of the same formula, while the possibility extracting information from the normalization process looks like a kind of by-product). In this line, proofs (finite sets of proofs) of strongly constructive formal systems are seen as an implicit amount of information to be made explicit (to be extracted, or executed, or handled) by suitable external mechanisms, called *generalized rules*, which do not introduce any essentially new pieces of information with respect to the ones involved in the proofs to which they are applied. By means of our generalized rules, whose behavior is ruled by bounds on the logical complexity of the information they can extract, we also aim to introduce, even if in a very initial stage, some tools to measure a kind of *abstract complexity involved in the extraction of information from proofs*. In this framework our results, according to which there are recursively axiomatizable formal systems which satisfy the disjunction property and the explicit definability property but are not strongly constructive, seems to be of some interest; surely, much more interesting than the circumstance that there are formal systems with the disjunction property and the explicit definability property which cannot be presented in the form of normalizable calculi.

The Thesis is organized as follows. In the remainder of this Chapter we present the preliminary definitions, and the two main calculi we will use in the following Chapters, that is, an Hilbert-style calculus and a Natural Deduction calculus for intuitionistic logic. In Chapter 2, we will develop the fundamental notions of *formal system* (Section 2.1), of *proof* and *calculus* (Section 2.2). This material is not standard, since we aim to abstract from the usual definitions of proof and calculus referred to a particular proof-theory, and to develop the notions of strongly constructive formal system and of calculus from an abstract point of view. Central notions of the Thesis are the ones of *generalized rule* $\mathcal{R}$ and the connected notion of $\mathcal{R}$-*subcalculus*, developed in Section 2.3. Generalized rules will constitute the *information extracting mechanism* of our approach, and the notions of *uniformity* developed in Section 2.4 intend to formalize the, so to say, *effectiveness* of the information extracting mechanism. Finally, in Section 2.5 the definitions of strongly constructive calculus and of strongly constructive formal system are provided. In Chapter 3 and in Chapter 4 we develop several examples of strongly constructive logics (formal systems without mathematical axioms) and theories respectively. The examples we provide are not intended to be an exhaustive presentation of strongly constructive formal systems, but only an illustrative presentation of some formal systems. The choice has been made so as to present several different techniques for proving that a system is strongly constructive. In particular, we remark that, for some of the examples of Chapter 4, some model-theoretic considerations on the systems in hand are required to prove their strong constructiveness. Finally, in Chapter

5, to complete our foundational analysis of the notion of strongly constructive formal system, we provide an example of a system with the disjunction and the explicit definability properties but failing to be strongly constructive.

1.2 Preliminaries

The set theoretic notation applied in this thesis is standard. In particular the symbols $\in$, $\notin$, $\subseteq$, $\subset$, $\times$, $\cap$, $\cup$ have their customary meaning. The difference of two sets $\mathbf{X}$ and $\mathbf{Y}$ will be denoted by $\mathbf{X} \setminus \mathbf{Y}$; the Cartesian power $\mathbf{X} \times \mathbf{X} \times \ldots \times \mathbf{X}$, with the set $\mathbf{X}$ being taken n times, is written $\mathbf{X}^n$. We shall use $\subset$ as the proper part of $\subseteq$. Occasionally we shall write $\mathbf{X} \supseteq \mathbf{Y}$ and $\mathbf{X} \supset \mathbf{Y}$ instead of $\mathbf{Y} \subseteq \mathbf{Y}$ and $\mathbf{Y} \subset \mathbf{Y}$. A similar notational convention will be applied to partial ordering relations denoted by $\leq$ and $<$. We will denote with $\emptyset$ the empty set and with $\mathrm{Pow}(\mathbf{X})$ the power set of $\mathbf{X}$. A distinguished notation will be used for finite sets, namely $\mathrm{Pow}_{\mathrm{fin}}(\mathbf{X})$ will denote the set of all the finite sets belonging to $\mathrm{Pow}(\mathbf{X})$ and $\mathbf{Y} \subseteq_{\mathrm{fin}} \mathbf{X}$ will mean that $\mathbf{Y} \subseteq \mathbf{X}$ and $\mathbf{Y}$ is finite. Finally, we denote with $\mathbf{N}$ the set of the natural numbers.

In this thesis we will consider (first order) languages built on the logical alphabet consisting of the propositional constant $\bot$, the propositional connectives $\neg, \wedge, \vee, \Rightarrow$, the quantifiers $\forall$ and $\exists$, and a denumerable set of individual variables, denoted by $x, y, z, \ldots$. The set of terms and the set of well formed formulas (formulas for short) of the language $\mathcal{L}_{\mathcal{A}}$, based on the extra-logical alphabet $\mathcal{A}$, are defined as usual. We will also write $A \in \mathcal{L}_{\mathcal{A}}$ to mean that A is a formula of the language $\mathcal{L}_{\mathcal{A}}$. We denote with $\mathcal{L}$ the *pure first order language* based on the extra-logical alphabet consisting, for every $n \geq 0$, of a denumerable set of n-ary predicate variables $p_j^{(n)}, q_j^{(n)}, \ldots$. We will use lower case Latin letters t, s, t', s' and upper case Latin letters $A, B, C, \ldots$, possibly with indexes, to denote terms and formulas respectively. Upper case Greek letters $\Gamma, \Delta, \Theta, \ldots$ (possibly with indexes) will be used to denote sets of formulas.

The notions of *free* and *bounded* individual variable, of *closed* and *open* term and formula are defined as usual. Notations such as $A(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n)$ (with $n \geq 1$) will indicate that $x_1, \ldots, x_n$ may occur free in the formula A and in the term t respectively, while FV(A) will indicate the set of all the free variables occurring in the formula A.

A *substitution of individual variables* is any function θ from the set of all the individual variables to the set of terms of the language in use. Given any individual substitution θ and any formula A (any term t), we will denote with θA (θt) the expression obtained by substituting each free occurrence of any free variable in A (in t) with the term associated with it by the substitution θ. If $A(x_1, \ldots, x_n)$ is a formula and $t_1, \ldots, t_n$ are terms of the language, we write $A(t_1/x_1, \ldots, t_n/x_n)$ to mean the formula obtained by simultaneously substituting any occurrence of x_1 in A with the term t_1 and ...and any occurrence of x_n in A with the term t_n. Using the substitution applied to formulas, we tacitly assume the terms to be free for the variables of the formula (similarly, since we regard formulas differing only in the name of the bounded variables as isomorphic, we can assume that a suitable renaming of bounded variables is carried on). If Γ is a set of formulas, $\theta\Gamma$ will denote the set containing the formula θA for any $A \in \Gamma$. Finally, if θ associates with every variable a closed term of the language, we say that θA is a *closed instance* of A.

We define the *degree of a formula* A as usual. That is:

1. $\mathrm{dg}(A) = 1$ if either A is atomic or $A \equiv \bot$;
2. $\mathrm{dg}(\neg A) = \mathrm{dg}(A) + 1$;
3. $\mathrm{dg}(A) = \mathrm{Max}\{\mathrm{dg}(B), \mathrm{dg}(C)\} + 1$ if A is $B \wedge C$, $B \vee C$ or $B \Rightarrow C$;
4. $\mathrm{dg}(A) = \mathrm{dg}(B) + 1$ if A is either $\exists x B(x)$ or $\forall x B(x)$.

The *degree of a set of formulas* Γ is:

$$\mathrm{dg}(\Gamma) = \mathrm{Max}\{\mathrm{dg}(A) : A \in \Gamma\} .$$

Finally, we remark that in Chapter 2 we will use the notions of recursive set and recursively enumerable set, and some fundamental results of recursion theory, without giving an explicit definition of these concepts. However, the reader can find a detailed presentation of all this material in Section 5.1.

Now, we present the calculi which we will use in this thesis.

1.2.1 Hilbert calculi

Here, we present the Hilbert-style calculi for intuitionistic and classical first-order logics. A detailed description of these kinds of calculi can be found in [Kleene, 1952, Troelstra and van Dalen, 1988a]. The Hilbert-style calculus $\mathcal{H}_{\mathbf{INT}}$ for intuitionistic logic consists of the following axioms and rules.

Axioms for conjunction:

$Ax1$ $A \wedge B \Rightarrow A$
$Ax2$ $A \wedge B \Rightarrow B$
$Ax3$ $A \Rightarrow (B \Rightarrow (A \wedge B))$

Axioms for disjunction:

$Ax4$ $A \Rightarrow A \vee B$
$Ax5$ $B \Rightarrow A \vee B$
$Ax6$ $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C))$

Axioms for implication:

$Ax7$ $\quad A \Rightarrow (B \Rightarrow A)$
$Ax8$ $\quad (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

Axiom for intuitionistic contradiction:

$Ax9$ $\quad \bot \Rightarrow A$
$Ax10$ $\quad A \wedge \neg A \Rightarrow \bot$

Axioms for existential quantifier:

$Ax11$ $\quad A(t/x) \Rightarrow \exists x A(x)$
$Ax12$ $\quad \forall x(A(x) \Rightarrow B) \Rightarrow (\exists x A(x) \Rightarrow B)$

where in the last axiom $x \notin$ FV(B).

Axioms for universal quantifier:

$Ax13$ $\quad \forall x A(x) \Rightarrow A(t/x)$
$Ax14$ $\quad \forall x(B \Rightarrow A(x)) \Rightarrow (B \Rightarrow \forall x A(x))$

where in the last axiom $x \notin$ FV(B).

Rules:

$$\frac{A \qquad A \Rightarrow B}{B}\,\textsc{Mp} \qquad\qquad \frac{A(x)}{\forall x A(x)}\,\textsc{Gen}$$

An Hilbert-style calculus $\mathcal{H}_{\mathbf{CL}}$ for classical logic can be obtained by replacing the axiom for intuitionistic contradiction (A9) with the axiom

Axiom for elimination of double negation:

$Ax15$ $\quad \neg\neg A \Rightarrow A$

A *proof* of $\mathcal{H}_{\mathbf{INT}}$ (of $\mathcal{H}_{\mathbf{CL}}$) is any finite sequence of formulas $B_1, \dots, B_n$ such that, for any i with $i = 1, \dots, n$, either B_i is an instance of an axiom scheme or it is obtained by applying Mp to two formulas $A \Rightarrow B$ and A which occur before in the sequence, or it is obtained by applying the rule Gen to a formula $A(x)$ which occurs before in the sequence. If $B_1, \dots, B_n$ is a proof in $\mathcal{H}_{\mathbf{INT}}$ ($\mathcal{H}_{\mathbf{CL}}$) we say that the formula B_n is *provable in $\mathcal{H}_{\mathbf{INT}}$ ($\mathcal{H}_{\mathbf{CL}}$)* and we write $\vdash_{\mathcal{H}_{\mathbf{INT}}} B_n$ ($\vdash_{\mathcal{H}_{\mathbf{CL}}} B_n$).

If we also permit assumptions, we say that A *is provable in $\mathcal{H}_{\mathbf{INT}}$ ($\mathcal{H}$) from* Γ (where Γ is a finite or infinite set of formulas), and we write $\Gamma \vdash_{\mathcal{H}_{\mathbf{INT}}} A$ ($\Gamma \vdash_{\mathcal{H}_{\mathbf{CL}}} A$), if there exists a finite sequence $B_1, \dots, B_n$ of formulas such that: for any i with $i = 1, \dots, n$, either B_i is an instance of an axiom scheme, or $B_i \in \Gamma$, or or it is obtained by applying Mp to two formulas $A \Rightarrow B$ and A which occur before in the sequence, or it is obtained by applying the rule GEN to a formula $A(x)$ which occurs before in the sequence, with the restriction that $x \notin \mathrm{FV}(\Gamma)$.

A *subproof* of a proof $\lambda \equiv B_1, \ldots, B_n$ in $\mathcal{H}_{\mathbf{INT}}$ (with assumptions or not) is any subsequence $C_1, \ldots, C_m$ of λ which is a proof of $\mathcal{H}_{\mathbf{INT}}$ ($\mathcal{H}_{\mathbf{CL}}$). The *degree* $\mathrm{dg}(\lambda)$ *of an Hilbert-style proof* $\lambda \equiv B_1, \ldots, B_n$ of the formula B_n from Γ is the maximum between the degrees of the formulas in λ. That is

$$\mathrm{dg}(\lambda) = \mathrm{Max}\{\mathrm{dg}(B_1), \ldots, \mathrm{dg}(B_n)\} \ .$$

1.2.2 Natural Deduction

We present here a syntactic variant (inspired to the one presented in [Gallier, 1991]) of the natural deduction systems for intuitionistic and classical first order logic due to Gentzen [Gentzen, 1969] and Prawitz [Prawitz, 1965]. Here, the logical alphabet does not include the connective $\neg$, that is $\neg A$ is taken as an abbreviation for $A \Rightarrow \bot$.

We call *sequent* an expression of the kind $\Gamma \vdash A$, where A is a formula and Γ is a finite set of formulas. For the sake of simplicity, we use the following conventions: $\Gamma, \Delta \vdash A$ abbreviates $\Gamma \cup \Delta \vdash A$, $A \vdash B$ abbreviates $\{A\} \vdash B$ and $\vdash A$ abbreviates $\emptyset \vdash A$. Moreover, we call *initial sequent* or *axiom sequent* any sequent of the form $A \vdash A$.

An *inference* is an expression of the form

$$\frac{\sigma_1, \ldots, \sigma_n}{\sigma}$$

where $\sigma_1, \ldots, \sigma_n, \sigma$ (with $n \geq 0$) are sequents. We say that inference rules with $n > 0$ are *proper* inference rules, while the ones with $n = 0$ are *improper* inference rules. We call $\sigma_1, \ldots, \sigma_n$ the *upper sequents* and σ the *lower sequent* of the inference. Intuitively, this means that, when $\sigma_1, \ldots, \sigma_n$ are asserted, we can infer σ from them.

Now, let $A, B, C, A(x)$ be any formulas and let Γ, Δ and Θ be *finite* sets of formulas. Here, we introduce the inference rules of the *natural deduction calculus* $\mathcal{ND}_{\mathbf{INT}}$ *for intuitionistic logic* :

Assumption Introduction:

$$\frac{}{A \vdash A}$$

Weakening Left:

$$\frac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}\,\scriptstyle{\mathrm{W-l}}$$

Logical Rules:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\scriptstyle{\mathrm{I}\wedge} \qquad\qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\scriptstyle{\mathrm{E}\wedge} \qquad\qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\scriptstyle{\mathrm{E}\wedge}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\mathrm{I}\vee \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\mathrm{I}\vee \qquad \frac{\Gamma \vdash A \vee B \qquad \Delta, A \vdash C \qquad \Theta, B \vdash C}{\Gamma, \Delta, \Theta \vdash C}\,\mathrm{E}\vee$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\,\mathrm{I}\Rightarrow \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash A \Rightarrow B}{\Gamma, \Delta \vdash B}\,\mathrm{E}\Rightarrow$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\,\bot_{\mathrm{INT}}$$

$$\frac{\Gamma \vdash A(y/x)}{\Gamma \vdash \forall x A(x)}\,\mathrm{I}\forall \qquad \frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(t/x)}\,\mathrm{E}\forall$$

where, in I$\forall$, y does not occur free in Γ or $\forall x A(x)$. We call y the *proper parameter of the* I$\forall$ *rule* .

$$\frac{\Gamma \vdash A(t/x)}{\Gamma \vdash \exists x A(x)}\,\mathrm{I}\exists \qquad \frac{\Gamma \vdash \exists x A(x) \qquad \Delta, A(y/x) \vdash C}{\Gamma, \Delta \vdash C}\,\mathrm{E}\exists$$

where, in E$\exists$, y does not occur free in Δ, $\exists x A(x)$ or C. We call y the *proper parameter of the* E$\exists$ *rule*.

We say that π is a *proof in* $\mathcal{ND}_{\mathbf{INT}}$ if π is a tree of sequents satisfying the following conditions:

1. The topmost sequents of π are initial sequents (introduced by means of an assumption introduction);

2. Every sequent in π except the lowest one is an upper sequent of one of the proper inference rules listed above, whose lower sequent is also in π.

We call the lowest sequent of a proof π the *end-sequent*. We say that π is a proof of A from Γ in $\mathcal{ND}_{\mathbf{INT}}$ and we write

$$\Gamma \vdash_{\mathcal{ND}_{\mathbf{INT}}} A\ ,$$

if the end-sequent of π is $\Gamma \vdash A$.

We define the notions of *depth* of a proof-tree π , we denote with depth(π), and the notion of *subproof* of a proof-tree in the obvious usual way. Moreover, we will denote with dg(π) the *degree* of a proof π of $\mathcal{ND}_{\mathbf{INT}}$, defined as the maximum between the degrees of the formulas which belong to sequents of the proof.

The *natural deduction calculus $\mathcal{ND}_{\mathbf{CL}}$ for classical first-order logic* is obtained by replacing the rule $\bot_{\mathrm{INT}}$ of the calculus $\mathcal{ND}_{\mathbf{INT}}$ with the rule:

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}\,\bot_{\mathrm{CL}}$$

The notion of proof in $\mathcal{ND}_{\mathbf{CL}}$ and the meaning $\Gamma \vdash_{\mathcal{ND}_{\mathbf{CL}}} A$ are defined in a way quite similar to the corresponding cases for $\mathcal{ND}_{\mathbf{INT}}$.

Now, we call *free variables of a proof* all the variables which occur free in some formula of the proof and which are not proper parameters. It is well known that proper parameters can always be chosen in such a way that the following two conditions are satisfied:

(I). Every proper parameter in a proof π is a proper parameter of exactly one rule.

(II). The set of proper parameters is disjoint from the set of free variables of a proof.

These conventions on proper parameters will hold also when we will introduce other rules with proper parameters, such as the Grzegorczyck Principle, Induction Rule and the Descending Chain Rule of Chapter 4.

For any proof satisfying the above conventions, the tree-structure obtained by replacing some of the free variables of the proof with terms is a well defined proof. We will write $\pi(t/x)$ to denote the proof obtained by substituting all the occurrences of the free variable x in the proof π with the term t, and $\theta\pi$ to denote the proof obtained by applying the substitution θ to the proof π.

Now, it is a well known fact that the calculi $\mathcal{ND}_{\mathbf{INT}}$ and $\mathcal{H}_{\mathbf{INT}}$ are equivalent (see e.g. [Troelstra, 1973a, Troelstra and van Dalen, 1988a]). That is, the following theorem holds:

1.2.1 Theorem $\Gamma \vdash_{\mathcal{H}_{\mathbf{INT}}} A$ *iff* $\Gamma \vdash_{\mathcal{ND}_{\mathbf{INT}}} A$. □

The proof of this theorem amounts to exhibit a map translating any proof of one of the two calculi into a proof of the other. We are particularly interested in the map translating proofs of $\mathcal{ND}_{\mathbf{INT}}$ into proofs of $\mathcal{H}_{\mathbf{INT}}$ and in the relation between the degrees of the proofs involved in this translation. For this reason we prove only one half of the previous theorem in the following form:

1.2.2 Theorem *There exist a map Ξ associating, with any proof of $\mathcal{ND}_{\mathbf{INT}}$, a proof of $\mathcal{H}_{\mathbf{INT}}$, and a linear function $\phi : \mathbf{N} \to \mathbf{N}$ such that:*

(i). For any $\pi \in \mathcal{ND}_{\mathbf{INT}}$, if $\Gamma \vdash A$ is the end-sequent of π and $\Xi(\pi) = \lambda$, then λ is an $\mathcal{H}_{\mathbf{INT}}$-proof of A from Γ;

(ii). For any $\pi \in \mathcal{ND}_{\mathbf{INT}}$, $\mathrm{dg}(\Xi(\pi)) \leq \phi(\mathrm{dg}(\pi))$.

Proof: Let us assume $\phi(x) = x + 4$. We begin to prove that there exists a map Ξ' associating, with any proof $\pi : \Gamma \vdash A$ of $\mathcal{ND}_{\mathbf{INT}}$ and with any formula H, a $\mathcal{H}_{\mathbf{INT}}$-proof λ of $H \Rightarrow A$ from $\Gamma' = \Gamma \setminus \{H\}$, such that $\mathrm{dg}(\lambda) \leq \mathrm{Max}\{\mathrm{dg}(H), \mathrm{dg}(\pi)\} + 4$. The proof proceeds by induction on the depth of the proof-tree π with end-sequent $\Gamma \vdash A$.

Basis: If $\mathrm{depth}(\pi) = 0$, then the only rule applied in π is an assumption introduction and hence $\Gamma \vdash A \equiv A \vdash A$. We have two cases. If $H \equiv A$, then the proof $\Xi'(\pi, H)$ is

1	$(A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))$	$Ax8$
2	$A \Rightarrow ((A \Rightarrow A) \Rightarrow A)$	$Ax7$
3	$(A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)$	Mp 1, 2
4	$A \Rightarrow (A \Rightarrow A)$	$Ax7$
5	$A \Rightarrow A$	Mp 4, 5

with $\mathrm{dg}(\lambda) = \mathrm{dg}(A) + 4 = \mathrm{dg}(\pi) + 4$. In the other case $H \not\equiv A$ and the proof is the following:

1	A	
2	$A \Rightarrow (H \Rightarrow A)$	$Ax7$
3	$H \Rightarrow A$	Mp 1, 2

with $\mathrm{dg}(\lambda) = \mathrm{Max}\{\mathrm{dg}(H), \mathrm{dg}(\pi)\} + 2$.

Step: If $\mathrm{depth}(\pi) = h + 1$, then the proof-tree π is of the kind:

$$\frac{\pi_1 \ \ldots \ \pi_n}{\Gamma \vdash A} \ ,$$

where $\pi_1, \ldots, \pi_n$ are proof-trees such that, for any i with $i = 1, \ldots, n$, $\mathrm{depth}(\pi_i) \leq h$. If $\Gamma_i \vdash A_i$ is the end-sequent of the proof-tree π_i, we have, by induction hypothesis, that, for any formula H' $\lambda_i = \Xi'(\pi_i, H')$ is a $\mathcal{H}_{\mathbf{INT}}$-proof of $H' \Rightarrow A_i$ from $\Gamma_i \setminus \{H'\}$ with $\mathrm{dg}(\lambda_i) \leq \mathrm{Max}\{\mathrm{dg}(H'), \mathrm{dg}(\pi_i)\} + 4$. Now, the proof goes by cases according to the last rule applied in π. Since in full details the proof is rather cumbersome, we only develop an illustrative case:

- **disjunction elimination**: That is, π is of the kind

$$\frac{\pi_1 : \Gamma_1 \vdash A \vee B \qquad \pi_2 : \Gamma_2, A \vdash C \qquad \pi_3 : \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash C} \mathrm{E}\vee$$

  The induction hypothesis provides us with the $\mathcal{H}_{\mathbf{INT}}$-proofs

  - $\lambda_1 = \Xi'(\pi_1, H)$ of $H \Rightarrow A \vee B$ from $\Gamma_1 \setminus \{H\}$;
  - $\lambda_2 = \Xi'(\pi_2, A)$ of $A \Rightarrow C$ from Γ_2;
  - $\lambda_3 = \Xi'(\pi_3, B)$ of $B \Rightarrow C$ from Γ_3.

with the appropriate relations on the degrees. Now, let $\lambda = \Xi'(\pi, H)$ be the following proof:

	λ_1	
1	$H \Rightarrow A \vee B$	
	λ_2	
2	$A \Rightarrow C$	
	λ_3	
3	$B \Rightarrow C$	
4	$(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C))$	$Ax6$
5	$(B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C)$	MP $2, 4$
6	$A \vee B \Rightarrow C$	MP $3, 5$
7	$(A \vee B \Rightarrow C) \Rightarrow (H \Rightarrow (A \vee B \Rightarrow C))$	$Ax7$
8	$H \Rightarrow (A \vee B \Rightarrow C)$	MP $6, 7$
9	$(H \Rightarrow (A \vee B \Rightarrow C)) \Rightarrow ((H \Rightarrow A \vee B) \Rightarrow (H \Rightarrow C))$	$Ax8$
10	$(H \Rightarrow A \vee B) \Rightarrow (H \Rightarrow C)$	MP $8, 9$
11	$H \Rightarrow C$	MP $1, 10$

Moreover, it is easy to verify that

$$\begin{aligned} \mathrm{dg}(\lambda) &= \mathrm{Max}\{\mathrm{dg}(\lambda_1), \mathrm{dg}(\lambda_2), \mathrm{dg}(\lambda_3), \mathrm{Max}\{\mathrm{dg}(H), \mathrm{dg}(A \vee B), \mathrm{dg}(C)\} + 3\} \\ &\leq \mathrm{Max}\{\mathrm{dg}(H), \mathrm{dg}(\pi)\} + 4 \end{aligned}$$

Now, we can prove the theorem. Let $\pi : \Gamma \vdash A$ be any proof in $\mathcal{ND}_{\mathbf{INT}}$. By the previous proof we can build on an $\mathcal{H}_{\mathbf{INT}}$-proof λ' of $(A \Rightarrow A) \Rightarrow A$ from $\Gamma \setminus \{A \Rightarrow A\}$ such that

$$\mathrm{dg}(\lambda') \leq \mathrm{Max}\{\mathrm{dg}(A \Rightarrow A), \mathrm{dg}(\pi)\} + 4 \ .$$

But it is easy to verify, by an inspection of the first of the three $\mathcal{H}_{\mathbf{INT}}$-proofs constructed above, that there exists an $\mathcal{H}_{\mathbf{INT}}$-proof λ'' of $A \Rightarrow A$ such that $\mathrm{dg}(\lambda'') = \mathrm{dg}(A) + 4$. Thus, the following is a proof of A from Γ:

	λ'	
1	$(A \Rightarrow A) \Rightarrow A$	
	λ''	
2	$A \Rightarrow A$	
3	A	MP $1, 2$

and

$$\begin{aligned} \mathrm{dg}(\lambda) &= \mathrm{Max}\{\mathrm{dg}(\lambda'), \mathrm{dg}(\lambda'')\} \\ &\leq \mathrm{Max}\{\mathrm{dg}(\pi) + 4, \mathrm{dg}(A) + 4\} \ . \end{aligned}$$

Since A occurs in π, this means $\mathrm{dg}(\lambda) \leq \phi(\mathrm{dg}(\pi))$.

□

We conclude this section by remarking that the previous proof guarantees that there exists an effective procedure Ξ for translating proofs of natural deduction into proofs of the Hilbert-style calculus for intuitionistic logic. Moreover, this translation has the property that there exists a linear function $\phi : \mathbf{N} \to \mathbf{N}$ relating the degree of a proof π of $\mathcal{ND}_{\mathbf{INT}}$ to the degree of the corresponding proof $\Xi(\pi)$ of $\mathcal{H}_{\mathbf{INT}}$. A map with the same properties can also be given to translate proofs of $\mathcal{H}_{\mathbf{INT}}$ into proofs of $\mathcal{ND}_{\mathbf{INT}}$.

Similarly, there exist translations of this kind between the calculi $\mathcal{ND}_{\mathbf{INT}}$ and $\mathcal{SEQ}_{\mathbf{INT}}$, where $\mathcal{SEQ}_{\mathbf{INT}}$ is the standard Gentzen sequent calculus for intuitionistic logic (see [Takeuti, 1975]). A translation between these two calculi is exhibited for example in [Gallier, 1991]. Translations with the same properties can also be given for the classical variants of these calculi.

Chapter 2

Fundamentals

2.1 Formal systems

Traditionally, a *formal system* (or what is usually called in literature *logical system*) is defined by means of a pair $(\mathcal{L}_\mathcal{A}, \mid\!\sim)$, where $\mathcal{L}_\mathcal{A}$ is a (first order) language with extra-logical alphabet $\mathcal{A}$ and $\mid\!\sim$ is a consequence relation satisfying some minimal properties. In this Thesis we are interested in studying formal systems determined by proof systems, being concerned with the problem of extracting information from proofs. To this aim we distinguish a *formal system* from a *calculus* for it. Our characterization of formal systems has something similar to the one given in [Gabbay, 1994]. We will characterize formal systems as triples $(\mathcal{L}_\mathcal{A}, \mid\!\sim, \mathbf{H}_{\mid\sim})$, where $\mathcal{L}_\mathcal{A}$ is a first order language, $\mid\!\sim$ is a mathematically defined consequence relation satisfying certain minimal conditions, and $\mathbf{H}_{\mid\sim}$ is an algorithmic theory (a proof theory) for generating it. We have in mind the following *intuitive interpretation* for this triple: the relation $\mid\!\sim$ establishes a relation between sets of formulas and formulas of $\mathcal{L}_\mathcal{A}$. It determines the information contained in the whole formal system. $\mathbf{H}_{\mid\sim}$ is a strategy to obtain the information in an algorithmic way. Differently from [Gabbay, 1994], we are not interested in $\mathbf{H}_{\mid\sim}$ *as a tool to get efficient Theorem Provers, but as a tool to effectively extract the information from proofs.* Here we introduce the notion of derivability relation:

2.1.1 Definition (Derivability relation) *Let $\mathcal{L}_\mathcal{A}$ be a (first order language). A* derivability relation *over $\mathcal{L}_\mathcal{A}$ is a relation $\mid\!\sim \subseteq \mathrm{Pow}(\mathcal{L}_\mathcal{A}) \times \mathcal{L}_\mathcal{A}$ satisfying the following properties:*

1. **Reflexivity**: *If $A \in \Gamma$, then $\Gamma \mid\!\sim A$;*

2. **Transitivity**: *If* $\Gamma \mid\!\sim A$ *for any* $A \in \Delta$ *and* $\Delta \mid\!\sim B$, *then* $\Gamma \mid\!\sim B$;

3. **Compactness**: *If* $\Gamma \mid\!\sim A$, *then there exists* $\Gamma_0 \subseteq_{\text{fin}} \Gamma$ *such that* $\Gamma_0 \mid\!\sim A$.

We call $\mid\!\sim$ *derivability relation* instead of *consequence relation* , how usually done in literature, to point out the fact that we will be interested in studying consequence relations generated by proof systems. This is the reason why we assume compactness as a fundamental property of derivability relations. This property is not considered fundamental for the general notion of a consequence relation, see e.g. [Gabbay, 1994, Wójcicki, 1988], because compactness rules out many formal systems defined by model theoretic methods.

2.1.2 Remark In literature derivability relations are often required to satisfy the following conditions:

1'. **Reflexivity**: If $A \in \Gamma$ then $\Gamma \mid\!\sim A$;

2'. **Monotonicity**: $\Gamma \mid\!\sim A$ implies $\Gamma, \Delta \mid\!\sim A$;

3'. **Cut**: $\Gamma \mid\!\sim A$ and $\Gamma, A \mid\!\sim B$ implies $\Gamma \mid\!\sim B$.

It is trivial to verify that reflexivity and transitivity allow to derive monotonicity and cut. Moreover, if the derivability relation is finitary (compact) and reflexivity holds, it is easy to prove that also the converse holds.

Given a derivability relation $\mid\!\sim$ over $\mathcal{L}_\mathcal{A}$, we say that $\Gamma \subseteq \mathcal{L}_\mathcal{A}$ is a $\mid\!\sim$-*theory*, and we write $\Gamma \in \mathrm{TH}(\mid\!\sim)$, if it is closed under the derivability relation $\mid\!\sim$; i.e. $A \in \Gamma$ whenever $\Delta \mid\!\sim A$ with $\Delta \subseteq \Gamma$. It is trivial to prove that, given a set $\Gamma \subseteq \mathcal{L}_\mathcal{A}$, the set $\{A : \Gamma \mid\!\sim A\}$ is a $\mid\!\sim$-theory. In particular, the set

$$\mathrm{Theo}(\mid\!\sim) = \{A : \ \mid\!\sim A\} \ ,$$

is a $\mid\!\sim$-theory, called the *base theory of* $\mid\!\sim$.

A useful fact about $\mathrm{TH}(\mid\!\sim)$ is that it is a *closure system*, that is, for any subset $\mathbf{X}$ of $\mathrm{TH}(\mid\!\sim)$, $\cap\mathbf{X} \in \mathrm{TH}(\mid\!\sim)$. This implies that $\mathrm{TH}(\mid\!\sim)$ is a complete lattice under $\subseteq$ with respect to:

$$\begin{aligned} \inf \mathbf{X} &= \cap\mathbf{X} \\ \sup \mathbf{X} &= \inf\{\Gamma \in \mathrm{TH}(\mid\!\sim) : \cup\mathbf{X} \subseteq \Gamma\} \end{aligned}$$

Note that $\mathrm{Theo}(\mid\!\sim)$ and $\mathcal{L}_\mathcal{A}$ are the least and the greatest element of $\mathrm{TH}(\mid\!\sim)$ respectively.

It is important to notice that a derivability relation is uniquely determined by the set of its theories:

2.1.3 Proposition *If* $\mid\!\sim_1$ *and* $\mid\!\sim_2$ *are two derivability relations over* $\mathcal{L}_\mathcal{A}$, *then*

$$\mid\!\sim_1 = \mid\!\sim_2 \quad \textit{iff} \quad \mathrm{TH}(\mid\!\sim_1) = \mathrm{TH}(\mid\!\sim_2) \ .$$

Proof: The "if" part is trivial. For the converse, let us assume that $\Gamma \mid\!\sim_1 A$ but $\Gamma \not\mid\!\sim_2 A$. Then, the set $\Delta = \{A : \Gamma \mid\!\sim_2 A\}$ is a $\mid\!\sim_2$-theory; moreover, $\Gamma \subseteq \Delta$ and $A \notin \Delta$, which implies that $\Delta \notin \mathrm{TH}(\mid\!\sim_1)$. □

Now, given a derivability relation $\mid\!\sim$ over the language $\mathcal{L}_{\mathcal{A}}$, following [Avron, 1991], we say that:

- $\mid\!\sim$ has an *internal implication* if there exists a binary connective $\#$ in $\mathcal{L}_{\mathcal{A}}$ such that, for all $A, B \in \mathcal{L}_{\mathcal{A}}$ and for every $\Gamma \subseteq \mathcal{L}_{\mathcal{A}}$:

$$\Gamma \mid\!\sim A\#B \quad \text{iff} \quad \Gamma, A \mid\!\sim B \ .$$

- $\mid\!\sim$ has a *combining conjunction* if there exists a binary connective & in $\mathcal{L}_{\mathcal{A}}$ such that, for all $A, B \in \mathcal{L}_{\mathcal{A}}$ and any $\Gamma \subseteq \mathcal{L}_{\mathcal{A}}$:

$$\Gamma \mid\!\sim A\&B \quad \text{iff} \quad \Gamma \mid\!\sim A \quad \text{and} \quad \Gamma \mid\!\sim B \ .$$

A consequence relation having an internal implication is also said to have the *deduction property*. This is obviously an important property, because it establishes a bridge between our understanding of the implication inner to the derivability relation and the implication of the language. The main property of the derivability relations with an internal implication is that they are uniquely determined by their base theories.

2.1.4 Proposition *For any two derivability relations $\mid\!\sim_1$, $\mid\!\sim_2$ over a language $\mathcal{L}_{\mathcal{A}}$ with an internal implication $\#$,*

$$\mid\!\sim_1 = \mid\!\sim_2 \quad \textit{iff} \quad \mathrm{Theo}(\mid\!\sim_1) = \mathrm{Theo}(\mid\!\sim_2) \ .$$

Proof: The "only if" part is trivial. For the converse, let us assume that $\Gamma \mid\!\sim_1 A$. Then, by compactness, there exists $\{B_1, \ldots, B_n\} \subseteq \Gamma$ such that $\{B_1, \ldots, B_n\} \mid\!\sim_1 A$. Now, applying the deduction property, it is easy to verify that

$$B_1\#(B_1\# \ldots \#(B_n\#A)) \in \mathrm{Theo}(\mid\!\sim_1) \ .$$

But $\mathrm{Theo}(\mid\!\sim_1) = \mathrm{Theo}(\mid\!\sim_2)$, and hence

$$\mid\!\sim_2 B_1\#(B_1\# \ldots \#(B_n\#A)) \ .$$

Since $\#$ is an internal implication for $\mid\!\sim_2$, it is easy to deduce $\{B_1, \ldots, B_n\} \mid\!\sim_2 A$, and, applying reflexivity and transitivity, we deduce $\Gamma \mid\!\sim_2 A$. □

We are interested in characterizing formal systems by means of proof systems (or calculi). A usual way to present formal systems is by means of axiomatic or Hilbert-style systems. An *Hilbert-style system* (or also *axiomatic system* over a language $\mathcal{L}_{\mathcal{A}}$ is a pair

$$\mathcal{H} = (\mathbf{H}_0, \mathbf{H}_1)$$

where

- $\mathbf{H}_0$ is a recursive set of formulas, called *axioms*;

- $\mathbf{H}_1$ is a recursive set of rules of the form $A_1, \dots, A_n/B$, called *inference rules*.

The *provability relation* determined by an Hilbert-style system $\mathcal{H}$ over a language $\mathcal{L}_{\mathcal{A}}$ is the unary relation $\vdash_{\mathcal{H}} \subseteq \mathcal{L}_{\mathcal{A}}$ defined as follows:

$\vdash_{\mathcal{H}} A$ iff there exists a finite sequence $B_1, \dots, B_n$ of formulas of $\mathcal{L}_{\mathcal{A}}$ such that $B_k \equiv A$ and, for any i with $i = 1, \dots, n$, B_i is either an axiom of $\mathbf{H}_0$ or, for some formulas $C_1, \dots, C_m \in \{B_1, \dots, B_{i-1}\}$, $C_1, \dots, C_m/B_i$ is a rule in $\mathbf{H}_1$.

We call the sequence $B_1, \dots, B_n$ a *proof of A in $\mathcal{H}$*; moreover, if $\vdash_{\mathcal{H}} A$, we say that A *is provable* in $\mathcal{H}$. The set of theorems of $\mathcal{H}$ is the set of all the formulas of the language $\mathcal{L}_{\mathcal{A}}$ which are provable in $\mathcal{H}$. That is:

$$\mathrm{Theo}(\mathcal{H}) = \{A \in \mathcal{L}_{\mathcal{A}} : \vdash_{\mathcal{H}} A\} .$$

There is more than one way to designate the derivability relation associated with an axiomatic system; here we use the *extension method* (see [Avron, 1991]). Given an Hilbert-style system $\mathcal{H} = (\mathbf{H}_0, \mathbf{H}_1)$ over $\mathcal{L}_{\mathcal{A}}$, we define the derivability relation $\mid\!\sim_{\mathcal{H}}$ (over $\mathcal{L}_{\mathcal{A}}$) as follows:

$\Gamma \mid\!\sim_{\mathcal{H}} A$ iff there exists a sequence of formulas $B_1, \dots, B_n$ of $\mathcal{L}_{\mathcal{A}}$ such that $B_n \equiv A$ and, for any i with $i = 1, \dots, n$, either (a) $B_i \in \Gamma$ or (b) $\vdash_{\mathcal{H}} B_i$ or (c) there exist $C_1, \dots, C_m \in \{B_1, \dots, B_{i-1}\}$ such that $C_1, \dots, C_m/B_i$ is a rule of $\mathbf{H}_1$.

It is trivial to prove that $\mid\!\sim_{\mathcal{H}}$ is a derivability relation over $\mathcal{L}_{\mathcal{A}}$. Now we will study under which conditions a derivability relation $\mid\!\sim$ can be characterized by an Hilbert-style system.

2.1.5 Proposition *Let $\mid\!\sim$ be a derivability relation over $\mathcal{L}_{\mathcal{A}}$ having an internal implication $\#$ and a combining conjunction $\&$. If* $\mathrm{Theo}(\mid\!\sim)$ *is recursively enumerable, then there exists an axiomatic system $\mathcal{H}^{\mid\sim}$ such that:*

1. *$\&$ is a combining conjunction for $\mid\!\sim_{\mathcal{H}^{\mid\sim}}$;*
2. *$\#$ is an internal implication for $\mid\!\sim_{\mathcal{H}^{\mid\sim}}$;*
3. *$\mid\!\sim_{\mathcal{H}^{\mid\sim}} = \mid\!\sim$.*

Proof: Let us consider a recursive enumeration $\{B_i\}_{i\in\omega}$ of $\mathrm{Theo}(\mid\!\sim)$, and let us consider the sequence of formulas $\{A_i\}_{i\in\omega}$ where:

$$\begin{aligned} A_0 &\equiv B_0 \\ A_{i+1} &\equiv A_i \& B_{i+1} \end{aligned}$$

It is easy to prove that the set of the formulas of the sequence $\{A_i\}_{i\in\omega}$ is recursive. Now, let:

- $\mathbf{H}_0^{\succ} = \{B : \text{ there exists } j \in \omega \text{ such that } B \equiv A_j\}$;
- $\mathbf{H}_1^{\succ}$ be the set of all rules of the form:

$$\begin{array}{ll} (r1) & A\&B/A \\ (r2) & A\&B/B \\ (r3) & A\#B, A/B \ . \end{array}$$

$\mathbf{H}_1^{\succ}$ is evidently recursive. Let

$$\mathcal{H}^{\succ} = (\mathbf{H}_0^{\succ}, \mathbf{H}_1^{\succ}) \ .$$

First of all, we prove that $\mathrm{Theo}(\succ) = \mathrm{Theo}(\succ_{\mathcal{H}^{\succ}})$. If $\emptyset \succ B$, then the formula B belongs to the enumeration $\{B_i\}_{i\in\omega}$. Let j be the positive integer such that $B \equiv B_j$; if $j = 0$ then $B \in \mathbf{H}_0^{\succ}$. Otherwise the axiom $A_j \equiv A_{j-1}\&B_j$ belongs to $\mathbf{H}_0^{\succ}$, hence the sequence A_j, B_j, where B_j is obtained by applying the rule $A_{j-1}\&B_j/B_j$, is a proof of B is $\mathcal{H}^{\succ}$. Therefore, $B \in \mathrm{Theo}(\succ_{\mathcal{H}^{\succ}})$. The converse follows from a straightforward induction on the sequence of formulas which determines $\vdash_{\mathcal{H}^{\succ}} B$. Moreover, still by a straightforward induction on the sequence of formulas determining $\Gamma \succ_{\mathcal{H}^{\succ}} B$, we have that $\Gamma \succ_{\mathcal{H}^{\succ}} B$ implies $\Gamma \succ B$. Now, we prove the assertions.
1) Let $\Gamma \succ_{\mathcal{H}^{\succ}} A$ and $\Gamma \succ_{\mathcal{H}^{\succ}} B$. Then, since $\succ_{\mathcal{H}^{\succ}}$ is a derivability relation, there exists $\Gamma_0 \subseteq_{\mathrm{fin}} \Gamma$ such that $\Gamma_0 \succ_{\mathcal{H}^{\succ}} A$ and $\Gamma_0 \succ_{\mathcal{H}^{\succ}} B$. This implies, by the above discussion, that $\Gamma_0 \succ A$ and $\Gamma_0 \succ B$. Since & is a combining conjunction for $\succ$, we have that $\Gamma_0 \succ A\&B$. Moreover, if $\Gamma_0 = \{C_1, \ldots, C_k\}$, by the fact that # is an internal implication for $\succ$, we deduce that

$$\succ C_1\#(C_2\#\ldots\#(C_k\#(A\&B)) \ .$$

This implies that the considered formula belongs to $\mathrm{Theo}(\succ_{\mathcal{H}^{\succ}})$. But, by reflexivity, $\Gamma \succ_{\mathcal{H}^{\succ}} C_i$ for any i such that $i = 1, \ldots, k$, and, by transitivity,

$$\Gamma \succ_{\mathcal{H}^{\succ}} C_1\#(C_2\#\ldots\#(C_k\#(A\&B)) \ .$$

Now, using the rule $(r3)$, it is easy to prove that $\Gamma \succ_{\mathcal{H}^{\succ}} A\&B$. The proof that $\Gamma \succ_{\mathcal{H}^{\succ}} A\&B$ implies $\Gamma \succ_{\mathcal{H}^{\succ}} A$ and $\Gamma \succ_{\mathcal{H}^{\succ}} B$ is trivial.

2) The proof that $\Gamma \succ_{\mathcal{H}^{\succ}} A\#B$ implies $\Gamma, A \succ_{\mathcal{H}^{\succ}} B$ is trivial. For the converse, the proof is analogous to the one of Point 1).
3) Immediate from $\mathrm{Theo}(\succ) = \mathrm{Theo}(\succ_{\mathcal{H}^{\succ}})$, Point 2) and Proposition 2.1.4. □

We remark that in general there is more than one Hilbert-style system $\mathcal{H}$ such that $\succ = \succ_{\mathcal{H}}$. We say that $\mathcal{H}$ is an *Hilbert-style system for generating* $\succ$ if $\succ = \succ_{\mathcal{H}}$.

Since in this thesis we are interested in logics and theories with a recursively enumerable set of theorems for which the connectives $\Rightarrow$ and $\wedge$ are an internal implication and a combining conjunction respectively, Proposition 2.1.5 allows us to restrict ourselves to the family of derivability relations which can be defined by means of Hilbert-style systems.

We say that a derivability relation $\mid\!\sim$ over $\mathcal{L}_\mathcal{A}$ is *regular* if Theo($\mid\!\sim$) is recursively enumerable and $\Rightarrow$ and $\wedge$ are respectively an internal implication and a combining conjunction for it. In a similar way, we say that an Hilbert-style system $\mathcal{H}$ over $\mathcal{L}_\mathcal{A}$ is *regular* if the corresponding derivability relation $\mid\!\sim_{\mathcal{H}\mid\sim}$ is regular.

The previous discussion justifies the following definition of *formal system.*

2.1.6 Definition (Formal system) *A* formal system *is a triple* $(\mathcal{L}_\mathcal{A}, \mid\!\sim, \mathcal{H})$ *such that* $\mathcal{L}_\mathcal{A}$ *is a (first order) language,* $\mid\!\sim$ *is a regular derivability relation (over* $\mathcal{L}_\mathcal{A}$*) and* $\mathcal{H}$ *is a (regular) Hilbert-style system (over* $\mathcal{L}_\mathcal{A}$*) for generating* $\mid\!\sim$*.*

According to our definition, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two differently Hilbert-style systems generating the same derivability relation $\mid\!\sim$, we consider the two formal systems $(\mathcal{L}_\mathcal{A}, \mid\!\sim, \mathcal{H}_1)$ and $(\mathcal{L}_\mathcal{A}, \mid\!\sim, \mathcal{H}_2)$ as different. This corresponds to the following intuitive interpretation of the ingredients of the formal system. The derivability relation specifies the relation between sets of formulas and formulas. We can think about $\Gamma\mid\!\sim A$ as: Γ *contains enough information to prove* A. But the derivability relation does not give any hint on the process we must perform to derive A form Γ. On the contrary, the Hilbert-style system specifies the way according to which we can obtain A starting from Γ. In this sense, different Hilbert-style systems corresponds to *different ways to organize and coordinate the involved information.* To define the process of *extraction of information from proofs* we must know in which way the formal system correlates information. This justifies the introduction of a proof theory for $\mid\!\sim$ in the definition of formal system.

Now, since our aim is to extract in an effective way information from a formal system, we need some machinery which allows us to control the logical complexity of the information to be extracted. More precisely, we require that the process extracting information from a proof (or a set of proofs) needs only to look at a share of the formal system bounded in logical complexity. The Hilbert-style systems allow us to define a notion of *formula proved with a bounded complexity* which is very natural.

First of all, given a first order language $\mathcal{L}_\mathcal{A}$ and a positive integer k, we denote with $\mathcal{L}^k_\mathcal{A}$ the set of all the formulas of $\mathcal{L}_\mathcal{A}$ with complexity less than or equal to k. That is

$$\mathcal{L}^k_\mathcal{A} = \{A \in \mathcal{L}_\mathcal{A} : \mathrm{dg}(A) \leq k\} \ .$$

Given an Hilbert-style system $\mathcal{H} = (\mathbf{H}_0, \mathbf{H}_1)$, we define, for any positive integer k, the k*-bounded provability relation* $\vdash^k_\mathcal{H}$ of $\mathcal{H}$ as follows:

$\vdash^k_\mathcal{H} A$ iff there exists a finite sequence $B_1, \ldots, B_n$ of formulas of $\mathcal{L}^k_\mathcal{A}$ such that $B_n \equiv A$, and for any i with $i = 1, \ldots, n$, B_i is either an axiom of $\mathbf{H}_0$ or there exist $C_1, \ldots, C_m \in \{B_1, \ldots, B_{i-1}\}$ such that $C_1, \ldots, C_m/B_i$ is a rule in $\mathbf{H}_1$.

If $\vdash^k_{\mathcal{H}} A$ we say that A *is k-provable in $\mathcal{H}$*. The set of all the formulas of $\mathcal{L}_{\mathcal{A}}$ which are k-provable in $\mathcal{H}$ will be denoted by $\text{Theo}^k(\mathcal{H})$, i.e.,

$$\text{Theo}^k(\mathcal{H}) = \{A \in \mathcal{L}_{\mathcal{A}} : \vdash^k_{\mathcal{H}} A\} .$$

The set $\text{Theo}^k(\mathcal{H})$ represents the amount of information of the formal system generated by $\mathcal{H}$ that the Hilbert-style system is able to generate within a fixed logical complexity k.

We can define the *k-bounded derivability relation* $\mid\!\sim^k_{\mathcal{H}}$ as follows:

$\Gamma \mid\!\sim^k_{\mathcal{H}} A$ iff there exists a finite sequence $B_1, \dots, B_n$ of formulas of $\mathcal{L}^k_{\mathcal{A}}$, such that $B_n \equiv A$ and, for any i with $i = 1, \dots, n$, either $B_i \in \Gamma$ or $\vdash^k_{\mathcal{H}} B_i$ or there exist $C_1, \dots, C_m \subseteq \{B_1, \dots, B_{i-1}\}$ such that $C_1, \dots, C_m / B_i$ is a rule of $\mathbf{H}_1$.

It is easy to see that, for any $k \geq 0$, $\mid\!\sim^k_{\mathcal{H}}$ is a derivability relation over $\mathcal{L}_{\mathcal{A}}$, but it is also easy to verify that $\mid\!\sim^k_{\mathcal{H}}$ cannot have an internal implication. We will call $\{\mid\!\sim^i_{\mathcal{H}}\}_{i\in\omega}$ the *stratification of* $\mid\!\sim_{\mathcal{H}}$. We remark that $\mid\!\sim_{\mathcal{H}} = \mid\!\sim^*_{\mathcal{H}}$, where

$$\mid\!\sim^*_{\mathcal{H}} = \bigcup_{i\in\omega} \mid\!\sim^i_{\mathcal{H}} .$$

2.2 Proofs and calculi

Usually a calculus is meant to be a system of rules to build up proofs. There are various proof-theoretical formalisms to define calculi. Here, in order to obtain an abstraction level adequate to our purposes, we axiomatize the notions of *proof* and *calculus* in such a way as to capture all the various proof-theoretical formalisms used to define the usual calculi for classical and intuitionistic logics. With this aim, we identify only the properties we consider fundamentals for these notions.

2.2.1 Definition (Proof) *A* proof *in a language $\mathcal{L}_{\mathcal{A}}$ is any finite object π such that:*

1. *The set of formulas of $\mathcal{L}_{\mathcal{A}}$ occurring in π is uniquely determined and non-empty;*

2. *There are two uniquely determined sets of formulas Γ and Δ occurring in π such that: Γ (possibly empty) is the set of* assumptions *of π and Δ, which must be non empty, is the set of* consequences *of π.*

2.2.2 Remark There is a natural restriction of the above definition, which amounts to assume a proof to be a pair (M, f), where M is an element of a class of structures $\mathcal{M}$ (partial orders, lists, trees,...) and f is a map associating, with every element of the structure M, a sequent of $\mathcal{L}_{\mathcal{A}}$. This definition might be useful to study various aspects of calculi. ■

We will characterize the proofs by suitable attributes, such as assumptions, consequences, formulas, complexity, and so on; the choice of the attributes depends on the level of abstraction we want. We only require for an attribute to be uniquely and effectively determined by the definition of a proof. Here, we consider the following attributes of a proof:

$\mathrm{Seq}(\pi)$: it is the sequent $\Gamma \vdash \Delta$, called the *sequent proved by* π, where Γ is the (finite and possibly empty) set of assumptions of the proof π, while Δ is the (non-empty and finite) set of consequences of π. To indicate that $\mathrm{Seq}(\pi) = \Gamma \vdash \Delta$, we use the compact notation

$$\pi : \Gamma \vdash \Delta \ .$$

$\mathrm{Wffs}(\pi)$: it is a set of formulas called the *formulas of* π; it contains all the formulas which occur in π.

$\mathrm{dg}(\pi)$: it is the *degree of the proof* π defined as

$$\mathrm{dg}(\pi) = \mathrm{Max}\{\mathrm{dg}(A) : A \in \mathrm{Wffs}(\pi)\} \ .$$

Given a sequent $\Gamma \vdash \Delta$, we define the degree of $\Gamma \vdash \Delta$ as

$$\mathrm{dg}(\Gamma \vdash \Delta) = \mathrm{Max}\{\mathrm{dg}(A) : A \in \Gamma \cup \Delta\} \ .$$

2.2.3 Definition (Calculus) *A calculus (over $\mathcal{L}_{\mathcal{A}}$) is a pair $\mathbf{C} = (C, \mathrm{SubPr})$, where:*

1. *C is a recursive set of proofs in the language $\mathcal{L}_{\mathcal{A}}$;*
2. *SubPr is a map from C into $\mathrm{Pow}_{\mathrm{fin}}(C)$ with the following properties:*
   (a) *For any $\pi \in C$, if $\tau \in \mathrm{SubPr}(\pi)$ then $\mathrm{SubPr}(\tau) \subseteq \mathrm{SubPr}(\pi)$;*
   (b) *For any $\pi' \in \mathrm{SubPr}(\pi)$, $\mathrm{dg}(\pi') \leq \mathrm{dg}(\pi)$.*

This definition of calculus does not refer to any particular inference system, but any known inference system (Hilbert-style, Gentzen-style, ...) is a calculus according to our definition.

The map SubPr associates with any proof of the calculus the finite set of its subproofs. We require the notion of subproof to depend on the calculus because we are looking at subproofs of a proof π as the fragment of the calculus needed to *extract the constructive content* of π. We remark that conditions (2a) and (2b) are natural: the former requires that the set of subproofs of a proof also contains the subproofs of its elements. The latter requires that the degree of the subproofs of a proof must not exceed the degree of the proof.

Now, given a set of proofs Π of a calculus $\mathbf{C} = (C, \mathrm{SubPr})$, we denote with $[\Pi]_{\mathbf{C}}$ the *closure under subproofs* of Π in the calculus $\mathbf{C}$. Namely,

$$[\Pi]_{\mathbf{C}} = \Pi \cup \mathrm{SubPr}(\Pi) \ .$$

Whenever the calculus $\mathbf{C}$ will be clear from context, we simply write $[\Pi]$ instead of $[\Pi]_{\mathbf{C}}$.

In general, $[\Pi]$ is not a recursive set of proofs. If Π is finite then, of course, $[\Pi]$ is recursive, and hence $([\Pi], \mathrm{SubPr}_{\restriction[\Pi]})$ is a calculus, where $\mathrm{SubPr}_{\restriction[\Pi]}$ is the restriction of SubPr to $[\Pi]$.

We will use the following notational conventions: calculi will be denoted by $\mathbf{C}, \mathbf{C}', \mathbf{C}_1, \ldots$, proofs by $\pi, \pi', \pi_1, \ldots$, sets of proofs by Π, Λ possibly with indexes, sequents by $\sigma, \sigma', \sigma_1, \ldots$, and sets of sequents by $\Sigma, \Sigma', \Sigma_1, \ldots$. Moreover, to simplify the notation we write $\vdash \Delta$ to indicate the sequent $\emptyset \vdash \Delta$ with an empty set of premises; given a calculus $\mathbf{C} = (C, \mathrm{SubPr})$ we write $\pi \in \mathbf{C}$ to mean $\pi \in C$ and $\pi : \sigma \in \mathbf{C}$ to mean that $\pi \in \mathbf{C}$ and $\mathrm{Seq}(\pi) = \sigma$ (indeed, for sake of simplicity, in the following, we will often identify the calculus $\mathbf{C}$ with the set of its proofs C).

Given a calculus $\mathbf{C}$, let $\Pi \subseteq \mathbf{C}$ be a set of proofs (possibly the whole calculus); we define the following attributes of Π.

$\mathrm{Seq}(\Pi)$: it is the set of all the *sequents proved in* Π, i.e.

$$\mathrm{Seq}(\Pi) = \cup_{\pi \in \Pi} \mathrm{Seq}(\pi) \ .$$

$\mathrm{Wffs}(\Pi)$: it is the set of all the formulas occurring in some proof of Π, i.e.

$$\mathrm{Wffs}(\Pi) = \cup_{\pi \in \Pi} \mathrm{Wffs}(\pi) \ .$$

$\mathrm{dg}(\Pi)$: it is the *degree of the set of proofs* Π defined as:

$$\mathrm{dg}(\Pi) = \mathrm{Max}\{\mathrm{dg}(\pi) : \pi \in \Pi\} \ ,$$

where we assume $\mathrm{dg}(\Pi) = \infty$ if Π contains proofs of any complexity.

$\mathrm{Theo}(\Pi)$: it is the set of *theorems proved in* Π, i.e.

$$\mathrm{Theo}(\Pi) = \{A : \vdash A \in \mathrm{Seq}(\Pi)\} \ .$$

Given two sets of proofs Π_1 and Π_2 over the same language $\mathcal{L}_{\mathcal{A}}$, but possibly belonging to different calculi, we write

$$\Pi_1 \approx \Pi_2 \quad \text{iff} \quad \mathrm{Seq}(\Pi_1) = \mathrm{Seq}(\Pi_2) \ .$$

We say that two calculi $\mathbf{C}_1 = (C_1, \mathrm{SubPr}_1)$ and $\mathbf{C}_2 = (C_2, \mathrm{SubPr}_2)$ (over the same language) are *equivalent* iff $C_1 \approx C_1$. Hereafter, we will always write $\mathbf{C}_1 \approx \mathbf{C}_2$ for $C_1 \approx C_1$.

Formal systems and calculi are related by the following obvious definition:

2.2.4 Definition (Presentation of a formal system) *Let* $\mathbf{S} = (\mathcal{L}_{\mathcal{A}}, \mid\!\sim, \mathcal{H})$ *be a formal system and* $\mathbf{C} = (C, \mathrm{SubPr})$ *be a calculus over* $\mathcal{L}_{\mathcal{A}}$. *We say that* $\mathbf{C}$ *is a* calculus *(or a* presentation*) for* $\mathbf{S}$ *iff* $\mathrm{Theo}(\mathbf{C}) = \mathrm{Theo}(\mid\!\sim)$.

This definition relates calculi and formal systems only looking at global properties of the two systems; namely, the set of theorems they prove. Now, as already discussed, we consider the Hilbert-style calculus involved in the definition of a formal system as a way to coordinate and organize pieces of information. In this sense, we intend to consider a presentation (calculus) for a formal system as a kind of simulation of the main deductive features of the Hilbert-style system generating it. Thus, we will consider "reasonable calculi" for a formal system only those calculi which, so to say, do not affect in an essential way the organization of the information involved in the Hilbert-style system on which the formal system is based. The measure we will use to distinguish "reasonable calculi" form "unreasonable ones" depends on the degree of the proofs of the calculus in hand, and is formalized by the following definition.

2.2.5 Definition *Let* $\mathbf{S} = (\mathcal{L}_{\mathcal{A}}, \mid\!\sim, \mathcal{H})$ *be a formal system and let* $\mathbf{C} = (C, \mathrm{SubPr})$ *a presentation for it. We say that* $\mathbf{C}$ agrees with $\mathbf{S}$ *if, for any set* $\Pi \subseteq \mathbf{C}$ *such that* $\mathrm{dg}(\Pi) \leq k$ *(*$k > 0$*), there exists a positive integer* h *such that* $\mathrm{Theo}(\Pi) \subseteq \mathrm{Theo}(\vdash^{h}_{\mathcal{H}})$.

Thus, according to the above definition, a calculus for a formal system can be seen as an alternative way to organize its information. The "local information" in this calculus may be greater than the one of the Hilbert-style system generating the formal system. This means that the set of formulas the calculus can prove within a given complexity may be greater than the set of formulas provable in the Hilbert-style system within the same complexity. But we do not accept presentations for a formal system allowing to prove within a fixed complexity sets of formulas which in the Hilbert-style system involve proofs of any complexity.

2.3 Generalized rules

In the following we will be interested in characterizing subsets of a calculus which have some closure properties. To this aim we introduce the notion of generalized rule.

2.3.1 Definition (Generalized rule) *Let* Σ *be the set of all the sequents in the language* $\mathcal{L}_{\mathcal{A}}$ *and let* Σ^* *be the set of all finite sequences of sequents in* Σ. *A* generalized rule (over $\mathcal{L}_{\mathcal{A}}$) *is a relation* $\mathcal{R} \subseteq \Sigma^* \times \Sigma$.

We denote with ϵ the empty sequence of sequents. If σ^* is a finite sequence of sequents in Σ, we write $\sigma \in \mathcal{R}(\sigma^*)$ to mean that $(\sigma^*, \sigma) \in \mathcal{R}$. The intuitive reading of $\sigma \in \mathcal{R}(\sigma^*)$ is: σ is obtained by applying $\mathcal{R}$ to σ^*. We call *domain* of $\mathcal{R}$ the set

$$\mathrm{dom}(\mathcal{R}) = \{\sigma^* \in \Sigma^* \ : \ \text{there exists } \sigma \text{ such that } \sigma \in \mathcal{R}(\sigma^*)\} \ .$$

Examples of generalized rules are:

Substitution rule (SUBST).

The domain of SUBST is the set of all the sequents, and, for every substitution

θ of terms for individual variables:

$$\theta\Gamma \vdash \theta\Delta \in \textsc{Subst}(\Gamma \vdash \Delta) \ .$$

Cut rule ($\textsc{Cut}^+$).
The domain of $\textsc{Cut}^+$ contains all the sequences of sequents which have the form $\Gamma_1 \vdash H, \Delta_1; \Gamma_2, H \vdash \Delta_2$, and:

$$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \in \textsc{Cut}^+(\Gamma_1 \vdash H, \Delta_1; \Gamma_2, H \vdash \Delta_2) \ .$$

Modus Ponens ($\textsc{Mp}$).
The domain of $\textsc{Mp}$ contains any sequence of sequents of the form $\Gamma_1 \vdash A \Rightarrow B; \Gamma_2 \vdash A$, and:

$$\Gamma_1, \Gamma_2 \vdash B \in \textsc{Mp}(\Gamma_1 \vdash A \Rightarrow B; \Gamma_2 \vdash A) \ .$$

Weakening rules ($\textsc{Wk-l}$, $\textsc{Wk-r}$) .
The domains of $\textsc{Wk-l}$ and $\textsc{Wk-r}$ contain any sequent $\Gamma \vdash \Delta$ and, for any formula A:

$$\Gamma, A \vdash \Delta \in \textsc{Wk-l}(\Gamma \vdash \Delta) \quad \text{and} \quad \Gamma \vdash \Delta, A \in \textsc{Wk-r}(\Gamma \vdash \Delta) \ .$$

We will say that a generalized rule $\mathcal{R}$ is *standard* iff it includes $\textsc{Subst}$, that is $\textsc{Subst} \subseteq \mathcal{R}$. Hereafter, we will always consider standard generalized rules, and hence we will say generalized rule to mean a standard generalized rule.

2.3.2 Definition *Let $\mathcal{R}$ be a generalized rule over $\mathcal{L}_\mathcal{A}$. A set of sequents Δ in the language $\mathcal{L}_\mathcal{A}$ is* closed under $\mathcal{R}$ *(shortly* $\mathcal{R}$-closed*) iff, for any $\sigma, \sigma_1, \ldots, \sigma_n \in \Sigma$, if $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$ and $\sigma_1, \ldots, \sigma_n \in \Delta$ then $\sigma \in \Delta$. A set of proofs Π over $\mathcal{L}_\mathcal{A}$ is* $\mathcal{R}$-closed *iff* $\mathrm{Seq}(\Pi)$ *is $\mathcal{R}$-closed.*

That is, a set of proofs Π is $\mathcal{R}$-closed iff, for every $\sigma, \sigma_1, \ldots, \sigma_n \in \Sigma$ such that $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$, if $\pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n \in \Pi$, then Π contains at least a proof $\pi : \sigma$. By an abuse of notation, given proofs $\pi : \sigma$ and $\pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n$ we will write $\pi : \sigma \in \mathcal{R}(\pi_1 : \sigma_1; \ldots; \pi_n : \sigma_n)$ and $\pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n \in \mathrm{dom}(\mathcal{R})$.

Since the union of generalized rules is a generalized rule, we can restrict our analysis of the closure properties to single rules.

2.3.3 Definition ($\mathcal{R}$-subcalculus) *Let Π be a set of proofs of a calculus* **C** *over $\mathcal{L}_\mathcal{A}$. Given a generalized rule $\mathcal{R}$ over $\mathcal{L}_\mathcal{A}$, we call* $\mathcal{R}$-subcalculus of Π *any subset Π' of Π which is $\mathcal{R}$-closed. We write $\Pi' \sqsubseteq_\mathcal{R} \Pi$ to mean that Π' is an $\mathcal{R}$-subcalculus of* Π.

According to the previous definition, different kinds of generalized rules $\mathcal{R}$ characterize different kinds of $\mathcal{R}$-subcalculi.

We notice that an $\mathcal{R}$-subcalculus is not required to be a calculus, that is, neither we require that it is a recursive set of proofs, nor we require that it is closed with respect to subproofs.

Now, let $\mathcal{R} \subseteq \Sigma^* \times \Sigma$ be a generalized rule, ad let $\mathbf{C}$ be an $\mathcal{R}$-closed calculus. We define the operator $\overline{\mathcal{R}}$ on subsets of $\mathbf{C}$ associated with $\mathcal{R}$ as follows: for any $\Pi \subseteq \mathbf{C}$,

$$\overline{\mathcal{R}}(\Pi) = \Pi \cup \{\pi : \sigma \in \mathbf{C} : \ \sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n) \text{ and } \pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n \in \Pi\} \ .$$

We define $\overline{\mathcal{R}}^n(\Pi)$, for $n \geq 0$, in the usual way, that is:

$$\begin{array}{rcl} \overline{\mathcal{R}}^0(\Pi) & = & \Pi \\ \overline{\mathcal{R}}^{n+1}(\Pi) & = & \overline{\mathcal{R}}(\overline{\mathcal{R}}^n(\Pi)) \end{array}$$

The *closure of* $\mathcal{R}$ *over* Π is

$$\overline{\mathcal{R}}^*(\Pi) = \bigcup_{k<\omega} \overline{\mathcal{R}}^k(\Pi)$$

According to the previous definition, one can see $\overline{\mathcal{R}}$ as an operator on the power set of $\mathbf{C}$. It is easy to verify that $\overline{\mathcal{R}}$ is a monotone and continuous operator on the complete partial order $(\text{Pow}(\mathbf{C}), \subseteq)$, and hence $\overline{\mathcal{R}}^*(\Pi)$ is the least fix point of $\overline{\mathcal{R}}$ containing Π.

2.3.4 Remark Let us consider the structure $P = (\text{Pow}(\mathbf{C}), \subseteq)$. Now, given $X \subseteq \text{Pow}(\mathbf{C})$, let $\sup(X)$, the *least upper bound* of X, and $\inf(X)$, the *greatest lower bound* of X, be defined respectively as the union set $\cup X$ of X and the intersection set $\cap X$ of X. Since, in P, $\sup(X)$ and $\inf(X)$ exist for any $X \subseteq \text{Pow}(\mathbf{C})$, P is a complete lattice and hence a *complete partial order* . Moreover, $\overline{\mathcal{R}}$ is order preserving and hence, by the Knaster-Tarsky Theorem (see [Davey and Priestley, 1990]), $\overline{\mathcal{R}}$ has a least fix point in $\text{Pow}(\mathbf{C})$. Now, we recall that a set X is *directed* if every finite subset of X has an upper bound in X. To prove that $\overline{\mathcal{R}}$ is continuous we have to prove that, for any $X \subseteq \text{Pow}(\mathbf{C})$ which is directed

$$\overline{\mathcal{R}}(\sup(X)) = \sup(\overline{\mathcal{R}}(X)) \ .$$

First of all we notice that, if $\{\pi_1, \ldots, \pi_n\} \subseteq \sup(X)$, then there exists a set of proofs Π' in X such that $\{\pi_1, \ldots, \pi_n\} \subseteq \Pi'$. Now, $\pi : \sigma \in \overline{\mathcal{R}}(\sup(X))$

iff [by definition of $\overline{\mathcal{R}}$]
$\pi \in \sup(X)$ or $\pi : \sigma \in \mathcal{R}(\pi_1 : \sigma_1; \ldots; \pi_n : \sigma_n)$ with $\pi \in \mathbf{C}$ and $\pi_1, \ldots, \pi_n \in \sup(X)$

iff [by the previous remark]
$\pi \in \Pi'$ or $\pi : \sigma \in \mathcal{R}(\pi_1 : \sigma_1; \ldots; \pi_n : \sigma_n)$ and $\pi_1, \ldots, \pi_n \in \Pi'$ for some $\Pi' \in X$

iff [by definition of $\overline{\mathcal{R}}$]
$\pi \in \overline{\mathcal{R}}(\Pi')$ for some $\Pi' \in X$

iff $\pi \in \sup(\overline{\mathcal{R}}(X))$.

Therefore, $\overline{\mathcal{R}}$ is a continuous operator on $\text{Pow}(\mathbf{C})$. Hence, by a well-known Theorem (see CPO Fix Point Theorem I in [Davey and Priestley, 1990]), $\overline{\mathcal{R}}^*(\Pi)$ is the least fix point of $\overline{\mathcal{R}}$ containing Π. ■

Hereafter, given a set of proofs Π we will denote simply with $\mathcal{R}^*(\Pi)$ the set $\overline{\mathcal{R}}^*(\Pi)$. In particular, in the following sections we will use the closure under SUBST, SUBST$^*(\Pi)$, of a set of proofs Π. $\overline{\mathcal{R}}^*(\Pi)$ is related to $\mathcal{R}$-subcalculi by the following proposition:

2.3.5 Proposition *Let $\mathcal{R}$ be a generalized rule and let $\mathbf{C}$ be an $\mathcal{R}$-closed calculus. Then, for any $\mathcal{R}$-subcalculus Π of $\mathbf{C}$:*

1. *If $\Lambda \subseteq \Pi$ then $\mathrm{Seq}(\mathcal{R}^*(\Lambda)) \subseteq \mathrm{Seq}(\Pi)$;*

2. *$\Pi \approx \mathcal{R}^*(\Pi)$, but in general $\Pi \subseteq \mathcal{R}^*(\Pi)$.*

Proof: 1) If $\sigma \in \mathrm{Seq}(\mathcal{R}^*(\Lambda))$, then there exist $k \geq 0$ and $\pi \in \mathcal{R}^k(\Lambda)$ such that $\mathrm{Seq}(\pi) = \sigma$. Now, we prove by induction on k that $\sigma \in \mathrm{Seq}(\mathcal{R}^k(\Lambda))$ entails $\sigma \in \mathrm{Seq}(\Pi)$. If $k = 0$, then the assertion follows from $\mathrm{Seq}(\Lambda) \subseteq \mathrm{Seq}(\Pi)$. Let us suppose that $\sigma \in \mathrm{Seq}(\mathcal{R}^{k+1}(\Lambda))$ with $\pi : \sigma \in \mathcal{R}^{k+1}(\Lambda)$. Then either $\pi : \sigma \in \mathcal{R}^k(\Lambda)$ or there exist $\pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n \in \mathcal{R}^k(\Lambda)$ such that $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$. In the first case the assertion immediately follows from induction hypothesis. In the second case, by induction hypothesis, $\sigma_1, \ldots, \sigma_n \in \mathrm{Seq}(\Pi)$ and hence, since Π is $\mathcal{R}$-closed, $\sigma \in \mathrm{Seq}(\Pi)$.
2) Since Π is included in $\mathcal{R}^*(\Pi)$ ($\Pi = \mathcal{R}^0(\Pi)$), $\mathrm{Seq}(\Pi) \subseteq \mathrm{Seq}(\mathcal{R}^*(\Pi))$ is immediate, while the converse holds by Point (1) of this proposition. On the other hand, by definition of $\mathcal{R}^*(\Pi)$, any proof π of a sequent σ such that $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$ and $\sigma_1, \ldots, \sigma_n \in \mathrm{Seq}(\Pi)$ is in $\mathcal{R}^*(\Pi)$, while at least one proof of such a sequent is required to belong to Π to assure it is $\mathcal{R}$-closed. □

2.3.6 Definition (Generalized $\mathcal{R}$-subcalculus) *Let $\mathcal{R}$ be a generalized rule over $\mathcal{L}_{\mathcal{A}}$ and let $\mathbf{C}$ be an $\mathcal{R}$-closed calculus (over $\mathcal{L}_{\mathcal{A}}$). We say that a set of proofs Π is a* generalized $\mathcal{R}$-subcalculus *of $\mathbf{C}$, written $\Pi \preceq_{\mathcal{R}} \mathbf{C}$, if there is an $\mathcal{R}$-subcalculus Π' of $\mathbf{C}$ such that $\Pi \approx \Pi'$.*

The notion of generalized $\mathcal{R}$-subcalculus is important because, passing from $\mathcal{R}$-subcalculi to generalized $\mathcal{R}$-subcalculi, we are not obliged to work with subsets of $\mathbf{C}$, but we can work with sets of proofs equivalent (in the sense of $\approx$) to some $\mathcal{R}$-subcalculus of $\mathbf{C}$. The notion is particularly interesting when the generalized $\mathcal{R}$-subcalculus is itself a calculus. To characterize this notion, we fix a convenient representation of generalized $\mathcal{R}$-subcacluli.

2.3.7 Definition (The abstract calculus $\mathbb{D}(\mathcal{R}, \Sigma)$) *Let $\mathcal{R}$ be a generalized rule over $\mathcal{L}_{\mathcal{A}}$ and let Σ be any set of sequents in the same language. The* deductive sequent-system $\mathbb{D}(\mathcal{R}, \Sigma)$ *is the set of proof-trees τ inductively defined as follows:*

Basis : For every $\sigma \in \Sigma$, $\tau \equiv \sigma$ is a proof-tree of $\mathbb{D}(\mathcal{R}, \Sigma)$ with root σ and with $\mathrm{depth}(\tau) = 1$.

Step : *If* $\tau_1 : \sigma_1, \dots, \tau_n : \sigma_n$ *are proof-trees of* $\mathbb{D}(\mathcal{R}, \Sigma)$ *(where* σ_i *is the root of* τ_i*) then, for every* $\sigma \in \mathcal{R}(\sigma_1; \dots; \sigma_n)$*,*

$$\tau \equiv \frac{\tau_1 : \sigma_1 \;\; \dots \;\; \tau_n : \sigma_n}{\sigma}\mathcal{R}$$

is a proof-tree of $\mathbb{D}(\mathcal{R}, \Sigma)$ *with root* σ *and*

$$\mathrm{depth}(\tau) = \mathrm{Max}\{\mathrm{depth}(\tau_1), \dots, \mathrm{depth}(\tau_n)\} + 1 \; .$$

We remark that, if both $\mathcal{R}$ and Σ are recursive, then $\mathbb{D}(\mathcal{R}, \Sigma)$ is a calculus, where we consider the obvious function SubPr determined by the inductive definition of $\mathbb{D}(\mathcal{R}, \Sigma)$. Namely,

- If $\mathrm{depth}(\tau) = 1$, then $\mathrm{SubPr}(\tau) = \{\tau\}$;
- If $\mathrm{depth}(\tau) = h + 1$, with

$$\tau \equiv \frac{\tau_1 : \sigma_1 \;\; \dots \;\; \tau_n : \sigma_n}{\sigma}\mathcal{R}$$

  then, $\mathrm{SubPr}(\tau) = \mathrm{SubPr}(\tau_1) \cup \dots \cup \mathrm{SubPr}(\tau_n) \cup \{\tau\}$.

If $\mathbb{D}(\mathcal{R}, \Sigma)$ is a calculus, we can consider Σ as the set of *axiom-sequents* and $\mathcal{R}$ as the set of *inference rules* of $\mathbb{D}(\mathcal{R}, \Sigma)$.

Given a generalized rule $\mathcal{R}'$ such that $\mathcal{R}' \subseteq \mathcal{R}$, by the $\mathcal{R}'$*-depth* of a proof $\tau \in \mathbb{D}(\mathcal{R}, \Sigma)$ we mean the number of applications of the generalized rules $\mathcal{R}'$ which occurs in τ. If $\mathcal{R}'$ and $\mathcal{R}''$ are two generalized rules included in $\mathcal{R}$, we will use simply $(\mathcal{R}', \mathcal{R}'')$-depth to indicate the $(\mathcal{R}' \cup \mathcal{R}'')$-depth of a proof.

Definition 2.3.7 allows us to recover the meaning of the generalized rules as inference rules, but abstracting from the particular inference system.

2.3.8 Proposition *Let* $\mathcal{R}$ *be a generalized rule, let* $\mathbf{C}$ *be an* $\mathcal{R}$*-closed calculus, and let* Λ *be a set of proofs of* $\mathbf{C}$*. Then:*

1. $\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Lambda)) \approx \mathcal{R}^*(\Lambda)$.
2. $\mathrm{Seq}(\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Lambda)) \subseteq \mathrm{Seq}(\Pi)$ *for every* $\mathcal{R}$*-subcalculus* Π *such that* $\mathrm{Seq}(\Lambda) \subseteq \mathrm{Seq}(\Pi)$.

Proof: 1) The fact that $\sigma \in \mathrm{Seq}(\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Lambda))$ implies $\sigma \in \mathrm{Seq}(\mathcal{R}^*(\Lambda))$ follows from a straightforward induction on the depth of the proof of σ in $\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Lambda))$. On the other hand, $\mathrm{Seq}(\mathcal{R}^k(\Lambda)) \subseteq \mathrm{Seq}(\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Lambda)))$ follows from a straightforward induction on k.

2) First of all we notice that $\mathbb{D}(\mathcal{R}, .)$ is a monotone operator on sets of sequents. Hence $\mathrm{Seq}(\Lambda) \subseteq \mathrm{Seq}(\Pi)$ implies $\mathrm{Seq}(\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Lambda))) \subseteq \mathrm{Seq}(\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi)))$. But, by the previous point of this proposition, $\mathrm{Seq}(\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))) = \mathrm{Seq}(\mathcal{R}^*(\Pi))$ and, by Point (2) of Proposition 2.3.5, $\mathrm{Seq}(\mathcal{R}^*(\Pi)) = \mathrm{Seq}(\Pi)$. Hence the assertion. □

Now, by Point (2) of Proposition 2.3.5, we have that, if Π is an $\mathcal{R}$-subcalculus of an $\mathcal{R}$-closed calculus $\mathbf{C}$, then

$$\Pi \approx \mathcal{R}^*(\Pi)$$

and hence, by Point (1) of Proposition 2.3.8,

$$\Pi \approx \mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi)) \ .$$

Hence $\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$ is a generalized $\mathcal{R}$-subcalculus of $\mathbf{C}$. More generally, since for any set of proofs Π, $\mathcal{R}^*(\Pi)$ is an $\mathcal{R}$-subcalculus of $\mathbf{C}$, by Point (1) of Proposition 2.3.8, $\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi)) \approx \mathcal{R}^*(\Pi)$ and hence $\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$ is a generalized $\mathcal{R}$-subcalculus of $\mathbf{C}$. Summarizing, the following results hold:

2.3.9 Theorem *Let $\mathcal{R}$ be a generalized rule and let $\mathbf{C}$ be an $\mathcal{R}$-closed calculus.*

(i). If Π is an $\mathcal{R}$-subcalculus of $\mathbf{C}$ then $\Pi \approx \mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$;

(ii). If $\Pi \subseteq \mathbf{C}$, then $\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$ is a generalized $\mathcal{R}$-subcalculus of $\mathbf{C}$.

□

By Point (2) of Proposition 2.3.8, $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$ represents the information contained in a set of proofs Π and nothing else in the following sense: for every $\mathcal{R}$-subcalculus Π' of $\mathbf{C}$, if Π' is sufficient to prove all the sequents proved by $[\Pi]$, then Π' is sufficient to prove all the sequents proved by $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$. It is also an abstract representation, since it is independent of the calculus used to construct the proofs in Π. This assigns the following role to $\mathcal{R}$. Given any proof π, the pieces of information contained in it are seen as contained only implicitly in $[\pi]$. To extract them in an explicit way, we use the closure under a generalized rule $\mathcal{R}$. Here $\mathcal{R}$ works as a mechanism to *extract information from proofs*, where the extracted information is the set of the sequents and theorems proved by $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi]))$. However, it is obvious that for this extraction procedure to be effective, we need the rule $\mathcal{R}$ and the set of sequents determined by $[\pi]$ to be recursive. In Chapters 3 and 4, we will show that the sets of theorems extracted from proofs in some constructive calculi by suitable rules $\mathcal{R}$ satisfy the *disjunction property* and the *explicit definability property*. The extraction mechanism works as well for classical proofs, even if in this case the disjunction property and the explicit definability property cannot be guaranteed.

2.4 Uniformity

A representation function is any function mapping proofs of a calculus into proofs of another calculus preserving the proved sequents. Formally:

2.4.1 Definition (Representation function) *Given two sets of proofs Π_1 and Π_2 in a language $\mathcal{L}_{\mathcal{A}}$ (possibly with Π_1 and Π_2 calculi), a* representation function of Π_1 in Π_2 *is any function $r : \Pi_1 \to \Pi_2$ such that, for every $\pi \in \Pi_1$, if $\mathrm{Seq}(\pi) = \sigma$ then $r(\pi) : \sigma$.*

One immediately sees that a way to prove that $\mathrm{Seq}(\Pi_1) \subseteq \mathrm{Seq}(\Pi_2)$ is to exhibit a representation function $r : \Pi_1 \to \Pi_2$. If also exists a representation function $r' : \Pi_2 \to \Pi_1$ then $\mathrm{Seq}(\Pi_2) \subseteq \mathrm{Seq}(\Pi_1)$, that is $\Pi_1 \approx \Pi_2$. Moreover, if $r(\Pi_1)$ (the image of Π_1 under r) is an $\mathcal{R}$-subcalculus of Π_2, then Π_1 is a generalized $\mathcal{R}$-subcalculus of Π_2.

Now, we define *uniform representations* as follows:

2.4.2 Definition *Let $\phi : \mathbf{N} \to \mathbf{N}$. We say that :*

(i). A representation function $r : \Pi_1 \to \Pi_2$ is uniform w.r.t. ϕ *iff, for every $\pi \in \Pi_1$, $\mathrm{dg}(r(\pi)) \leq \phi(\mathrm{dg}(\pi))$.*

(ii). Π_1 is uniformly embedded in Π_2 w.r.t. ϕ*, written $\Pi_1 \overset{\phi}{\hookrightarrow} \Pi_2$, if there exists a representation function $r : \Pi_1 \to \Pi_2$ uniform w.r.t. ϕ.*

(iii). If $r : \Pi_1 \to \Pi_2$ is a uniform representation w.r.t. ϕ and $r(\Pi_1)$ (the image of Π_1 under r) is an $\mathcal{R}$-subcalculus of Π_2, then we say that Π_1 is a uniform generalized $\mathcal{R}$-subcalculus of Π_2 w.r.t. ϕ*, written $\Pi_1 \preceq_\phi \Pi_2$.*

(iv). We say that Π_1 and Π_2 are uniformly equivalent w.r.t. ϕ *, written $\Pi_1 \approx_\phi \Pi_2$, if $\Pi_1 \overset{\phi}{\hookrightarrow} \Pi_2$ and $\Pi_2 \overset{\phi}{\hookrightarrow} \Pi_1$.*

The above definition can be justified in the following way. Let us consider a syntactical proof of the fact that a deductive system $\mathbf{C}_1$ is equivalent to another deductive system $\mathbf{C}_2$; in the most cases, such a proof implicitly or explicitly defines two representation functions $r : \mathbf{C}_1 \to \mathbf{C}_2$ and $r' : \mathbf{C}_2 \to \mathbf{C}_1$. As an example, see the sketch of the proof of equivalence between $\mathcal{H}_{\mathbf{INT}}$ and $\mathcal{ND}_{\mathbf{INT}}$ we gave in Section 1.2.2, or the proof of equivalence between $\mathcal{SEQ}_{\mathbf{INT}}$ and $\mathcal{ND}_{\mathbf{INT}}$ exhibited in [Gallier, 1991]. In general, r and r' are uniform w.r.t. a suitable function ϕ. In particular, the usual calculi for classical (for intuitionistic) logic are uniformly equivalent with respect to a linear function. The notions of uniform representation and of generalized uniform $\mathcal{R}$-subcalculus play a fundamental role also in the framework of logic program synthesis. For instance, in [Lau and Ornaghi, 1992] the authors show that the SLD-derivations are uniformly representable in the usual calculi for minimal logic, and in [Momigliano and Ornaghi, 1994] that SLDNF-derivations are uniformly representable in the same calculi; these results are used in [Lau and Ornaghi, 1992] to prove an incompleteness result for deductive program synthesis from first-order specifications.

In this Thesis we will use the following result which is a consequence of Theorem 1.2.2 and of the discussions at the end of Section 1.2.2:

2.4.3 Theorem *There exists a linear function $\phi : \mathbf{N} \to \mathbf{N}$ such that $\mathcal{H}_{\mathbf{INT}} \approx_\phi \mathcal{ND}_{\mathbf{INT}}$.* □

2.4.4 Definition *Let $\mathcal{R}$ be a generalized rule (over $\mathcal{L}_\mathcal{A}$), let $\mathbf{C}$ be calculus (over $\mathcal{L}_\mathcal{A}$) and let $\phi : \mathbf{N} \to \mathbf{N}$. We say that $\mathbf{C}$ is* uniformly $\mathcal{R}$-closed w.r.t. ϕ *iff:*

1. *$\mathbf{C}$ is $\mathcal{R}$-closed;*

2. *For any $\sigma \in \mathrm{Seq}(\mathbf{C})$ and any $\pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n \in \mathbf{C}$ such that $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$, there exists at least a proof $\pi : \sigma \in \mathbf{C}$ such that*

$$\mathrm{dg}(\pi) \leq \mathrm{Max}\,\{\mathrm{dg}(\pi_1), \ldots, \mathrm{dg}(\pi_n), \phi(\mathrm{dg}(\sigma_1)), \ldots, \phi(\mathrm{dg}(\sigma_n)), \phi(\mathrm{dg}(\sigma))\}\ .$$

We say that $\mathbf{C}$ is *uniformly $\mathcal{R}$-closed* if there exists a function $\phi : \mathbf{N} \to \mathbf{N}$ such that $\mathbf{C}$ is uniformly $\mathcal{R}$-closed w.r.t. ϕ.

2.4.5 Example The natural deduction calculus $\mathcal{ND}_{\mathbf{INT}}$ is uniformly closed under the intuitionistic version of the $\textsc{Cut}^+$ generalized rule, that is the generalized rule $\textsc{Cut}$ obtained by restricting the right-hand side of the sequents in hand to contain only one formula. Starting from the proofs $\pi_1 : \Gamma \vdash H$ and $\pi_2 : \Delta, H \vdash A$, we can build the proof $\pi : \Gamma, \Delta \vdash A$ as follows:

$$\dfrac{\pi_1 : \Gamma \vdash H \qquad \dfrac{\pi_2 : \Delta, H \vdash A}{\Delta \vdash H \Rightarrow A}\,{\scriptstyle \mathrm{I}\Rightarrow}}{\Gamma, \Delta \vdash A}\,{\scriptstyle \mathrm{E}\Rightarrow}$$

such that

$$\mathrm{dg}(\pi) \leq \mathrm{Max}\{\mathrm{dg}(\pi_1), \mathrm{dg}(\pi_2), \mathrm{dg}(\Delta, H \vdash A) + 1\}\ .$$

2.4.6 Proposition *Let $\mathcal{R}$ be a generalized rule (over $\mathcal{L}_{\mathcal{A}}$), let $\phi : \mathbf{N} \to \mathbf{N}$ be a non decreasing function, let $\mathbf{C}$ be a calculus (over $\mathcal{L}_{\mathcal{A}}$) uniformly $\mathcal{R}$-closed w.r.t. ϕ, and let Π be any set of proofs of $\mathbf{C}$ such that, for every $\pi : \sigma \in \Pi$, $\mathrm{dg}(\pi) \leq \phi(\mathrm{dg}(\sigma))$. Then $\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi)) \overset{\phi}{\hookrightarrow} \mathbf{C}$.*

Proof: Let $\tau, \tau_1, \tau_2, \ldots$ proofs of $\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$ and let $\pi, \pi_1, \pi_2, \ldots$ proofs in $\mathbf{C}$. A representation $r : \mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi)) \to \mathbf{C}$ uniform w.r.t. ϕ can be inductively defined as follows:

Basis: Let $\tau : \sigma \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$, with $\mathrm{depth}(\tau) = 1$. We have two cases: either $\sigma \in \mathrm{Seq}(\Pi)$ or $\sigma \in \mathcal{R}(\epsilon)$. In the former case, by hypothesis, there exists a proof $\pi \in \Pi$ such that $\mathrm{dg}(\pi) \leq \phi(\mathrm{dg}(\sigma))$. In the latter case, the existence of a proof π satisfying this property directly follows from uniform closure of $\mathbf{C}$ under $\mathcal{R}$. In both cases we set $r(\tau) = \pi$.

Step: Let us consider a proof $\tau \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$ with $\mathrm{depth}(\tau) = h + 1$:

$$\tau \equiv \dfrac{\tau_1 : \sigma_1 \ldots \tau_n : \sigma_n}{\sigma}\,{\scriptstyle \mathcal{R}}\ .$$

By definition $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$; hence, by induction hypothesis, we deduce that for any i with $1 \leq i \leq n$, there exists a proof $\pi_i = r(\tau_i) \in \mathbf{C}$ such that $\pi_i : \sigma_i$ and $\mathrm{dg}(\pi_i) \leq \phi(\mathrm{dg}(\tau_i))$. But $\mathbf{C}$ is uniformly $\mathcal{R}$-closed with respect to ϕ, hence there must exist a proof $\pi : \sigma \in \mathcal{R}(\pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n)$ with

$$\mathrm{dg}(\pi) \quad \leq \quad \mathrm{Max}\{\mathrm{dg}(\pi_1), \ldots, \mathrm{dg}(\pi_n), \phi(\mathrm{dg}(\sigma_1)), \ldots, \phi(\mathrm{dg}(\sigma_n)), \phi(\mathrm{dg}(\sigma))\}\ .$$

Since, for any i with $i = 1, \ldots, n$, $\mathrm{dg}(\sigma_i) \leq \mathrm{dg}(\tau_i)$ and ϕ is non decreasing, we have that $\phi(\mathrm{dg}(\sigma_i)) \leq \phi(\mathrm{dg}(\tau_i))$. Hence,

$$\begin{aligned}\mathrm{dg}(\pi) &\leq \mathrm{Max}\{\phi(\mathrm{dg}(\tau_1)), \ldots, \phi(\mathrm{dg}(\tau_n)), \phi(\mathrm{dg}(\sigma))\} \\ &\leq \phi(\mathrm{dg}(\tau)) \ .\end{aligned}$$

We set $r(\tau) = \pi$ and this concludes the proof. □

It is easy to verify that, if **C** is a calculus uniformly $\mathcal{R}$-closed with respect to a function ϕ, then there exists a non decreasing function ϕ' such that **C** is uniformly $\mathcal{R}$-closed with respect to ϕ'. For, consider the function ϕ' inductively defined as follows:

$$\begin{cases} \phi'(0) = \phi(0) \\ \phi'(i) = \mathrm{Max}\{\phi(i), \phi'(i-1)\} \end{cases}$$

Uniform representations will be used according to the following points.

2.4.7 Definition *A generalized rule $\mathcal{R}$ (over $\mathcal{L}_{\mathcal{A}}$) is* non increasing *iff:*

1. *For any $\sigma_1, \ldots, \sigma_n \in \mathrm{dom}(\mathcal{R})$ with $n > 0$, if $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$, then*

$$\mathrm{dg}(\sigma) \leq \mathrm{Max}\{\mathrm{dg}(\sigma_1), \ldots, \mathrm{dg}(\sigma_n)\} \ ;$$

2. *There exists a positive integer k such that, for any $\sigma \in \mathcal{R}(\epsilon)$, $\mathrm{dg}(\sigma) \leq k$.*

Given a non-increasing generalized rule $\mathcal{R}$, we say that it is *k-bounded* if k is the minimum integer for which condition (2) of the previous definition is satisfied. We remark that, if $\epsilon \notin \mathrm{dom}(\mathcal{R})$, then $\mathcal{R}$ is 0-bounded.

2.4.8 Proposition *Let $\mathcal{R}$ be a non increasing (k-bounded) generalized rule (over$\mathcal{L}_{\mathcal{A}}$), let $\phi : \mathbf{N} \to \mathbf{N}$ be a non decreasing function and let **C** be a calculus (over $\mathcal{L}_{\mathcal{A}}$) which is uniformly $\mathcal{R}$-closed w.r.t. ϕ. If Π is a subset of **C** such that the degree of any proof in Π is not greater than a constant k_Π, then there exists $h \in \mathbf{N}$ such that, for any sequent σ provable in $\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$, there is a proof $\pi : \sigma \in \mathbf{C}$ such that $\mathrm{dg}(\pi) \leq h$.*

Proof: Since $\mathcal{R}$ is non-increasing, it follows from a straightforward induction on $\mathrm{depth}(\tau)$ that every proof $\tau \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$ has degree

$$\mathrm{dg}(\tau) \leq k' = \mathrm{Max}\{k, k_\Pi\} \ . \tag{2.1}$$

Now, since, for any $\pi : \sigma \in \Pi$, $\mathrm{dg}(\pi : \sigma) \leq k_\Pi$, we also have that

$$\mathrm{dg}(\pi : \sigma) \leq \mathrm{Max}\{k_\Pi, \phi(\mathrm{dg}(\sigma))\} \ . \tag{2.2}$$

Let us define $\phi' : \mathbf{N} \to \mathbf{N}$ as the function associating, with any $x \in \mathbf{N}$, the value $\phi'(x) = \mathrm{Max}\{k_\Pi, \phi(x)\}$. Since $\phi \leq \phi'$ and **C** is uniformly $\mathcal{R}$-closed w.r.t. ϕ, then **C** is also uniformly $\mathcal{R}$-closed w.r.t. ϕ', where ϕ' is a non decreasing function. Since (2.2) implies that, for any $\pi : \sigma \in \Pi$, $\mathrm{dg}(\pi : \sigma) \leq \phi'(\mathrm{dg}(\sigma))$, we can apply Proposition 2.4.6 to deduce that $\mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$ is uniformly embedded in $\mathbf{C}$ w.r.t. ϕ'. That is, there exists a representation $r : \mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi)) \to \mathbf{C}$ such that, for any $\tau : \sigma \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}(\Pi))$,

$$\mathrm{dg}(r(\tau)) \leq \phi'(\mathrm{dg}(\tau)) = \mathrm{Max}\{k_\Pi, \phi(\mathrm{dg}(\tau))\} \tag{2.3}$$

But, since ϕ is a non decreasing function, from (2.1) and (2.3) we deduce

$$\mathrm{dg}(r(\tau)) \leq h = \mathrm{Max}\{k_\Pi, \phi(k')\} \ .$$

□

We remark that the notion of non-increasing rule is a generalization of the *subformula property*. Namely, we say that a formula H is a *subformula* of a formula A iff one of the following conditions is satisfied:

1. $A \equiv \theta H$ for some substitution of individual variables θ;
2. A is $\neg B$ and H is a subformula of B;
3. A is either $B \wedge C$ or $B \vee C$ or $B \Rightarrow C$, and H is a subformula of B or a subformula of C;
4. A is either $\forall x B(x)$ or $\exists x B(x)$, and H is a subformula of $B(x)$.

Now, we define what we mean by a *generalized rule satisfying the subformula property*.

2.4.9 Definition *A generalized rule $\mathcal{R}$ (over $\mathcal{L}_\mathcal{A}$) has the* subformula property *iff, for any $\sigma, \sigma_1, \ldots, \sigma_n$, if $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$, then every formula occurring in σ is a subformula of some formula occurring in $\sigma_1, \ldots, \sigma_n$.*

It should be evident that the subformula property corresponds to a particular case of non increasing rule. The following proposition can be easily proved:

2.4.10 Proposition *Let $\mathcal{R}$ be generalized rule (over $\mathcal{L}_\mathcal{A}$) with the subformula property and let $\mathbf{C}$ be an $\mathcal{R}$-closed calculus (over $\mathcal{L}_\mathcal{A}$). If $\Pi \subseteq \mathbf{C}$, then, for any $\tau : \sigma \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$, all the formulas of σ are subformulas of some formula occurring in* $\mathrm{Seq}([\Pi])$. □

To conclude this section we quote a result which states a sufficient condition for a calculus to agree with a formal system.

2.4.11 Proposition *Let $\mathcal{H}$ be an Hilbert-style system over $\mathcal{L}_\mathcal{A}$ and let $\mathbf{S}_\mathcal{H}$ be the formal system $(\mathcal{L}_\mathcal{A}, \mid\!\sim_\mathcal{H}, \mathcal{H})$ generated by $\mathcal{H}$. Moreover, let the calculus $\mathbf{C}$ (over $\mathcal{L}_\mathcal{A}$) be a presentation for $\mathbf{S}_\mathcal{H}$ and let Π be the set of all the proofs $\pi : \vdash A \in \mathbf{C}$ (that is Π contains the proofs of all the theorems of $\mathbf{C}$). If Π is uniformly embedded in $\mathcal{H}$ w.r.t. a function $\phi : \mathbf{N} \to \mathbf{N}$, then $\mathbf{C}$ agrees with $\mathbf{S}_\mathcal{H}$.*

Proof: Since $\Pi \overset{\phi}{\hookrightarrow} \mathcal{H}$, there must exist a representation function $r : \Pi \to \mathcal{H}$ uniform w.r.t. ϕ. Now, let Λ be any subset of **C** such that $\mathrm{dg}(\Lambda) \leq h$ (for some $h \geq 1$), and let $\Lambda' = \Lambda \cap \Pi$. Since r is uniform w.r.t. ϕ and $\mathrm{dg}(\Lambda') \leq h$, we have $\mathrm{dg}(r(\Lambda')) \leq \phi(h)$. This implies $\mathrm{Theo}(\Lambda') \subseteq \mathrm{Theo}(\vdash_{\mathcal{H}}^{\phi(h)})$, and since $\mathrm{Theo}(\Lambda) = \mathrm{Theo}(\Lambda')$ this immediately yields the assertion. □

Hence, by Theorem 2.4.3 and the previous proposition, we deduce:

2.4.12 Theorem *$\mathcal{ND}_{\mathbf{INT}}$ agrees with the formal system generated by $\mathcal{H}_{\mathbf{INT}}$.* □

2.5 Strongly constructive formal systems

Now, we have all the ingredients needed to define the notions of strongly constructive calculus and strongly constructive formal system.

First of all, given a set of formulas Γ, we say that it has the disjunction property and the explicit definability property (for closed or open formulas) iff the following properties hold:

Disjunction property:

(DP) : if $A \vee B \in \Gamma$ and $A \vee B$ is a closed formula, then either $A \in \Gamma$ or $B \in \Gamma$.

(DP$_{\mathrm{open}}$) : if $A \vee B \in \Gamma$, then either $A \in \Gamma$ or $B \in \Gamma$.

Explicit definability property:

(ED) : if $\exists x A(x) \in \Gamma$ and $\exists x A(x)$ is a closed formula, then $A(t/x) \in \Gamma$ for some closed term t of the language.

(ED$_{\mathrm{open}}$) : if $\exists x A(x) \in \Gamma$, then $A(t/x) \in \Gamma$ for some term t of the language.

In the following we will be faced with two different kinds of constructive formal systems: the first is based on the properties (DP$_{\mathrm{open}}$) and (ED$_{\mathrm{open}}$), and include pure logics or systems with weak extra-logical portions:, the second kind of constructive formal system depends on the properties (DP) and (ED) and involves formal systems with strong extra-logical principles such as the induction ones. We will call constructive, without any further qualification both kinds of formal system: the context will clearly determine, for the various systems taken into account, which constructive properties are involved. Thus, we simply say that a set of formulas is *constructive* if it has either the properties (DP) and (ED) or the properties (DP$_{\mathrm{open}}$) and (ED$_{\mathrm{open}}$).

Now, we define what we mean by a constructive formal system and a constructive calculus:

2.5.1 Definition (Constructive formal systems and calculi) *We say that a formal system* $\mathbf{S} = (\mathcal{L}, \mid\!\sim, \mathcal{H})$ *is* constructive *iff* $\mathrm{Theo}(\mid\!\sim)$ *in constructive. Analogously, we say that a calculus* $\mathbf{C} = (C, \mathrm{SubPr})$ *is constructive if the set* $\mathrm{Theo}(\mathbf{C})$ *is constructive.*

More generally, we say that a set of proofs Π is constructive if $\mathrm{Theo}(\Pi)$ is constructive.

We remark that the above definition involves properties which are global for the formal systems and calculi respectively. In particular, with respect to the notion of calculus, this definition does not give any hint on how to find the constructive content of a set of proofs in a subset of the calculus with a bounded logical complexity. On the contrary, our aim is to define *strong constructiveness* of a calculus so that the constructive content of a set of proofs can be extracted from a subset of the calculus with a bounded logical complexity, without need of using the whole calculus. This is the reason why we introduced the notions of generalized rule, $\mathcal{R}$-subcalculus, and uniformity.

Given a calculus $\mathbf{C}$, we will say that it is strongly constructive if we can identify a generalized rule allowing us, starting from a set of proofs $[\Pi]$, to recover in the calculus all the information needed to constructively "complete" the set $\mathrm{Theo}([\Pi])$, but *only looking at a subset of the calculus whose degree is bounded.* Hence, a key point is that this characterization of strong constructiveness is meaningful if the minimal $\mathcal{R}$-subcalculus containing a set of proofs $[\Pi]$ reasonably represents the set of formulas which can be proved using only the information contained in the proofs of $[\Pi]$ and nothing else.

To give an example known in literature, let us consider the case of a set Π consisting of a single proof π of $\mathcal{SEQ}_{\mathbf{INT}}$. In this case the constructive content of π can be directly found in the set $[\pi]$ of the subproofs of π (where the notion of subproof is the standard one). In fact, intuitionistic first order sequent calculus $\mathcal{SEQ}_{\mathbf{INT}}$ enjoys cut-elimination and cut-free proofs have the following properties (see e.g. [Takeuti, 1975, Girard, 1987]):

1. A cut-free proof $\pi : \vdash A \vee B$ directly contains a subproof of A or a subproof of B;

2. A cut-free proof of $\pi : \vdash \exists x A(x)$ directly contains a subproof of $A(t/x)$, for some term t.

Therefore, $[\pi]$ is a constructive calculus. If we only consider cut-free proofs, it is sufficient to require subcalculi to be closed under subproofs; in this case any subcalculus is constructive and the minimum subcalculus of $\mathcal{SEQ}_{\mathbf{INT}}$ containing the proof π is the set $[\pi]$ containing its subproofs.

The situation is slightly more complex when we introduce induction in first order (intuitionistic) arithmetic. A cut-free induction proof

$$\pi : \vdash A(x) \vee B(x)$$

in general does not directly contain neither a $\pi_1 : \vdash A(x)$ nor a $\pi_2 : \vdash B(x)$. But, for every numeral $\mathsf{s}^n\mathsf{0}$, we can extract either a proof $\pi_1^{\mathsf{s}^n\mathsf{0}} : \vdash A(\mathsf{s}^n\mathsf{0})$ or a proof $\pi_2^{\mathsf{s}^n\mathsf{0}} : \vdash B(\mathsf{s}^n\mathsf{0})$ in the following way. Let $\pi : \vdash A(x) \vee B(x)$ be our proof; for the sake of simplicity, we assume that the induction rule has been applied only once, as the last rule. Therefore π is as follows:

$$\pi \equiv \frac{\pi^{\mathsf{0}} : \vdash A(\mathsf{0}) \vee B(\mathsf{0}) \qquad \pi^{\mathsf{s}i} : A(i) \vee B(i) \vdash A(\mathsf{s}i) \vee B(\mathsf{s}i)}{A(x) \vee B(x)}\,\mathsf{Ind}$$

By cut on $\pi^{\mathsf{0}}, \pi^{\mathsf{s0}}, \pi^{\mathsf{s}^2\mathsf{0}}, \ldots$, we can build a proof $\pi^* : \vdash A(\mathsf{s}^n\mathsf{0}) \vee B(\mathsf{s}^n\mathsf{0})$, which does not contain applications of Ind; and we can apply cut-elimination to $\pi^{\mathsf{s}^n\mathsf{0}}$; we obtain a cut-free proof $\pi^* : \vdash A(\mathsf{s}^n\mathsf{0}) \vee B(\mathsf{s}^n\mathsf{0})$, which directly contains a subproof of $A(\mathsf{s}^n\mathsf{0})$ or a subproof of $B(\mathsf{s}^n\mathsf{0})$. Therefore, to extract information from π, we need two operations:

1. *Substitution*: needed to obtain $\pi^{\mathsf{0}}, \pi^{\mathsf{s0}}, \pi^{\mathsf{s}^2\mathsf{0}}, \ldots$;

2. *Cut*: needed to obtain (for every n) $\pi^{\mathsf{s}^n\mathsf{0}} : \vdash A(\mathsf{s}^n\mathsf{0}) \vee B(\mathsf{s}^n\mathsf{0})$.

A similar discussion can be made for the natural deduction calculi $\mathcal{ND}_{\mathbf{INT}}$ (for intuitionistic first order logic) and $\mathcal{ND}_{\mathbf{HA}}$ (for first order intuitionistic arithmetic) using normalized proofs.

According to the above example, we have that the information needed to extract the constructive content of a proof π in $\mathcal{SEQ}_{\mathbf{HA}}$ can be found in the subcalculus of $\mathcal{SEQ}_{\mathbf{HA}}$ containing $\mathrm{SubPr}(\pi)$ and closed under the rules CUT and SUBST, that is, this information is contained in a $\textsc{Cut} \cup \textsc{Subst}$-subcalculus of $\mathcal{SEQ}_{\mathbf{HA}}$. This justifies the notion of $\mathcal{R}$-subcalculus we have introduced in the previous sections. On the other hand, we remark that, given a proof π, we would like to extract the information from a proof without introducing information more complex than the one already contained if π. This is why, in the following fundamental definition, we restrict ourselves to consider only non-increasing generalized rules.

2.5.2 Definition *Let* $\mathbf{C} = (C, \mathrm{SubPr})$ *be a calculus over* $\mathcal{L}_{\mathcal{A}}$. *We say that* $\mathbf{C}$ *is* strongly constructive *iff there exists a generalized rule* $\mathcal{R}$ *such that:*

1. $\mathcal{R}$ *is a non-increasing rule;*

2. $\mathbf{C}$ *is uniformly* $\mathcal{R}$*-closed;*

3. *For any* $\Lambda \subseteq \mathbf{C}$, $\mathrm{Theo}(\mathcal{R}^*([\Lambda]))$ *is constructive.*

2.5.3 Definition *A formal system* $\mathbf{S}$ *is* strongly constructive *iff there exists a calculus* $\mathbf{C}$ *which agrees with* $\mathbf{S}$ *and is strongly constructive.*

2.6 Relations with program synthesis

We believe that the notion of strongly constructive formal system is a key one in program synthesis. To discuss this, let us us consider, for instance, a calculus $\mathbf{C} = (C, \mathrm{SubPr})$ over the language $\mathcal{L}_\mathbf{A}$ of arithmetic, which is a strongly constructive presentation for the formal system of intuitionistic arithmetic; let $\mathcal{R}$ be the non-increasing generalized rule involved in the definition of strong constructiveness of $\mathbf{C}$. Moreover, let us suppose that this calculus contains a proof

$$\pi : \vdash \exists! z F(x_1, \ldots, x_n, z) \ ,$$

where $\exists! z F(x_1, \ldots, x_n, z)$ is a formula in which the only free variables are $x_1, \ldots, x_n$. Now, the strong constructiveness of the calculus allows us to guarantee that it is possible to *uniformly evaluate* the formula $\exists! z F(x_1, \ldots, x_n, z)$ in it; that is, there exists a constructive subset Π of proofs of the calculus $\mathbf{C}$, whose degree is bounded (this is a consequence of Proposition 2.4.8), where all the closed instances $\exists! z F(t_1, \ldots, t_n, z)$ of the formula $\exists! z F(x_1, \ldots, x_n, z)$ are provable. Since Π is a constructive set of proofs, it contains, for any $t_1, \ldots, t_n$, a proof of a closed formula $F(t_1, \ldots, t_, t')$ for some closed term t' of $\mathcal{L}_\mathbf{A}$. This allows us to define the total function f from n-tuples of closed terms of $\mathcal{L}_\mathbf{A}$ into closed terms of $\mathcal{L}_\mathbf{A}$ expressed by the formula

$$\exists! z F(x_1, \ldots, x_n, z) \tag{2.4}$$

as follows:

> For any n-tuple $t_1, \ldots, t_n$ of closed terms of $\mathcal{L}_\mathbf{A}$, the value of $f(t_1, \ldots, t_n)$ is the (only) closed term t' in normal form such that a proof of $F(t_1, \ldots, t_n, t')$ belongs to Π.

In a quite similar way, if Π contains a proof

$$\pi : \vdash P(x_1, \ldots, x_n) \vee \neg P(x_1, \ldots, x_n)$$

in which $x_1, \ldots, x_n$ are the only free variables of $P(x_1, \ldots, x_n)$, it is possible to uniformly evaluate the formula $P(x_1, \ldots, x_n) \vee \neg P(x_1, \ldots, x_n)$ in a $\mathcal{R}$-subcalculus of $\mathbf{C}$ including $[\pi]$. This allows to define a predicate p over n-tuples of closed terms of $\mathcal{L}_\mathbf{A}$, expressed by the formula

$$P(x_1, \ldots, x_n) \vee \neg P(x_1, \ldots, x_n) \ . \tag{2.5}$$

Now, for the sake of simplicity, we informally call *problem* any formula $\mathcal{P}(x_1, \ldots, x_n)$ with either the form (2.4) or the form (2.5) (other kinds of problems could be defined in a more extensive discussion). Hence, we formulate the following notion of *general* (logical) synthesis method:

> A general program synthesis method is any method allowing to extract from a strongly constructive calculus containing a proof of a problem $\mathcal{P}(x_1, \ldots, x_n)$, a subset of proofs in which the problem $\mathcal{P}(x_1, \ldots, x_n)$ is uniformly evaluated.

In our formal setting, the abstract calculus $\mathbb{D}(\mathcal{R}, .)$ can be considered such a general synthesis method. Indeed, $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi]))$ can be seen as a (non-deterministic) computational model and a proof of a problem

$$\pi : \vdash \mathcal{P}(x_1, \ldots, x_n)$$

can be seen as a (non-deterministic) algorithm to solve the problem $\mathcal{P}(x_1, \ldots, x_n)$. Namely, since, by Theorem 2.3.9,

$$\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi])) \preceq_{\mathcal{R}} \mathbf{C}$$

(i.e. it is a generalized $\mathcal{R}$-subcalculus of $\mathbf{C}$) and $\mathcal{R}$ is non-increasing, this set of proofs allows to uniformly evaluate all the closed instances of $\mathcal{P}(x_1, \ldots, x_n)$. We notice that, if $\mathbf{C}$ has the property that $\pi(t_1/x_1, \ldots, t_n/x_n)$ is a proof of the formula $\mathcal{P}(t_1/x_1, \ldots, t_n/x_n)$, then

$$\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi(t_1/x_1, \ldots, t_n/x_n)])) \preceq_{\mathcal{R}} \mathbf{C}$$

and hence this set contains sufficient information to solve the problem; in this sense, the set of proofs $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi(t_1/x_1, \ldots, t_n/x_n)]))$ can be seen as a family of non-deterministic (possibly unsuccessful) computations of the program (proof) π with input values $t_1, \ldots, t_n$, among which there is the successful one.

On the other hand, we notice that, even if the rule $\mathcal{R}$ is recursive, the set of theorems of $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi]))$ is recursively enumerable and hence search strategies are in order to use $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$ as a computational model.

An analysis of such search strategies is out of the scope of this Thesis, which is mainly devoted to the study of the very notion of strong constructiveness. This analysis will be the object of further studies.

Chapter 3

Exhibiting strongly constructive logics

3.1 Generalities

The aim of this chapter and of the next is to convince the reader that the notion of strongly constructive system is adequate for modeling the extraction of information from constructive proofs and is more general than the one based on Normalization and Cut-elimination. To this aim we will prove that a variety of formal systems is strongly constructive. For some of these systems the Normalization Theorem and the Cut-elimination Theorem hold, while others fail to meet these properties.

In particular, in this chapter we will prove that several interesting constructive logics, including the intuitionistic one, are strongly constructive. Here, with logics, we mean purely logical systems, without mathematical axioms; this allows us to consider the notion of constructiveness related to open formulas (that is w.r.t. ($\mathrm{DP_{open}}$) and ($\mathrm{ED_{open}}$)). For such logics we will prove strong constructiveness exhibiting strongly constructive calculi in pseudo-natural deduction-style. This choice has been made for reasons of uniformity and simplicity only; indeed, the same result can be proved using Gentzen's sequent calculi or Hilbert-style calculi.

Our proofs will exhibit the minimal generalized rules needed to get strong constructiveness. In particular, we will show that closure under CUT and SUBST is sufficient to guarantee strong constructiveness (w.r.t ($\mathrm{DP_{open}}$) and ($\mathrm{ED_{open}}$)) of intuitionistic logic and its extension with *Kuroda* principle. On the other hand, closure under CUT, SUBST and some restricted versions of ⇒-introduction, ∀-introduction is sufficient to guarantee strong constructiveness of *Grzegorczyck logic*, *Kreisel-Putnam logic*, *Scott logic*. Finally, we will show that to obtain strong constructiveness for systems obtained by adding a family of *Harrop formulas* to intuitionistic logic we need closure under restricted versions of $\wedge$, $\Rightarrow$ and $\forall$ elimination. However, we remark that no one of our proofs of strong constructiveness depends on Normalization (or Cut-elimination) Theorems.

Hereafter, we will use the following notational convention about the Hilbert-style calculi: given an Hilbert-style calculus $\mathcal{H} = (\mathbf{H}_0, \mathbf{H}_1)$ and a set of axioms-schemes Γ, we will denote with

$$\mathcal{H}_{\mathbf{INT}} + \Gamma$$

the Hilbert-style calculus obtained by adding the set of all the instances of the axiom schemes in Γ to the set $\mathbf{H}_0$.

Since the formal systems treated in this Chapter are predicate intermediate logics, and some discussions about their semantics are in order to give a detailed presentation, we briefly introduce the notions of intermediate logic and Kripke semantics, to help people not acquainted with this material. For a detailed discussion on these topics we refer the reader to [Ono, 1972, Troelstra, 1973a, Gabbay, 1981, **?**, **?**, Avellone et al., 1996].

An *intermediate (predicate) logic* is any set $\mathbf{L}$ of formulas of $\mathcal{L}$ such that:

1. Theo($\mathbf{INT}$) $\subseteq \mathbf{L} \subseteq$ Theo($\mathbf{CL}$), where $\mathbf{INT}$ and $\mathbf{CL}$ indicate respectively the formal system of intuitionistic and classical logic. (That is the set of all the intuitionistically valid formulas is included in $\mathbf{L}$ and any formula of $\mathbf{L}$ is classically valid.)

2. $\mathbf{L}$ is closed under detachment (i.e., $A \in \mathbf{L}$ and $A \Rightarrow B \in \mathbf{L}$ implies $B \in \mathbf{L}$).

3. $\mathbf{L}$ is closed under generalization (i.e., $A(x) \in \mathbf{L}$ implies $\forall x A(x) \in \mathbf{L}$).

4. $\mathbf{L}$ is closed under predicate substitution (i.e., $A \in \mathbf{L}$ implies $\xi A \in \mathbf{L}$ for every predicate substitution ξ). For a formal definition of predicate substitution we refer the reader to [Ono, 1972, Avellone et al., 1996].

Likewise, an *intermediate propositional logic* is any set $\mathbf{L}$ of formulas of the propositional language such that Theo($\mathbf{INT}_{\mathrm{prop}}$) $\subseteq \mathbf{L} \subseteq$ Theo($\mathbf{CL}_{\mathrm{prop}}$) (where $\mathbf{INT}_{\mathrm{prop}}$ and $\mathbf{CL}_{\mathrm{prop}}$ are the formal system of intuitionistic propositional logic and the formal system of classical propositional logics respectively), and $\mathbf{L}$ is closed under detachment and propositional substitution.

A *Kripke frame* is a triple $\langle P, \leq, D\rangle$, where $\langle P, \leq\rangle$ is a non-empty partially ordered set and D is a function associating, with every element $\alpha \in P$ a nonempty set $D(\alpha)$ in such a way that, if $\alpha \leq \beta$ in $\langle P, \leq\rangle$, then $D(\alpha) \subseteq D(\beta)$. We call $D(\alpha)$ the *domain of* α. We also associate with any element $\alpha \in P$ the extended language $\mathcal{L}_\alpha$ obtained by adding to the pure predicate language $\mathcal{L}$ the elements of $D(\alpha)$ as constant symbols.

Given a Kripke-frame $\langle P, \leq, D\rangle$, $\underline{K} = \langle P, \leq, D, \Vdash\rangle$ is a *Kripke model built on the frame* $\langle P, \leq, D\rangle$ if $\Vdash$ (called the *forcing* relation) is a binary relation between elements of P and closed atomic formulas of the corresponding extended languages, such that

- if $\alpha \Vdash A(c_1, \ldots, c_n)$ (where $c_1, \ldots, c_n$ are constant symbols of $\mathcal{L}_\alpha$) then, for any $\beta \in P$ such that $\alpha \leq \beta$, $\beta \Vdash A(c_1, \ldots, c_n)$.

In the propositional context a Kripke-frame is a partially ordered set $\langle P, \leq \rangle$, and a Kripke model characterized by such a frame is a triple $\underline{K} = \langle P, \leq, \Vdash \rangle$, where $\Vdash$ is a binary relation between elements of P and propositional variables such that, for any propositional variable p, if $\alpha \Vdash p$ then $\beta \Vdash p$ for any $\beta \in P$ such that $\alpha \leq \beta$.

The forcing relation is extended to arbitrary formulas of the language as follows:

1. $\alpha \Vdash B \wedge C$ iff $\alpha \Vdash B$ and $\alpha \Vdash C$;
2. $\alpha \Vdash B \vee C$ iff either $\alpha \Vdash B$ or $\alpha \Vdash C$;
3. $\alpha \Vdash B \Rightarrow C$ iff, for any $\beta \in P$ such that $\alpha \leq \beta$, $\beta \Vdash B$ implies $\beta \Vdash C$;
4. $\alpha \Vdash \exists x B(x)$ iff there exists $c \in D(\alpha)$ such that $\alpha \Vdash B(c/x)$;
5. $\alpha \Vdash \forall x B(x)$ iff, for any $\beta \in P$ such that $\alpha \leq \beta$ and for any $c \in D(\beta)$, $\beta \Vdash B(c/x)$.

We say that a formula A *holds in* $\underline{K} = \langle P, \leq, D, \Vdash \rangle$ iff, $\alpha \Vdash \forall A$ ($\alpha \Vdash A$ in the propositional case) for any $\alpha \in P$, where $\forall A$ is the universal closure of A. Now, given a family $\mathcal{F}$ of Kripke-frames, we denote with $\mathcal{K}(\mathcal{F})$ the family of all the Kripke-models built on the Kripke-frames in $\mathcal{F}$. Finally, let us denote with $\mathbf{L}(\mathcal{K})$ the set of all the formulas of $\mathcal{L}$ which hold in any Kripke model of $\mathcal{K}$. We say that an intermediate logic $\mathbf{L}$ *is characterized by the family of Kripke frames* $\mathcal{F}$ iff $\mathbf{L} = \mathbf{L}(\mathcal{K}(\mathcal{F}))$.

We conclude this presentation, recalling that intuitionistic predicate logic (that is the set of theorems of $\mathcal{H}_{\mathbf{INT}}$) is characterized by the family of all the Kripke-frames.

Now, to conclude this section we briefly sketch the general line of the proofs of strong constructiveness we provide in this Chapter. First of all, we will characterize the formal system in hand by means of an Hilbert-style calculus $\mathcal{H}$ including $\mathcal{H}_{\mathbf{INT}}$, then we will give a pseudo-natural deduction presentation $\mathcal{ND}$ for this formal system. (We say pseudo-natural deduction calculus to mean that, for the specific axioms of the logic in hand, we do not introduce a pair of rules, one of introduction and one of elimination, according to the paradigms of the natural deduction calculi.) At this point, we will introduce a suitable non-increasing generalized rule $\mathcal{R}$ and will prove that the calculus $\mathcal{ND}$ is uniformly $\mathcal{R}$-closed. Then we will consider the generalized $\mathcal{R}$-subcalculi $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$ generated by sets Π of proofs of the calculus $\mathcal{ND}$ in hand. All the constructiveness results will be proved by showing that, for every $\Pi \subseteq \mathcal{ND}$, $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$ satisfies ($\mathrm{DP_{open}}$) and ($\mathrm{ED_{open}}$). Thus, by Point (1) of Theorem 2.3.9, we will get that every $\mathcal{R}$-subcalculus of $\mathcal{ND}$ including $[\Pi]$ satisfies ($\mathrm{DP_{open}}$) and ($\mathrm{ED_{open}}$). The techniques we will use to prove that $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$

satisfies ($\mathrm{D}\mathrm{P}_{\mathrm{open}}$) and ($\mathrm{E}\mathrm{D}_{\mathrm{open}}$) will depend on the logics in hand; the choice of the systems presented in this Chapter is mainly devoted to provide a good illustration of the variety of such techniques. Finally, to prove the strong constructiveness of the formal system generated by the various Hilbert-style systems $\mathcal{H}$ in hand, we will use Proposition 2.4.11 and the fact that the involved calculi $\mathcal{ND}$ are uniformly embedded in $\mathcal{H}$.

3.2 Intuitionistic Logic

In the case of intuitionistic logic, to prove strong constructiveness it is enough to consider the generalized rules Subst and Cut so defined:

Substitution rule (Subst).
The domain of Subst is the set of all the sequents, and, for every substitution θ of terms for individual variables:

$$\theta\Gamma \vdash \theta\Delta \in \textsc{Subst}(\Gamma \vdash \Delta) \ .$$

Cut rule (Cut).
The domain of Cut contains all the sequences of sequents which have the form $\Gamma_1 \vdash H; \Gamma_2, H \vdash A$, and:

$$\Gamma_1, \Gamma_2 \vdash A \in \textsc{Cut}(\Gamma_1 \vdash H; \Gamma_2, H \vdash A) \ .$$

We will denote with $\mathcal{R}_{\mathbf{INT}}$ the union of these two generalized rules, that is:

$$\mathcal{R}_{\mathbf{INT}} = \textsc{Cut} \cup \textsc{Subst}$$

It is immediate to verify that this is a non-increasing rule. Our proof of strong constructiveness is based on the following notion of formula evaluated in a set of proofs.

3.2.1 Definition (Evaluation) *Let Π be a set of proofs and A be a formula. We say that A is evaluated in Π iff the following conditions hold:*

(i). There is a proof $\pi \in \Pi$ such that $\pi : \vdash A$;

(ii). According to the form of A, one of the following cases hold:

(a) A is atomic or negated;
(b) $A \equiv B \wedge C$, and both B and C are evaluated in Π;
(c) $A \equiv B \vee C$, and either B is evaluated in Π or C is evaluated in Π;
(d) $A \equiv B \Rightarrow C$, and either B is not evaluated in Π or C is evaluated in Π;
(e) $A \equiv \exists x B(x)$, and $B(t/x)$ is evaluated in Π for some term t;

(f) $A \equiv \forall x B(x)$, *and, for any term* t, $B(t/x)$ *is evaluated in* Π.

We remark that in the above definition no assumption is made on the form of the proofs in Π; this gives rise to a technique which is largely independent on the presentation of the calculi.

The following lemma is fundamental to prove strong constructiveness of several calculi by means of the above notion of formula evaluated in a set of proofs.

3.2.2 Lemma *Let* $\mathcal{R}$ *be any generalized rule including* Subst *and* Cut, *let* $\mathbf{C}$ *be an* $\mathcal{R}$*-closed calculus and let* Π *be a set of proofs of* $\mathbf{C}$. *For any* $\pi : \Gamma \vdash A \in \textsc{Subst}^*([\Pi])$, *if* $\Gamma \subseteq \mathrm{Theo}(\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi])))$ *then there exists* $\tau : \vdash A \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$.

Proof: Since, $\pi : \Gamma \vdash A \in \textsc{Subst}^*([\Pi])$ and the composition of substitutions is a substitution, there must exist a substitution for individual variables θ and a proof $\pi' \in [\Pi]$ such that $\Gamma \vdash A \equiv \theta\Gamma' \vdash \theta A'$ and $\pi' : \Gamma' \vdash A'$. Since, $\Gamma' \vdash A' \in \mathrm{Seq}([\Pi])$, by definition of $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$ there exists a proof tree

$$\tau' : \Gamma' \vdash A' \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi])) \ .$$

But, $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$ is Subst-closed, and hence there is a proof

$$\tau'' : \Gamma \vdash A \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi])) \ .$$

Now, let $\Gamma = \{H_1, \ldots, H_n\}$. Since $\Gamma \subseteq \mathrm{Theo}(\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$, there exist proofs

$$\tau_1 : \vdash H_1 \ , \ \ldots \ , \ \tau_n : \vdash H_n \ \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi])) \ .$$

But the calculus $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$ is Cut-closed, and hence, by applying the Cut-rule to the proofs

$$\tau_1 : \vdash H_1 \ , \ldots , \ \tau_n : \vdash H_n \ \text{ and } \ \tau'' : \{H_1, \ldots, H_n\} \vdash A$$

we obtain that $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$ also contains a proof $\tau : \vdash A$. □

Let **INT** be the formal system generated by the Hilbert-style calculus $\mathcal{H}_{\mathbf{INT}}$, that is

$$\mathbf{INT} = (\mathcal{L}, \mid\!\sim_{\mathcal{H}_{\mathbf{INT}}}, \mathcal{H}_{\mathbf{INT}}) \ .$$

Now, we give the proof of strong constructiveness for the formal system of intuitionistic logic **INT**, by proving that the calculus $\mathcal{ND}_{\mathbf{INT}}$, quoted in Section 1.2.2, which is a presentation for **INT** (see Theorem 1.2.1), is a strongly constructive calculus for it. However, we remark that the proof could be developed along the same lines for the sequent calculus $\mathcal{SEQ}_{\mathbf{INT}}$ and the Hilbert style calculus for intuitionistic logic quoted in [Kleene, 1952], as well as the Hilbert-style calculus $\mathcal{H}_{\mathbf{INT}}$ defining **INT**.

First of all, to apply the results of Chapter 2, we need to prove that $\mathcal{ND}_{\mathbf{INT}}$ is uniformly $\mathcal{R}_{\mathbf{INT}}$-closed.

3.2.3 Proposition $\mathcal{ND}_{\mathbf{INT}}$ *is uniformly* $\mathcal{R}_{\mathbf{INT}}$*-closed.*

Proof: To prove that $\mathcal{ND}_{\mathbf{INT}}$ is SUBST-closed let us consider a proof $\pi : \Gamma \vdash A \in \mathcal{ND}_{\mathbf{INT}}$, and a substitution on the individual variables θ. Since we can change the names of the proper parameters of π in such a way that their set is disjunct from the set of the free variables of π and from the set of the free variables occurring in the terms that θ associates with the free variables of π (see Section 1.2.2), the tree π' obtained by applying the substitution θ to all the formulas which occur in the proof π is a proof in $\mathcal{ND}_{\mathbf{INT}}$. Moreover, it is a proof of the sequent $\theta\Gamma \vdash \theta A$ and $\mathrm{dg}(\pi') = \mathrm{dg}(\pi)$. To prove that $\mathcal{ND}_{\mathbf{INT}}$ is CUT-closed, let us consider two proofs $\pi_1 : \Gamma \vdash H$ and $\pi_2 : \Delta, H \vdash A$ of $\mathcal{ND}_{\mathbf{INT}}$. The following is a proof of the sequent $\Gamma, \Delta \vdash A$ in $\mathcal{ND}_{\mathbf{INT}}$:

$$\pi' \equiv \cfrac{\pi_1 : \Gamma \vdash H \qquad \cfrac{\pi_2 : \Delta, H \vdash A}{\Delta \vdash H \Rightarrow A}\,{\scriptstyle \mathrm{I}\Rightarrow}}{\Gamma, \Delta \vdash A}\,{\scriptstyle \mathrm{E}\Rightarrow} \; .$$

Moreover,

$$\begin{aligned} \mathrm{dg}(\pi') &= \mathrm{Max}\{\mathrm{dg}(\pi_1), \mathrm{dg}(\pi_2), \mathrm{dg}(\Delta \vdash H \Rightarrow A)\} = \\ &= \mathrm{Max}\{\mathrm{dg}(\pi_1), \mathrm{dg}(\pi_2), \mathrm{dg}(\Delta, H \vdash A) + 1\} \; . \end{aligned}$$

Hence $\mathcal{ND}_{\mathbf{INT}}$ is uniformly $\mathcal{R}_{\mathbf{INT}}$-closed w.r.t. the function $\phi : \mathbf{N} \to \mathbf{N}$ such that $\phi(x) = x + 1$. □

Now, we will prove that, for any set Π of proofs of $\mathcal{ND}_{\mathbf{INT}}$, the information contained in the subproofs of Π is sufficient to obtain a constructive generalized $\mathcal{R}_{\mathbf{INT}}$-subcalculus of $\mathcal{ND}_{\mathbf{INT}}$ containing $\mathrm{Seq}([\Pi])$. This generalized $\mathcal{R}_{\mathbf{INT}}$-subcalculus is $\mathbb{D}(\mathcal{R}_{\mathbf{INT}}, \mathrm{Seq}([\Pi]))$. To simplify the notation we set:

$$\mathbb{D}_{\mathbf{INT}}(\Pi) \;= \mathbb{D}(\mathcal{R}_{\mathbf{INT}}, \mathrm{Seq}(\Pi))$$

Before going into the details of the proof of the strong constructiveness of $\mathcal{ND}_{\mathbf{INT}}$, let us consider the following trivial fact, which will be implicitly used in the following result and, in general, in all the proofs of strong constructiveness we will develop in this Thesis. Let Π be a set of proofs of a pseudo-natural deduction calculus. If $\pi \in [\Pi]$ and $\pi_1, \ldots, \pi_n$ are all the subproofs of π, then, for any substitution θ, $\theta\pi_1, \ldots, \theta\pi_n$ are all the subproofs of $\theta\pi$. This implies that, if $\pi \in \mathrm{SUBST}^*([\Pi])$, then all the subproofs of π belongs to $\mathrm{SUBST}^*([\Pi])$.

The following lemma is a key point of our proof of strong constructiveness of $\mathcal{ND}_{\mathbf{INT}}$.

3.2.4 Lemma *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{INT}}$. *For any* $\pi : \Gamma \vdash A \in \mathrm{SUBST}^*([\Pi])$, *if* Γ *is evaluated in* $\mathbb{D}_{\mathbf{INT}}([\Pi])$ *then* A *is evaluated in* $\mathbb{D}_{\mathbf{INT}}([\Pi])$.

Proof: We will denote with $\pi, \pi_1, \pi_2, \ldots$ the proofs of $\mathcal{ND}_{\mathbf{INT}}$ and with $\tau, \tau', \tau_1, \ldots$ the ones of $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Since $\Gamma = \{H_1, \ldots, H_n\}$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$, by Point (i) of Definition 3.2.1 we have that there exist proofs

$$\tau_1 : \vdash H_1 \; , \; \ldots \; , \; \tau_n : \vdash H_n \;\; \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi])) \; ,$$

and hence $\Gamma \subseteq \text{Theo}(\mathbb{D}_{\mathbf{INT}}([\Pi]))$. By Lemma 3.2.2, this implies that there exists a proof

$$\tau' : \vdash A \in \mathbb{D}_{\mathbf{INT}}([\Pi]) \ .$$

This proves that A satisfies Point (i) of Definition 3.2.1. To complete the proof, we must show that A also satisfies Point (ii) of this definition. We prove this point by induction on depth(π).

Basis: If depth(π) = 0 then the only rule which occurs in π must be an axiom, that is $A \in \Gamma$. Thus, A is trivially evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$.

Step: We assume that our assertion holds for any proof $\pi' : \Gamma' \vdash A' \in \text{SUBST}^*([\Pi])$ such that depth$(\pi') \leq h$ $(h \geq 0)$. We prove it for $\pi : \Gamma \vdash A$ with depth$(\pi) = h + 1$. The proof goes on by cases according to the last rule applied in π.

- **Conjunction Introduction**.

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma_1 \vdash B \qquad \pi_2 : \Gamma_2 \vdash C}{\Gamma \vdash B \wedge C}\,{\scriptstyle \text{I}\wedge}$$

  Since $\Gamma = \Gamma_1 \cup \Gamma_2$, depth$(\pi_1) \leq h$ and depth$(\pi_2) \leq h$, the application of the induction hypothesis to $\pi_1 : \Gamma_1 \vdash B$ and $\pi_2 : \Gamma_2 \vdash C$ immediately yields that both B and C are evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Hence, by definition, $B \wedge C$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$.

- **Conjunction Elimination**.

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma \vdash B \wedge C}{\Gamma \vdash B}\,{\scriptstyle \text{E}\wedge} \quad (\text{or} \ \ \pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma \vdash B \wedge C}{\Gamma \vdash C}\,{\scriptstyle \text{E}\wedge})$$

  Since depth$(\pi_1) \leq h$, we immediately have, by induction hypothesis, that $B \wedge C$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. This implies that both B and C are evaluated in this set of proofs.

- **Disjunction Introduction**.

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma \vdash B}{\Gamma \vdash B \vee C}\,{\scriptstyle \text{I}\vee} \quad (\text{or} \ \ \pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma \vdash C}{\Gamma \vdash B \vee C}\,{\scriptstyle \text{I}\vee})$$

  Since depth$(\pi_1) \leq h$, we immediately have that B (or C) is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$; this implies that $B \vee C$ is evaluated in this set of proofs.

- **Disjunction Elimination**.

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma_1 \vdash B \vee C \qquad \pi_2 : \Gamma_2, B \vdash A \qquad \pi_3 : \Gamma_3, C \vdash A}{\Gamma \vdash A}\,{\scriptstyle \text{E}\vee}$$

  Since Γ_1 is included in Γ and the depth of π_1 is less than or equal to h, we can apply the induction hypothesis to $\pi_1 : \Gamma_1 \vdash B \vee C$, to deduce that

$B \vee C$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Thus, either B is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$ or C is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Hence, we obtain that A is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$, in the former case by applying the induction hypothesis to the proof $\pi_2 : \Gamma_2, B \vdash A$, in the latter case by applying the induction hypothesis to the proof $\pi_3 : \Gamma_3, C \vdash A$.

- **Implication Introduction**.

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma, B \vdash C}{\Gamma \vdash B \Rightarrow C}\,{\scriptstyle \mathrm{I}\Rightarrow}$$

By induction hypothesis, if B is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$ then C is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Hence the assertion.

- **Implication Elimination**.

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma_1 \vdash B \Rightarrow C \qquad \pi_2 : \Gamma_2 \vdash B}{\Gamma \vdash C}\,{\scriptstyle \mathrm{E}\Rightarrow}$$

By induction hypothesis $B \Rightarrow C$ and B are evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$; by definition, this implies that C is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$.

- **Exists Introduction**.

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma \vdash B(t/x)}{\Gamma \vdash \exists x B(x)}\,{\scriptstyle \mathrm{I}\exists}$$

By induction hypothesis, $B(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$ and this immediately implies, by definition, that $\exists x B(x)$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$.

- **Exists Elimination**.

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma_1 \vdash \exists x B(x) \qquad \pi_s : \Gamma_2, B(s/x) \vdash A}{\Gamma \vdash A}\,{\scriptstyle \mathrm{E}\exists}$$

By applying the induction hypothesis to π_1, we deduce that there exists a term t such that $B(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Now, since s is the proper parameter of the exists elimination, it does not occur free in Γ_2 and in A. Thus, the tree $\pi_s(t/s)$, obtained by replacing any free occurrence of s in any formula of the proof π_s with t, is a proof of the sequent $\Gamma_2, B(t/x) \vdash A$ in $\mathcal{ND}_{\mathbf{INT}}$. Moreover, since $\pi_s \in [\Pi]$, we also have that

$$\pi_s(t/s) : \Gamma_2, B(t/x) \vdash A \in \textsc{Subst}^*([\Pi]) \ .$$

Now, since $\Gamma_2, B(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$, by induction hypothesis on $\pi_s[t/s]$, we obtain that A is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$.

- **For-all Introduction**.

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_s : \Gamma \vdash B(s/x)}{\Gamma \vdash \forall x B(x)}\,\text{I}\forall$$

Let t be any term. Since s is the proper parameter of the application of the I$\forall$–rule, by our assumptions on the proper parameters of a proof, we immediately have that the tree $\pi_s(t/s)$, obtained by replacing any free occurrence of s in any formula of π_s with t, is a proof of $\Gamma \vdash B(t/x)$ in $\mathcal{ND}_{\mathbf{INT}}$. Moreover,

$$\pi_s(t/s) : \Gamma \vdash B(t/x) \in \textsc{Subst}^*([\Pi]) \ .$$

Now, by induction hypothesis on the latter proof, we obtain that $B(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Since t is any term of $\mathcal{L}$, we have that $B(t/x)$ is evaluate in $\mathbb{D}_{\mathbf{INT}}([\Pi])$ for any term t. That is, $\forall x B(x)$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$.

- **For-all Elimination**

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma \vdash \forall x B(x)}{\Gamma \vdash B(t/x)}\,\text{E}\forall$$

By induction hypothesis, $\forall x B(x)$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. This immediately implies, by definition, that $B(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$.

□

3.2.5 Corollary *Let Π be a set of proofs of $\mathcal{ND}_{\mathbf{INT}}$. For any proof $\tau : \Gamma \vdash A \in \mathbb{D}_{\mathbf{INT}}([\Pi])$ and any substitution θ, if $\theta\Gamma$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$, then θA is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$.*

Proof: First of all, we must show that there exists a proof of the sequent $\vdash \theta A$ in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Let $\Gamma = \{H_1, \ldots, H_n\}$; since $\theta\Gamma$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$, by Point (i) of Definition 3.2.1, there exist proofs

$$\tau_1 : \vdash \theta H_1 \ , \ \ldots \ , \ \tau_n : \vdash \theta H_n \in \mathbb{D}_{\mathbf{INT}}([\Pi]) \ .$$

Moreover, since $\mathbb{D}_{\mathbf{INT}}([\Pi])$ is $\textsc{Subst}$-closed, it also contains a proof

$$\tau' : \theta\Gamma \vdash \theta A$$

and since $\mathbb{D}_{\mathbf{INT}}([\Pi])$ is $\textsc{Cut}$-closed, this implies that in this calculus a proof

$$\tau'' : \vdash \theta A$$

exists. This prove that A satisfies Point (i) of Definition 3.2.1. To prove Point (ii) we proceed by induction on the $\textsc{Cut}$-*depth* of $\tau : \Gamma \vdash A$, i.e. on the number of $\textsc{Cut}$-rules applied in τ.

Basis: If no CUT-rule is applied in τ, then, by definition of $\mathbb{D}_{\mathbf{INT}}([\Pi])$, $\tau : \Gamma \vdash A$ is obtained by applying a (possibly empty) sequence of SUBST-rules to a sequent in $\text{Seq}([\Pi])$. Thus, there exists a proof $\pi' : \Gamma' \vdash A' \in [\Pi]$ such that $\theta'\Gamma' \vdash \theta' A' \equiv \Gamma \vdash A$ for some substitution θ'. Then, $\text{SUBST}^*([\Pi])$ also contains a proof of the sequent $\theta\Gamma \vdash \theta A \equiv \theta\theta'\Gamma' \vdash \theta\theta' A'$. Since $\theta\Gamma$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$, by Lemma 3.2.4, we have that θA is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$.
Step: The CUT-depth of τ is $h+1$ ($h \geq 0$), namely $\tau : \Gamma \vdash A$ is:

$$\dfrac{\dfrac{\dfrac{\tau_1 : \Gamma_1' \vdash H \qquad \tau_2 : \Gamma_2', H \vdash A'}{\Gamma' \vdash A'}\ \text{Cut}}{\vdots}\ \text{Subst}}{\theta'\Gamma' \vdash \theta' A'}\ \text{Subst}$$

where $\Gamma' = \Gamma_1' \cup \Gamma_2'$, the CUT-depth of τ_1, τ_2 is less than or equal to h and τ ends with a (possibly empty) sequence of applications of SUBST. We prove that, if $\theta\Gamma \equiv \theta\theta'\Gamma'$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$, then $\theta A \equiv \theta\theta' A'$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Since $\theta\theta'\Gamma_1 \subseteq \theta\theta'\Gamma'$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$ by induction hypothesis on the proof

$$\dfrac{\tau_1 : \Gamma_1' \vdash H}{\theta\theta'\Gamma_1' \vdash \theta\theta' H}\ \text{Subst}$$

we get that, $\theta\theta' H$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Hence, $\theta\theta'\Gamma_2', \theta\theta' H$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$, and thus, by induction hypothesis on the proof

$$\dfrac{\tau_2 : \Gamma_2', H \vdash A'}{\theta\theta'\Gamma_2', \theta\theta' H \vdash \theta\theta' A'}\ \text{Subst}$$

we get that $\theta\theta' A'$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. □

3.2.6 Corollary *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{INT}}$. *Then* $\mathbb{D}_{\mathbf{INT}}([\Pi])$ *satisfies* (DP_{open}) *and* (ED_{open}).

Proof: Let $A \vee B \in \text{Theo}(\mathbb{D}_{\mathbf{INT}}([\Pi]))$; then there exists a proof $\tau : \vdash A \vee B$ in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Since the empty set of premises is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$, by Corollary 3.2.5 we immediately have that $A \vee B$ is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. By definition of evaluation, it follows that either A is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$ or B is evaluated in $\mathbb{D}_{\mathbf{INT}}([\Pi])$. Hence, by Point (i) of the definition of evaluation, we get that either $A \in \text{Theo}(\mathbb{D}_{\mathbf{INT}}([\Pi]))$ or $B \in \text{Theo}(\mathbb{D}_{\mathbf{INT}}([\Pi]))$. This means that $\text{Theo}(\mathbb{D}_{\mathbf{INT}}([\Pi]))$ satisfies (DP_{open}). The proof that $\text{Theo}(\mathbb{D}_{\mathbf{INT}}([\Pi]))$ satisfies (ED_{open}) is analogous. □

The previous corollary allows us to prove the strong constructiveness result for the calculus $\mathcal{ND}_{\mathbf{INT}}$.

3.2.7 Theorem (SCR-$\mathcal{ND}_{\mathbf{INT}}$) *$\mathcal{ND}_{\mathbf{INT}}$ is a strongly constructive calculus w.r.t.* (DP$_{\text{open}}$) *and* (ED$_{\text{open}}$).

Proof: We already know that $\mathcal{R}_{\mathbf{INT}}$ is a non increasing rule and, by Proposition 3.2.3, that $\mathcal{ND}_{\mathbf{INT}}$ is uniformly $\mathcal{R}_{\mathbf{INT}}$-closed. Hence, the proof of strong constructiveness of $\mathcal{ND}_{\mathbf{INT}}$ amounts to show that, for any set of proofs Π of $\mathcal{ND}_{\mathbf{INT}}$, $\mathcal{R}^*_{\mathbf{INT}}([\Pi])$ is constructive w.r.t. (DP$_{\text{open}}$) and (ED$_{\text{open}}$). But, by Point (i) of Proposition 2.3.8,

$$\mathcal{R}^*_{\mathbf{INT}}([\Pi]) \approx \mathbb{D}_{\mathbf{INT}}([\Pi])$$

and, by Corollary 3.2.6, $\mathbb{D}_{\mathbf{INT}}([\Pi])$ is constructive w.r.t. (DP$_{\text{open}}$) and (ED$_{\text{open}}$) and hence the assertion. □

The proof of the strong constructiveness of the formal system **INT** immediately follows from the previous theorem and from the fact that $\mathcal{ND}_{\mathbf{INT}}$ is uniformly embedded in $\mathcal{H}_{\mathbf{INT}}$. The latter fact has been stated in Theorem 2.4.3 and implies that $\mathcal{ND}_{\mathbf{INT}}$ agrees with the formal system **INT** according to Proposition 2.4.11. Therefore:

3.2.8 Theorem (SCR-INT) **INT** *is a strongly constructive formal system w.r.t.* (DP$_{\text{open}}$) *and* (ED$_{\text{open}}$). □

To conclude this section, we remark that any usual sequent calculus for **INT** can be proved to be a strongly constructive calculus, and this also holds for formulations of sequent calculi which do not meet the Cut-elimination property such as the one given in [Miglioli and Ornaghi, 1979].

3.3 Kuroda Logic

Now, we consider the *Kuroda Principle*, that is the axiom schema:

$$(\mathsf{Kur}) \qquad \forall x \neg\neg A(x) \Rightarrow \neg\neg \forall x A(x) \ .$$

This principle has an important role with respect to classical logic in the following sense: let **L** be any intermediate (predicate) logic, including all the instances of the axiom schema (Kur); let Γ be any set of formulas and let $\Gamma + \mathbf{L}$ and $\Gamma + \mathbf{CL}$ denote respectively the closure with respect to modus ponens and generalization of $\Gamma \cup \mathbf{L}$ and $\Gamma \cup \mathbf{CL}$ (**CL** being classical logic). Then $\Gamma + \mathbf{L}$ is consistent iff $\Gamma + \mathbf{CL}$ is (see [Gabbay, 1981, Avellone et al., 1996, Miglioli et al., 1997]).

(Kur) is often quoted in the following equivalent form:

$$(\mathsf{Kur}') \qquad \neg\neg \forall x (A(x) \vee \neg A(x)) \ .$$

As it is well known (see e.g. [Kleene, 1952, Troelstra, 1973a]), these principles are classically valid but not intuitionistically valid. A Kripke model for intuitionistic logic which is a counter-model for (Kur′) is shown in [Smorynski, 1973]. Now, let

$$\mathcal{H}_{\mathbf{Kur}} = \mathcal{H}_{\mathbf{INT}} + \{(\mathsf{Kur})\} \ ,$$

that is $\mathcal{H}_{\mathbf{Kur}}$ is the Hilbert-style calculus obtained by adding all the instances of the axiom schema (Kur) to the set of axioms of $\mathcal{H}_{\mathbf{INT}}$. We denote with **Kur** the formal system generated by $\mathcal{H}_{\mathbf{Kur}}$. The intermediate logic determined by **Kur**, that is the set of theorems of the formal system **Kur**, is characterized by the family of Kripke frames with final elements. That is the family consisting of all the Kripke frames $\langle P, \leq, D\rangle$ with the following property: for any $\alpha \in P$ there exists $\beta \in P$ such that, for any $\gamma \in P$, if $\beta \leq \gamma$ then $\beta = \gamma$.

The axiom schema (Kur) can be expressed with the following rule in the style of a pseudo-natural calculus:

$$\frac{\pi : \Gamma \vdash \forall x \neg\neg A(x)}{\Gamma \vdash \neg\neg\forall x A(x)}\,\mathsf{Kur}$$

$\mathcal{ND}_{\mathbf{Kur}}$ will denote the pseudo-natural deduction calculus obtained by adding the Kur-rule to $\mathcal{ND}_{\mathbf{INT}}$. It is easy to verify that $\mathcal{ND}_{\mathbf{Kur}}$ is a presentation for the formal system **Kur**. Moreover, the following result immediately follows from Proposition 3.2.3.

3.3.1 Proposition $\mathcal{ND}_{\mathbf{Kur}}$ *is uniformly* $\mathcal{R}_{\mathbf{INT}}$*-closed.* □

The proof of strong constructiveness for $\mathcal{ND}_{\mathbf{Kur}}$ goes on along the same lines as the ones for $\mathcal{ND}_{\mathbf{INT}}$. Only a slight modification of the main lemma is required.

3.3.2 Lemma *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{Kur}}$*. For any proof* $\pi : \Gamma \vdash A \in$ SUBST$^*([\Pi])$*, if* Γ *is evaluated in* $\mathbb{D}_{\mathbf{INT}}([\Pi])$*, then* A *is evaluated in* $\mathbb{D}_{\mathbf{INT}}([\Pi])$*.*

Proof: $\vdash A \in \mathrm{Seq}(\mathbb{D}_{\mathbf{INT}}([\Pi]))$ follows from Lemma 3.2.2. The proof of Point (ii) of Definition 3.2.1 is analogous to the one of Lemma 3.3.2. It goes on by induction on the depth of the proof $\pi \in \mathbf{C}$. We only have to treat the new case corresponding to the Kur-rule. But this proof is trivial; in fact, if $\pi : \Gamma \vdash A$ is a proof ending with an application of the Kur-rule, A must be a negated formula and negated formulas only need to be provable in $\mathbb{D}_{\mathbf{INT}}([\Pi])$ to be evaluated in this set. □

Hence, similarly to Section 3.2, we can prove the following results.

3.3.3 Theorem (SCR-$\mathcal{ND}_{\mathbf{Kur}}$) $\mathcal{ND}_{\mathbf{Kur}}$ *is a strongly constructive calculus w.r.t.* (DP$_{\mathrm{open}}$) *and* (ED$_{\mathrm{open}}$). □

3.3.4 Corollary (SCR-Kur) Kur *is a strongly constructive formal system w.r.t.* (DP$_{\mathrm{open}}$) *and* (ED$_{\mathrm{open}}$). □

3.4 Grzegorczyck Logic

Another interesting principle quoted in [Troelstra, 1973a] is the *Grzegorczyck Principle*:

$$(\mathsf{Grz}) \qquad \forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B \ ,$$

under the condition that x is not free in B.

It is immediate to verify that (Grz) is classically valid, but it is not intuitionistically valid. A Kripke model for intuitionistic logic which is a counter-model for it is quoted in [Smorynski, 1973]. (Grz) can be expressed with the following rule in the style of a pseudo-natural calculus:

$$\frac{\pi : \Gamma \vdash \forall x(A(x) \vee B)}{\Gamma \vdash \forall x A(x) \vee B}\,\mathsf{Grz}$$

with x not occurring free in B.

Grzegorczyck logic is the formal system **Grz** generated by the Hilbert-style calculus obtained by adding all instances of the axiom schema (Grz) to $\mathcal{H}_{\mathbf{INT}}$, that is

$$\mathcal{H}_{\mathbf{Grz}} = \mathcal{H}_{\mathbf{INT}} + \{(\mathsf{Grz})\} \ .$$

The intermediate logic determined by this formal system, that is the set of theorems of $\mathcal{H}_{\mathbf{Grz}}$, is characterized by the Kripke frames with constant domains (see [Görnemann, 1971, Smorynski, 1973, Gabbay, 1981]), that is Kripke frames $\langle P, \leq, D\rangle$ where D is a constant function. The pseudo-natural deduction calculus $\mathcal{ND}_{\mathbf{Grz}}$ is obtained by adding the Grz-rule to $\mathcal{ND}_{\mathbf{INT}}$. It is easy to verify that $\mathcal{ND}_{\mathbf{Grz}}$ is a presentation for the formal system **Grz**.

To prove the strong constructiveness of $\mathcal{ND}_{\mathbf{Grz}}$ we need to extend the generalized rule $\mathcal{R}_{\mathbf{INT}}$ with a restricted form of the generalization rule.

- R-Gen is the generalized rule whose domain consists of all the sequences of sequents of $\mathcal{L}$ of the form $\vdash A(x); \vdash \forall x A(x) \vee B$, and such that:

$$\vdash \forall x A(x) \in \text{R-Gen}(\vdash A(x); \vdash \forall x A(x) \vee B) \ .$$

Because of the presence of the sequent $\vdash \forall x A(x) \vee B$, this rule is non-increasing and hence also the compound rule

$$\mathcal{R}_{\mathbf{Grz}} = \text{Cut} \cup \text{Subst} \cup \text{R-Gen}$$

is non increasing. We remark that the non-restricted version of the generalized rule, Gen, whose domain consists of all the sequents of the form $A(x)$ and is such that $\vdash \forall x A(x) \in \text{Gen}(\vdash A(x))$, is evidently an increasing rule. This justifies the restriction we introduce.

Since $\mathcal{ND}_{\mathbf{Grz}}$ contains the rule of I$\forall$, the following proposition is an immediate consequence of Proposition 3.2.3.

3.4.1 Proposition *$\mathcal{ND}_{\mathbf{Grz}}$ is uniformly $\mathcal{R}_{\mathbf{Grz}}$-closed.* □

Now, we will prove that, for any set Π of proofs of $\mathcal{ND}_{\mathbf{Grz}}$, the information contained in the subproofs of Π is sufficient to obtain a constructive generalized $\mathcal{R}_{\mathbf{Grz}}$-subcalculus. This generalized subcalculus is:

$$\mathbb{D}_{\mathbf{Grz}}(\Pi) \;=\; \mathbb{D}(\mathcal{R}_{\mathbf{Grz}}, \mathrm{Seq}(\Pi))$$

3.4.2 Lemma *Let Π be any set of proofs of $\mathcal{ND}_{\mathbf{Grz}}$. For any proof $\pi : \Gamma \vdash A \in$ $\textsc{Subst}^*([\Pi])$, if Γ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$, then A is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$.*

Proof: $\vdash A \in \mathbb{D}_{\mathbf{Grz}}([\Pi])$ follows from Lemma 3.2.2. The proof of Point (ii) goes on by induction on depth(π). The basis step does not require any change, and the induction step only requires the treatment of the Grz-rule.

- **Grz-rule.**

$$\pi : \Gamma \vdash A \equiv \frac{\pi_1 : \Gamma \vdash \forall x(B(x) \vee C)}{\Gamma \vdash \forall x B(x) \vee C}\,\mathsf{Grz}$$

  Let us assume that Γ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$. Since we have proved Point (i) of the definition of evaluation for the formula $\forall x B(x) \vee C$, there exists a proof

$$\tau : \vdash \forall x B(x) \vee C \in \mathbb{D}_{\mathbf{Grz}}([\Pi]) \ .$$

  By applying the induction hypothesis to the proof $\pi_1 : \Gamma \vdash \forall x(B(x) \vee C)$, we get that $\forall x(B(x) \vee C)$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$. Let us consider any term t of the language $\mathcal{L}$; then, by definition, we deduce that $(B(t/x) \vee C)$ must be evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$. This implies that either $B(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$ or C is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$. In the latter case we immediately deduce that $\forall x B(x) \vee C$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$. If C is not evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$, from the former case we deduce that, for any term t, $B(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$. Hence, to prove that $\forall x B(x)$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$ we only have to show that this formula has a proof in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$. But $B(x)$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$, hence there exists a proof

$$\tau' : \vdash B(x) \in \mathbb{D}_{\mathbf{Grz}}([\Pi]) \ .$$

  By the existence of the proofs τ and τ' in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$, and by the closure of $\mathbb{D}_{\mathbf{Grz}}([\Pi])$ under R-$\textsc{Gen}$, we immediately deduce that there exists

$$\tau'' : \vdash \forall x B(x) \in \mathbb{D}_{\mathbf{Grz}}([\Pi]) \ .$$

  This concludes the proof.

□

3.4.3 Corollary *Let Π be any set of proofs of $\mathcal{ND}_{\mathbf{INT}}$. For any proof $\tau : \Gamma \vdash$ $A \in \mathbb{D}_{\mathbf{Grz}}([\Pi])$ and any substitution θ, if $\theta\Gamma$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$, then θA is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$.*

Proof: The proof of the fact that the formula θA satisfies Point (i) of the definition of evaluation coincides with the one given in Corollary 3.2.5. Therefore, there exists a proof

$$\tau' : \vdash \theta A \in \mathbb{D}_{\mathbf{Grz}}([\Pi]) \ . \tag{3.1}$$

To prove Point (ii) of the definition of evaluation we proceed by induction on the (Cut, R-Gen)-depth of $\tau : \Gamma \vdash A$, i.e. on the number of (Cut, R-Gen)-rules applied in τ.

Basis: If no Cut-rule and no R-Gen-rule occur in τ, then, by definition of the calculus $\mathbb{D}_{\mathbf{Grz}}([\Pi])$, $\tau : \Gamma \vdash A$ is obtained by applying a (possibly empty) sequence of Subst-rules to a sequent in $[\Pi]$. Thus, there exists a proof $\tau' : \Gamma' \vdash A' \in [\Pi]$ such that $\theta'\Gamma' \vdash \theta' A \equiv \Gamma \vdash A$ for some substitution θ'. This implies that also the sequent $\theta\Gamma \vdash \theta A \equiv \theta\theta'\Gamma' \vdash \theta\theta' A'$ has a proof in Subst$^*([\Pi])$. Since $\theta\Gamma$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$, by Lemma 3.4.2 we have that θA is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$.

Step: The (Cut, R-Gen)-depth of τ is $h+1$ ($h \geq 0$). We have two cases. If the last between the rules Cut and R-Gen applied in the proof τ is Cut, then the proof coincides with the one given for Corollary 3.2.5. Otherwise, if the last between the rules Cut and R-Gen applied in τ is R-Gen, then the proof τ has the following form:

$$\cfrac{\cfrac{\cfrac{\tau_1 : \vdash A'(x) \qquad \tau_2 : \vdash \forall x A'(x) \vee B'}{\vdash \forall x A'(x)}\ \text{R-Gen}}{\vdots}\ \text{Subst}}{\vdash \theta' \forall x A'(x)}\ \text{Subst}$$

where $\Gamma = \emptyset$, $\theta'\forall x A'(x) \equiv A$, the Cut, R-Gen-depth of τ_1, τ_2 is less than or equal to h and τ ends with a (possibly empty) sequence of applications of Subst. Therefore, we have to prove that $\theta\theta'\forall x A'(x)$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$. We already know, by (3.1), that $\theta A \equiv \theta\theta'\forall x A'(x)$ has a proof in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$, and so we only have to prove that, for any term t, $\theta\theta' A(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$. Let us consider the proof τ_1; since its (Cut, R-Gen)-depth is less than $h+1$, the (Cut, R-Gen)-depth of the proof

$$\cfrac{\cfrac{\tau_1 : \vdash A'(x)}{\vdash A'(t/x)}\ \text{Subst}}{\vdash \theta\theta' A'(t/x)}\ \text{Subst}$$

is still less than $h+1$. Hence, by induction hypothesis, $\theta\theta' A(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{Grz}}([\Pi])$. This concludes the proof. □

By Lemma 3.4.2 and the previous Corollary, we can prove, like the Theorems of strong constructiveness for $\mathcal{ND}_{\mathbf{INT}}$ and **INT**, the following facts:

3.4.4 Theorem (SCR-$\mathcal{ND}_{\mathbf{Grz}}$) *$\mathcal{ND}_{\mathbf{Grz}}$ is a strongly constructive calculus w.r.t.* (DP$_{\text{open}}$) *and* (ED$_{\text{open}}$). □

3.4.5 Theorem (SCR-Grz) Grz *is a strongly constructive formal system w.r.t.* (DP$_{\text{open}}$) *and* (ED$_{\text{open}}$). □

3.5 Kreisel-Putnam Logic

In this section we study the formal system we call *Kreisel-Putnam Logic*. This is the formal system **KP** generated by the Hilbert-style calculus

$$\mathcal{H}_{\mathbf{KP}} = \mathcal{H}_{\mathbf{INT}} + \{(\mathsf{KP}_\vee), (\mathsf{KP}_\exists)\}$$

obtained by adding to the axioms of $\mathcal{H}_{\mathbf{INT}}$ all the instances of the following axiom schemes:

$$(\mathsf{KP}_\vee) \qquad (\neg A \Rightarrow B \vee C) \Rightarrow (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C)$$
$$(\mathsf{KP}_\exists) \qquad (\neg A \Rightarrow \exists x B(x)) \Rightarrow \exists x(\neg A \Rightarrow B(x))$$

The first of these axiom-schemes is well known to every people working in intermediate propositional logics [Kreisel and Putnam, 1957, Gabbay, 1970, Gabbay, 1981]. Indeed the propositional logic obtained by adding the axiom schema $(\mathsf{KP}_\vee)$ to propositional intuitionistic logic has been the first counterexample to Lukasiewicz's conjecture of 1952 (see [Lukasiewicz, 1952]), asserting that intuitionistic propositional logic is the greatest consistent and constructive propositional system closed under substitution of propositional variables and modus-ponens. The second axiom schema, which is also known in the area of constructivism as (IP) ([Troelstra, 1973a]), naturally completes the meaning of the former at the predicate level. Both these principles are classically valid but not intuitionistically valid, and the formal system **KP** is a constructive intermediate logic (see e.g. [Avellone et al., 1996]).

$(\mathsf{KP}_\vee)$ and $(\mathsf{KP}_\exists)$ can be expressed with the following pseudo-natural deduction rules

$$\frac{\Gamma, \neg A \vdash B \vee C}{\Gamma \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C)}\,\mathsf{KP}_\vee \qquad \frac{\Gamma, \neg A \vdash \exists x B(x)}{\Gamma \vdash \exists x(\neg A \Rightarrow B(x))}\,\mathsf{KP}_\exists$$

$\mathcal{ND}_{\mathbf{KP}}$ is the pseudo-natural deduction calculus obtained by adding the rules $\mathsf{KP}_\exists$ and $\mathsf{KP}_\vee$ to $\mathcal{ND}_{\mathbf{INT}}$. It is easy to verify that $\mathcal{ND}_{\mathbf{KP}}$ is a presentation for the formal system **KP**.

Now, we introduce the generalized rules R-In$^{\Rightarrow\vee}$ and R-In$^{\Rightarrow\exists}$ corresponding to restricted version of $\Rightarrow$-introduction.

- R-In$^{\Rightarrow\vee}$ is the generalized rule whose domain is the set containing, for all formulas A, B, C and all sets of formulas Γ, Δ with $\neg A \notin \Gamma$, the sequences of sequents σ^* having one of the following forms:

$$\begin{array}{lcl}\sigma^* & \equiv & \Gamma, \neg A \vdash B; \Delta \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C) \\ \sigma^* & \equiv & \Gamma \vdash B; \Delta \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C) \\ \sigma^* & \equiv & \Gamma, \neg A \vdash C; \Delta \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C) \\ \sigma^* & \equiv & \Gamma \vdash C; \Delta \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C)\end{array}$$

and such that:

$$\begin{array}{lcl}\Gamma, \Delta \vdash \neg A \Rightarrow B & \in & \text{R-In}^{\Rightarrow\vee}(\Gamma \vdash B; \Delta \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C)) \\ \Gamma, \Delta \vdash \neg A \Rightarrow B & \in & \text{R-In}^{\Rightarrow\vee}(\Gamma, \neg A \vdash B; \Delta \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C)) \\ \Gamma, \Delta \vdash \neg A \Rightarrow C & \in & \text{R-In}^{\Rightarrow\vee}(\Gamma \vdash C; \Delta \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C)) \\ \Gamma, \Delta \vdash \neg A \Rightarrow C & \in & \text{R-In}^{\Rightarrow\vee}(\Gamma, \neg A \vdash C; \Delta \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C))\end{array}$$

- R-In$^{\Rightarrow\exists}$ is the the generalized rule whose domain is the set containing, for all formulas $A, B(x)$ and all sets of formulas Γ, Δ with $\neg A \notin \Gamma$, the sequences of sequents σ^* having one of the following forms:

$$\begin{array}{lcl}\sigma^* & \equiv & \Gamma, \neg A \vdash B(t); \Delta \vdash \neg A \Rightarrow \exists x B(x) \\ \sigma^* & \equiv & \Gamma \vdash B(t); \Delta \vdash \neg A \Rightarrow \exists x B(x)\end{array}$$

  and such that

$$\begin{array}{lcl}\Gamma, \Delta \vdash \neg A \Rightarrow B(t) & \in & \text{R-In}^{\Rightarrow\exists}(\Gamma \vdash B(t); \Delta \vdash \neg A \Rightarrow \exists x B(x)) \\ \Gamma, \Delta \vdash \neg A \Rightarrow B(t) & \in & \text{R-In}^{\Rightarrow\exists}(\Gamma, \neg A \vdash B(t); \Delta \vdash \neg A \Rightarrow \exists x B(x))\ .\end{array}$$

Now, we define

$$\mathcal{R}_{\mathbf{KP}} = \text{Cut} \cup \text{Subst} \cup \text{R-In}^{\Rightarrow\vee} \cup \text{R-In}^{\Rightarrow\exists}\ .$$

We remark that, the restriction on the domains of R-In$^{\Rightarrow\vee}$ and R-In$^{\Rightarrow\exists}$ prevents $\mathcal{R}_{\mathbf{KP}}$ from being an increasing rule. It is immediate to prove that $\mathcal{ND}_{\mathbf{KP}}$ is uniformly $\mathcal{R}_{\mathbf{KP}}$-closed.

3.5.1 Proposition *$\mathcal{ND}_{\mathbf{KP}}$ is uniformly $\mathcal{R}_{\mathbf{KP}}$-closed.* □

To prove the strong constructiveness of $\mathcal{ND}_{\mathbf{KP}}$, we need a more complex definition of evaluation in a set of proofs.

3.5.2 Definition (Neg-evaluation) *Let Π be a set of proofs, let* Neg *be a set of negated formulas and let A be a formula. We say that A* is Neg-evaluated in Π *iff the following properties hold:*

(i) Either $A \in \mathsf{Neg}$ *or there is a proof* $\pi \in \Pi$ *such that* $\pi : \Gamma \vdash A$ *and* $\Gamma \subseteq \mathsf{Neg}$*;*

(ii) According to the form of A*, one of the following cases holds:*

(a) A *is atomic or negated;*

(b) $A \equiv B \wedge C$*, and both* B *and* C *are* Neg*-evaluated in* Π*;*

(c) $A \equiv B \vee C$*, and either* B *is* Neg*-evaluated in* Π *or* C *is* Neg*-evaluated in* Π*;*

(d) $A \equiv B \Rightarrow C$*, and, for any set of negated formulas* Neg' *including* Neg*, if* B *is* Neg'*-evaluated in* Π *then* C *is* Neg'*-evaluated in* Π*;*

(e) $A \equiv \exists x B(x)$*, and* $B(t/x)$ *is* Neg*-evaluated in* Π *for some term* t*;*

(f) $A \equiv \forall x B(x)$*, and, for any term* t*,* $A(t/x)$ *is* Neg*-evaluated in* Π*.*

The following properties will be needed.

3.5.3 Proposition *Let* Neg *be a set of negated formulas, let* Π *be a set of proofs and let* A *be a formula. If* A *is* Neg*-evaluated in* Π *then* A *is* Neg'*-evaluated in* Π *for any set of negated formulas* Neg' *including* Neg*.* □

3.5.4 Proposition *Let* Π *be a* CUT*-closed set of proofs, let* Neg *be a set of negated formulas and let* A *and* H *be arbitrary formulas. If* A *is* $\mathsf{Neg} \cup \{\neg H\}$*-evaluated in* Π *and* $\neg H$ *is* Neg*-evaluated in* Π*, then* A *is* Neg*-evaluated in* Π*.*

Proof: First of all, we must prove that A satisfies condition (i) of Definition 3.5.2. Let us suppose that $A \notin \mathsf{Neg}$. Then, we must show that there exists a proof

$$\pi' : \Gamma \vdash A \in \Pi$$

with $\Gamma \subseteq \mathsf{Neg}$. Now, since A is $\mathsf{Neg} \cup \{\neg H\}$-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, there exists a proof

$$\pi_1 : \Delta_1 \cup \{\neg H\} \vdash A \in \Pi$$

with $\Delta_1 \subseteq \mathsf{Neg}$. Moreover, since the formula $\neg H$ is Neg-evaluated in Π, then either $\neg H \in \mathsf{Neg}$, and this implies that we can assume π' to be the proof π_1 itself, or there exists a proof

$$\pi_2 : \Delta_2 \vdash \neg H \in \Pi$$

such that $\Delta_2 \subseteq \mathsf{Neg}$. Now, since Π is CUT-closed, by the presence in Π of the proofs π_1 and π_2, we get that there exists a proof

$$\pi : \Delta_1, \Delta_2 \vdash A \in \Pi$$

with $\Delta_1 \cup \Delta_2 \subseteq \mathsf{Neg}$. This concludes the proof of Point (i).

To prove that A satisfies Point (ii) of Definition 3.5.2, we proceed by induction on the complexity of the formula A. If A is atomic or negated, then the existence of the proof π in Π is enough to guarantee that it is Neg-evaluated in Π. If A is

$B \wedge C$, $B \vee C$, $\forall x B(x)$ or $\exists x B(x)$, then the proof easily follows from the induction hypothesis. The only interesting case is $A \equiv B \Rightarrow C$. Let us suppose that Neg' is a set of negated formulas including Neg, and B is Neg'-evaluated in Π. By Proposition 3.5.3 we have that B is also $\mathsf{Neg}' \cup \{\neg H\}$-evaluated in Π. Now, since

$$\mathsf{Neg}' \cup \{\neg H\} \supseteq \mathsf{Neg} \cup \{\neg H\}$$

and $B \Rightarrow C$ is $\mathsf{Neg} \cup \{\neg H\}$-evaluated in Π, it is also $\mathsf{Neg}' \cup \{\neg H\}$-evaluated in Π. But $B \Rightarrow C$ and B $\mathsf{Neg}' \cup \{\neg H\}$-evaluated in Π imply, by definition, that C is $\mathsf{Neg}' \cup \{\neg H\}$-evaluated in Π. □

Now, let us consider the calculus:

$$\mathbb{D}_{\mathbf{KP}}(\Pi) \;=\; \mathbb{D}(\mathcal{R}_{\mathbf{KP}}, \mathrm{Seq}(\Pi)) \;.$$

3.5.5 Lemma *Let Π be a set of proofs of $\mathcal{ND}_{\mathbf{KP}}$. For every $\pi : \Gamma \vdash H \in$ $\textsc{Subst}^*([\Pi])$ and for every set of negated formulas Neg, if Γ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, then H is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$.*

Proof: Let us consider an arbitrary set of negated formulas Neg and let us suppose Γ to be Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. To prove Point (i) of Definition 3.5.2, let us suppose that $H \notin \mathsf{Neg}$. Then we must show that there exists a proof

$$\tau : \Delta \vdash H \in \mathbb{D}_{\mathbf{KP}}([\Pi])$$

with $\Delta \subseteq \mathsf{Neg}$. First of all, we notice that $\pi \in \textsc{Subst}^*([\Pi])$ implies that there exist $\pi' : \Gamma' \vdash A' \in [\Pi]$ and a substitution θ of individual variables such that $\theta\Gamma' \vdash \theta A' \equiv \Gamma \vdash A$. Thus, by its definition, the set of proofs $\mathbb{D}_{\mathbf{KP}}([\Pi])$ contains a proof of the sequent $\Gamma' \vdash A'$ and hence, since $\mathbb{D}_{\mathbf{KP}}([\Pi])$ is $\textsc{Subst}$-closed, there exists a proof

$$\tau' : \Gamma \vdash H \in \mathbb{D}_{\mathbf{KP}}([\Pi]). \tag{3.2}$$

Now, since Γ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ we can write $\Gamma = \Delta_0 \cup \{H_1, \ldots, H_n\}$ where Δ_0 and $\{H_1, \ldots, H_n\}$ are disjoint sets of formulas, $\Delta_0 \subseteq \mathsf{Neg}$ and there exist proofs

$$\tau_1 : \Delta_1 \vdash H_1 \;,\; \ldots \;,\; \tau_n : \Delta_n \vdash H_n \;\in \mathbb{D}_{\mathbf{KP}}([\Pi]) \tag{3.3}$$

with $\Delta_1 \subseteq \mathsf{Neg}, \ldots, \Delta_n \subseteq \mathsf{Neg}$. By the presence in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ of the proofs τ' (3.2) and $\tau_1, \ldots, \tau_n$ (3.3), and by the closure of $\mathbb{D}_{\mathbf{KP}}([\Pi])$ under the $\textsc{Cut}$-rule, we get that also the proof

$$\tau^* : \Delta_0 \cup \ldots \cup \Delta_n \vdash H$$

is in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, and since $\Delta_0 \cup \ldots \cup \Delta_n \subseteq \mathsf{Neg}$, we have the assertion.

Now, we prove by induction on depth(π) that H satisfies condition (ii) of Definition 3.5.2.

Basis: If depth$(\pi) = 0$ then the only rule which occurs in π must be an assumption introduction, that is $H \in \Gamma$. Thus, H is trivially Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$.

Step: We assume that our assertion holds for any proof $\pi' : \Gamma' \vdash A' \in \textsc{Subst}^*([\Pi])$ such that depth$(\pi') \leq h$ $(h \geq 0)$, and let us suppose that depth$(\pi) = h + 1$. The proof proceeds by cases on the last rule applied in π; here the interesting cases are the implication introduction and the KP-rules.

- **Implication Introduction**.

$$\pi : \Gamma \vdash H \equiv \cfrac{\pi_1 : \Gamma, B \vdash C}{\Gamma \vdash B \Rightarrow C}\,{\scriptstyle \mathrm{I}\Rightarrow}$$

with $H \equiv B \Rightarrow C$. Let Neg' be any set of negated formulas such that $\mathsf{Neg} \subseteq \mathsf{Neg}'$. Since Γ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, by Proposition 3.5.3 it follows that Γ is Neg'-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ too. Now, if B is Neg'-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, then all the premises of the proof π_1 are Neg'-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, therefore, by induction hypothesis, C is Neg'-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Hence $H \equiv B \Rightarrow C$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$.

- **Rule** $(\mathsf{KP}_\vee)$.

$$\pi : \Gamma \vdash H \equiv \cfrac{\pi_1 : \Gamma, \neg A \vdash B \vee C}{\Gamma \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C)}\,{\scriptstyle \mathsf{KP}_\vee}$$

We must prove that either $\neg A \Rightarrow B$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ or $\neg A \Rightarrow C$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Since Γ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, $\Gamma \cup \{\neg A\}$ is $\mathsf{Neg} \cup \{\neg A\}$-evaluated in this set. Then, by induction hypothesis on π_1, either B or C is $\mathsf{Neg} \cup \{\neg A\}$-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. For the sake of definiteness, let us assume that B is the evaluated formula. This implies that there exists a proof

$$\tau : \Delta \vdash \; B \in \mathbb{D}_{\mathbf{KP}}([\Pi])$$

with $\Delta \subseteq \mathsf{Neg} \cup \{\neg A\}$. By Point (i) of Definition 3.5.2, already proved in this Lemma, taking $H \equiv (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C)$ we get that there exists a proof

$$\tau' : \Delta' \vdash (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C) \in \mathbb{D}_{\mathbf{KP}}([\Pi])$$

with $\Delta' \subseteq \mathsf{Neg}$. By the existence of the proofs τ and τ' in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, and by the fact that $\mathbb{D}_{\mathbf{KP}}([\Pi])$ is R-$\textsc{In}^{\Rightarrow\vee}$-closed, we obtain the proof

$$\pi'' : \Delta'' \vdash \neg A \Rightarrow B \in \mathbb{D}_{\mathbf{KP}}([\Pi])$$

with $\Delta'' = (\Delta \cup \Delta') \setminus \{\neg A\}$ if $\neg A \in \Delta$ and $\Delta'' = \Delta \cup \Delta'$ otherwise. Hence, $\Delta'' \subseteq \mathsf{Neg}$, and Point (i) of Definition 3.5.2 is satisfied. Now, let us suppose that $\neg A$ is Neg'-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ with $\mathsf{Neg} \subseteq \mathsf{Neg}'$. We already know that B is $\mathsf{Neg} \cup \{\neg A\}$-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, and hence, by Proposition 3.5.3, it is also $\mathsf{Neg}' \cup \{\neg A\}$-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Since $\neg A$ is Neg'-evaluated, by Proposition 3.5.4 we have that B is Neg'-evaluated. This concludes the proof of this case.

- **Rule** ($\mathsf{KP}_\exists$).

$$\pi : \Gamma \vdash H \equiv \cfrac{\pi_1 : \Gamma, \neg A \vdash \exists x B(x)}{\Gamma \vdash \exists x(\neg A \Rightarrow B(x))}\,\mathsf{KP}_\exists \quad x \notin \mathrm{FV(A)}$$

We have to prove that $\neg A \Rightarrow B(t/x)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, for some term t of the language. Since Γ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, we easily deduce that the set $\Gamma \cup \{\neg A\}$ is $\mathsf{Neg} \cup \{\neg A\}$-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Thus, applying the induction hypothesis to the proof π_1, we obtain that there exists a term t such that $B(t/x)$ is $\mathsf{Neg} \cup \{\neg A\}$-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Then, there exists a proof

$$\tau : \Delta \vdash \; B(t/x) \in \mathbb{D}_{\mathbf{KP}}([\Pi])$$

with $\Delta \subseteq \mathsf{Neg} \cup \{\neg A\}$. Moreover, by Point (i) of Definition 3.5.2, already proved in this Lemma, taking $H \equiv \exists x(\neg A \Rightarrow B(x))$ we get that there exists a proof

$$\tau' : \Delta' \vdash \exists x(\neg A \Rightarrow B(x)) \in \mathbb{D}_{\mathbf{KP}}([\Pi])$$

with $\Delta' \subseteq \mathsf{Neg}$. By the existence of the proofs τ and τ' in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ and by the closure of $\mathbb{D}_{\mathbf{KP}}([\Pi])$ under the rule R-IN$^{\Rightarrow\exists}$, we easily obtain that the proof

$$\tau'' : \Delta'' \vdash \neg A \Rightarrow B(t/x) \in \mathbb{D}_{\mathbf{KP}}([\Pi])$$

with $\Delta'' = (\Delta \cup \Delta') \setminus \{\neg A\}$ if $\neg A \in \Delta$ and $\Delta'' = \Delta \cup \Delta'$ otherwise. Thus, $\Delta'' \subseteq \mathsf{Neg}$ and Point (i) of Definition 3.5.2 is satisfied. Now, let us suppose that $\neg A$ is Neg'-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ with $\mathsf{Neg} \subseteq \mathsf{Neg}'$. We already know that $B(t/x)$ is $\mathsf{Neg} \cup \{\neg A\}$-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ for some term t, and hence, by Proposition 3.5.3, it is also $\mathsf{Neg}' \cup \{\neg A\}$-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Since $\neg A$ is Neg'-evaluated, by Proposition 3.5.4 we have that $B(t/x)$ is Neg'-evaluated. This concludes the proof.

□

3.5.6 Corollary *Let Π be any set of proofs of $\mathcal{ND}_{\mathbf{KP}}$ and Neg be any set of negated formulas. For every proof $\tau : \Gamma \vdash A \in \mathbb{D}_{\mathbf{KP}}([\Pi])$ and every substitution θ, if $\theta\Gamma$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, then θA is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$.*

Proof: First of all, we must show that there exists a proof of the sequent $\Delta \vdash \theta A$ in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ with $\Delta \subseteq \mathsf{Neg}$. Since $\tau : \Gamma \vdash A \in \mathbb{D}_{\mathbf{KP}}([\Pi])$ and that this set of proofs is SUBST-closed, there exists a proof

$$\tau' : \theta\Gamma \vdash \theta A$$

in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Let $\theta\Gamma = \{H_1, \ldots, H_n\}$. Since $\theta\Gamma$ is Neg-evaluate din $\mathbb{D}_{\mathbf{KP}}([\Pi])$, there exist proofs

$$\tau_1 : \Delta_1 \vdash H_1, \ldots, \tau_n : \Delta_n \vdash H_n$$

in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ with $\Delta_1 \cup \ldots \Delta_n \subset \mathsf{Neg}$. Since $\mathbb{D}_{\mathbf{KP}}([\Pi])$ contains the proofs $\tau', \tau_1, \ldots, \tau_n$ and $\mathbb{D}_{\mathbf{KP}}([\Pi])$ is CUT-closed, it also contains the proof

$$\tau'' \vdash \Delta_1 \cup \ldots \Delta_n \vdash \theta A \ .$$

To prove Point (ii) of the Definition of Neg-evaluation, we proceed by induction on the number applications of the rules CUT, R-IN$^{\Rightarrow\vee}$ and R-IN$^{\Rightarrow\exists}$ in τ.

Basis: If none of these rules is applied in τ, then, by definition of $\mathbb{D}_{\mathbf{KP}}([\Pi])$, $\tau : \Gamma \vdash A$ is obtained by applying a (possibly empty) sequence of SUBST to a sequent in $[\Pi]$. Thus, there exists a proof $\tau' : \Gamma' \vdash A' \in \mathbb{D}_{\mathbf{KP}}([\Pi])$ such that $\theta'\Gamma' \vdash \theta' A \equiv \Gamma \vdash A$ for some substitution θ'. Then, also the sequent $\theta\Gamma \vdash \theta A \equiv \theta\theta'\Gamma' \vdash \theta\theta' A'$ has a proof in SUBST$^*([\Pi])$. Since $\theta\Gamma$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, by Lemma 3.5.5 we have that θA is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$.

Step: Let us suppose that for any proof $\tau'' : \Gamma'' \vdash A'' \in \mathbb{D}_{\mathbf{KP}}([\Pi])$ such that the CUT, R-IN$^{\Rightarrow\vee}$, R-IN$^{\Rightarrow\exists}$-depth is less than or equal to h ($h \geq 0$), for any substitution θ'' and for any set of formulas Neg$''$, if $\theta''\Gamma''$ is Neg$''$-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ then $\theta'' A$ is Neg$''$-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Now, we suppose that the (CUT, R-IN$^{\Rightarrow\vee}$, R-IN$^{\Rightarrow\exists}$)-depth of the proof τ is $h+1$. The proofs goes, as usual, by cases, taking into account the last between the rules CUT, R-IN$^{\Rightarrow\vee}$, R-IN$^{\Rightarrow\exists}$ occurring in τ.

- CUT**-rule**: then the proof $\tau : \Gamma \vdash A$ has the following form:

$$\dfrac{\dfrac{\dfrac{\tau_1 : \Gamma'_1 \vdash H \qquad \tau_2 : \Gamma'_2, H \vdash A'}{\Gamma' \vdash A'}\,\text{CUT}}{\vdots}\,\text{SUBST}}{\theta'\Gamma' \vdash \theta' A'}\,\text{SUBST}$$

where: $\Gamma' = \Gamma'_1 \cup \Gamma'_2$, the (CUT, R-IN$^{\Rightarrow\vee}$, R-IN$^{\Rightarrow\exists}$)-depth of τ_1, τ_2 is less than or equal to h, and τ ends with a (possibly empty) sequence of applications of SUBST. We prove that, if $\theta\Gamma \equiv \theta\theta'\Gamma'$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, then $\theta A \equiv \theta\theta' A'$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Since $\theta\theta'\Gamma'_1 \subseteq \theta\theta'\Gamma'$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, by induction hypothesis on the proof:

$$\dfrac{\tau_1 : \Gamma'_1 \vdash H}{\theta\theta'\Gamma'_1 \vdash \theta\theta' H}\,\text{SUBST}$$

$\theta\theta' H$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Hence, $\theta\theta'\Gamma'_2, \theta\theta' H$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, and therefore, by induction hypothesis on the proof

$$\dfrac{\tau_2 : \Gamma'_2, H \vdash A'}{\theta\theta'\Gamma'_2, \theta\theta' H \vdash \theta\theta' A'}\,\text{SUBST}$$

we get that $\theta\theta' A$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$.

- R-In$^{\Rightarrow\vee}$**-rule**: We consider the case where the rule is applied to the sequence of sequents
$$\Gamma_1 \vdash C'; \Gamma_2 \vdash (\neg B' \Rightarrow C') \vee (\neg B' \Rightarrow D')$$
or to the sequence of sequents
$$\Gamma_1, \neg B' \vdash C'; \Gamma_2 \vdash (\neg B' \Rightarrow C') \vee (\neg B' \Rightarrow D')\ ,$$
the other cases being symmetrical. Then, the proof $\tau : \Gamma \vdash A$ with $A = \neg B \Rightarrow C$ has one of the following forms
$$\dfrac{\dfrac{\dfrac{\tau_1 : \Gamma_1 \vdash C' \qquad \tau_2 : \Gamma_2 \vdash (\neg B' \Rightarrow C') \vee (\neg B' \Rightarrow D')}{\Gamma' \vdash \neg B' \Rightarrow C'}\,\text{R-In}^{\Rightarrow\vee}}{\vdots}\,\text{Subst}}{\theta'\Gamma' \vdash \theta'(\neg B' \Rightarrow C')}\,\text{Subst} \tag{3.4}$$
$$\dfrac{\dfrac{\dfrac{\tau_1 : \Gamma_1, \neg B' \vdash C' \qquad \tau_2 : \Gamma_2 \vdash (\neg B' \Rightarrow C') \vee (\neg B' \Rightarrow D')}{\Gamma' \vdash \neg B' \Rightarrow C'}\,\text{R-In}^{\Rightarrow\vee}}{\vdots}\,\text{Subst}}{\theta'\Gamma' \vdash \theta'(\neg B' \Rightarrow C')}\,\text{Subst} \tag{3.5}$$
where $\Gamma' = \Gamma_1 \cup \Gamma_2$, $B \Rightarrow C \equiv \theta'(B' \Rightarrow C')$, the number of occurrences of Cut, R-In$^{\Rightarrow\vee}$ and R-In$^{\Rightarrow\exists}$ applications in τ_1 and τ_2 is less than or equal to h, and τ ends with a (possibly empty) sequence of applications of Subst. The case of the proof (3.4) is trivial, since, being $\theta\theta'\Gamma_1$ Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, by induction hypothesis on the proof
$$\dfrac{\tau_1 : \Gamma_1 \vdash C'}{\theta\theta'\Gamma_1 \vdash \theta\theta' C'}\,\text{Subst}$$
we immediately get that $\theta\theta' C'$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Now, let us consider the proof (3.5). We have to prove that, if $\theta\theta' B'$ is Neg'-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$ for some set of negated formulas Neg' including Neg, then $\theta\theta' C'$ is Neg'-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$. Since $\theta\theta'\Gamma'$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, $\theta\theta'\Gamma'$ it is also Neg'-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$, by Proposition 3.5.3. Hence, by induction hypothesis on the proof
$$\dfrac{\tau_1 : \Gamma_1, \neg B' \vdash C'}{\theta\theta'\Gamma_1, \theta\theta'\neg B' \vdash \theta\theta' C'}\,\text{Subst}$$
we get $\theta\theta' C'$ is Neg-evaluated in $\mathbb{D}_{\mathbf{KP}}([\Pi])$.

- R-IN$^{\Rightarrow\exists}$**-rule**: The proof is similar to the one developed for the previous case.

□

By the previous Lemma, we can prove, as seen for Theorem 3.2.7 and Corollary 3.2.8, the following facts:

3.5.7 Theorem (SCR-$\mathcal{ND}_{\mathbf{KP}}$) $\mathcal{ND}_{\mathbf{KP}}$ *is a strongly constructive calculus w.r.t.* (DP$_{\text{open}}$) *and* (ED$_{\text{open}}$). □

3.5.8 Corollary (SCR-KP) KP *is a strongly constructive formal system w.r.t.* (DP$_{\text{open}}$) *and* (ED$_{\text{open}}$). □

3.6 Scott Logic

Now, we consider the formal system **St**, we call *Scott Logic*, generated by the Hilbert style calculus:

$$\mathcal{H}_{\mathbf{St}} = \mathcal{H}_{\mathbf{INT}} + \{(\mathsf{St})\}$$

where (St) is the *Scott Principle* defined as follows:

$$(\mathsf{St}) \qquad ((\neg\neg A \Rightarrow A) \Rightarrow A \vee \neg A) \Rightarrow \neg A \vee \neg\neg A\ .$$

In the propositional frame, Scott Principle is the extra-intuitionistic axiom schema of a well known intermediate constructive propositional logic quoted in the paper [Kreisel and Putnam, 1957] as another example contradicting Lukasiewicz's conjecture. This principle has been extensively studied by people working in intermediate propositional logics, see e.g. [Anderson, 1972] (where it is quoted in an equivalent version named F_9), or [Minari, 1986, Miglioli, 1992, **?**]. In [**?**] (where a Kripke-frame semantics formerly introduced in [Miglioli, 1992] is shown to be valid and complete for such propositional logic) a logic extending it and maximal in the family of intermediate constructive propositional logics is exhibited (where a maximal propositional intermediate constructive logic is a propositional intermediate logics with the disjunction property which does not admit any constructive extension). In the framework of intermediate propositional logics, Scott Logic is maximal in the fragment in one variable; moreover, as shown in [**?**], the fragment in one variable of any propositional constructive logic is either contained in the fragment in one variable of propositional **St**, or it is contained in the fragment in one variable of the propositional intermediate logic **ASt**, the latter being the set of theorems generated by the propositional Hilbert-style calculus obtained by adding to propositional Hilbert-style calculus $\mathcal{H}_{\mathbf{INT}}$ the following propositional axiom schema:

$$(\mathsf{ASt}) \qquad (((\neg\neg A \Rightarrow A) \Rightarrow A \vee \neg A) \Rightarrow \neg A \vee \neg\neg A) \Rightarrow \neg\neg A \vee (\neg\neg A \Rightarrow A)\ .$$

The name **ASt** means "anti" Scott: this refers to the fact that the logics **St** and **ASt** are constructively incompatible, that is, the union of these two formal systems gives rise to an intermediate propositional logic which does not admit a constructive extension (in [**?**] also a maximal intermediate constructive logic is exhibited which includes **ASt**).

In the following we will consider Scott Principle in the predicate frame, that is, **St** will indicate a first order formal system. Note that in this context (as far as we know) no Kripke-frame semantics has been given for the intermediate predicate logic defined by **St**; thus (as it happens for the intermediate predicate logic **KP**, for which, in turn, no Kripke-frame semantics has been provided), we lack the typical constructivity proof of the formal system based on the Kripke models (we also remark that the techniques based on the Kripke models provide proofs of constructivity, not of strong constructivity).

Now, coming to our treatment, Scott Principle can be expressed by the following rule in a pseudo-natural calculus style:

$$\frac{\pi_1 : \Gamma, \neg\neg A \Rightarrow A \vdash A \vee \neg A}{\neg A \vee \neg\neg A}\,\textsf{St}\ ;$$

the calculus $\mathcal{ND}_{\mathbf{St}}$ is the calculus obtained by adding the rule St to the natural deduction calculus $\mathcal{ND}_{\mathbf{INT}}$. This calculus is easily seen to be a presentation for the formal system **St**.

The generalized rule we need to study the strong constructiveness of $\mathcal{ND}_{\mathbf{St}}$ is the generalized rule containing Cut, Subst and the following restricted version of $\neg$-introduction.

- R-In$^{\neg}$ is the generalized rule whose domain consists, for all the sets Γ, Δ of formulas, and for all the formulas A such that $\neg\neg A \Rightarrow A \notin \Gamma$, of all the sequences of sequents with one of following forms:

$$\begin{aligned} \sigma^* &\equiv \Gamma, \neg\neg A \Rightarrow A \vdash A; \Delta \vdash \neg A \vee \neg\neg A \\ \sigma^* &\equiv \Gamma, \neg\neg A \Rightarrow A \vdash \neg A \end{aligned}$$

  and such that:

$$\begin{aligned} \Gamma, \Delta \vdash \neg\neg A &\in \text{R-In}^{\neg}(\Gamma \vdash A; \Delta \vdash \neg A \vee \neg\neg A) \\ \Gamma \vdash \neg A &\in \text{R-In}^{\neg}(\Gamma, \neg\neg A \Rightarrow A \vdash \neg A) \end{aligned}$$

It is immediate to verify that R-In$^{\neg}$ is a non-increasing rule. Using this generalized rule we define

$$\mathcal{R}_{\mathbf{St}} = \text{Cut} \cup \text{Subst} \cup \text{R-In}^{\neg}$$

which is non-increasing, since it is the union of non-increasing rules.

3.6.1 Proposition *$\mathcal{ND}_{\mathbf{St}}$ is uniformly $\mathcal{R}_{\mathbf{St}}$-closed.*

Proof: Since $\mathcal{ND}_{\mathbf{INT}}$ is uniformly CUT-closed and SUBST-closed, and $\mathcal{ND}_{\mathbf{St}}$ is an extension of $\mathcal{ND}_{\mathbf{INT}}$, it is sufficient to prove that $\mathcal{ND}_{\mathbf{St}}$ is uniformly R-IN$^{\neg}$-closed. We only treat the case where a proof

$$\pi_1 : \Gamma, \neg\neg A \Rightarrow A \vdash \neg A$$

belongs to $\mathcal{ND}_{\mathbf{St}}$. Then, we have to prove that there is a proof of $\pi : \Gamma \vdash \neg\neg A$ in $\mathcal{ND}_{\mathbf{St}}$. The following proof uses the fact that there exists a proof $\pi_3 : \vdash \neg\neg(\neg\neg A \Rightarrow A)$ in $\mathcal{ND}_{\mathbf{INT}}$ such that $\mathrm{dg}(\pi_3) = \mathrm{dg}(\neg\neg(\neg\neg A \Rightarrow A))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1 : \Gamma, \neg\neg A \Rightarrow A \vdash \neg A \qquad A \vdash A}{\Gamma, \neg\neg A \Rightarrow A, A \vdash \bot}\,E\Rightarrow}{\Gamma, A \vdash \neg(\neg\neg A \Rightarrow A)}\,I\Rightarrow \qquad \pi_3 : \vdash \neg\neg(\neg\neg A \Rightarrow A)}{\Gamma, A \vdash \bot}\,E\Rightarrow}{\Gamma \vdash \neg A}\,I\Rightarrow$$

The degree of the above proof is the maximum between the degree of π_1 and the degree of $\neg\neg(\neg\neg A \Rightarrow A)$, which is $\mathrm{dg}(\neg\neg A) + 3$. □

We remark that the presence of the sequent $\Delta \vdash \neg A \vee \neg\neg A$ is needed only to make R-IN$^{\neg}$ a non-increasing rule, and plays no other role in the results we are going to prove (similar aspects can be found in the previous treatments of **Grz** and **KP**).

To get the proof of the strong constructiveness of $\mathcal{ND}_{\mathbf{St}}$ we need another notion of evaluation of a formula in a set of proofs.

3.6.2 Definition (Λ-evaluation) *Let Π be a set of proofs, Λ be a set of formulas of the kind $\neg\neg K \Rightarrow K$ and let A be a formula. We say that A is* Λ-evaluated in Π *iff the following condition hold:*

(i) There is a proof $\pi \in \Pi$ such that $\pi : \Gamma \vdash A$ and $\Gamma \subseteq \Lambda$;

(ii) According to the form of A, one of the following cases hold:

- *(a) A is atomic or negated;*
- *(b) $A \equiv B \wedge C$, and both B and C are Λ-evaluated in Π;*
- *(c) $A \equiv B \vee C$, and either B is Λ-evaluated in Π or C is Λ-evaluated in Π;*
- *(d) $A \equiv B \Rightarrow C$, and, for every $\Lambda' \supseteq \Lambda$ with Λ' a set of formulas of the kind $\neg\neg K \Rightarrow K$, if B is Λ'-evaluated in Π then C is Λ'-evaluated in Π;*
- *(e) $A \equiv \exists x B(x)$, and $B(t/x)$ is Λ-evaluated in Π for some term t;*
- *(f) $A \equiv \forall x B(x)$, and, for any term t, $A(t/x)$ is Λ-evaluated in Π.*

The following properties will be needed.

3.6.3 Proposition *Let Λ be a set of formulas of the kind $\neg\neg K \Rightarrow K$, let Π be a set of proofs and let A be a formula. If A is Λ-evaluated in Π, then A is Λ'-evaluated in Π for any set Λ' of formulas of the kind $\neg\neg K \Rightarrow K$ such that $\Lambda \subseteq \Lambda'$.* □

3.6.4 Proposition *Let* Π *be a* Cut*-closed set of proofs, let* Λ *be a set of formulas of the form* $\neg\neg K \Rightarrow K$*, and let* A *and* H *be arbitrary formulas. If* A *is* $\Lambda \cup \{\neg\neg H \Rightarrow H\}$*-evaluated in* Π *and* $\neg\neg H \Rightarrow H$ *is* Λ*-evaluated in* Π*, then* A *is* Λ*-evaluated in* Π.

Proof: First of all we have to prove Point (i) of Definition 3.6.2, i.e. that there exists a proof

$$\pi : \Gamma \vdash A \in \Pi$$

with $\Gamma \subseteq \Lambda$. Since the formula $\neg\neg H \Rightarrow H$ is Λ-evaluated in Π, there exists a proof

$$\pi' : \Delta' \vdash \neg\neg H \Rightarrow H \in \Pi$$

such that $\Delta' \subseteq \Lambda$. Moreover, since A is $\Lambda \cup \{\neg\neg H \Rightarrow H\}$-evaluated in Π, there exists a proof

$$\pi'' : \Delta'', \neg\neg H \Rightarrow H \vdash A \in \Pi$$

with $\Delta'' \subseteq \Lambda$ (we remark that the case $\pi'' : \Delta'' \vdash A \in \Pi$ with $\neg\neg H \Rightarrow H \notin \Delta''$ is trivial). Now, since by hypothesis Π is a Cut-closed set of proofs, we obtain, by the presence of the proofs π' and π'' in Π, that there exists also a proof

$$\pi : \Delta', \Delta'' \vdash A \in \Pi \ .$$

Since $\Delta' \cup \Delta'' \subseteq \Lambda$, this proves Point (i) of Definition 3.6.2.

To prove Point (ii), we proceed by induction on the complexity of the formula A. If A is atomic or negated then the existence of the proof π in Π immediately entails that A is Λ-evaluated in Π. If A is $B \wedge C$, $B \vee C$, $\forall x B(x)$ or $\exists x B(x)$ then the proof easily follows from the induction hypothesis. The only interesting case is $A \equiv B \Rightarrow C$. Let us suppose that Λ' is a set of formulas of the form $\neg\neg K \Rightarrow K$ including Λ, and that B is Λ'-evaluated in Π. By Proposition 3.6.3 we obtain that B is also $\Lambda' \cup \{\neg\neg H \Rightarrow H\}$-evaluated in Π. But, since both $B \Rightarrow C$ and B are $\Lambda \cup \{\neg\neg H \Rightarrow H\}$-evaluated in Π, we have that also C is $\Lambda \cup \{\neg\neg H \Rightarrow H\}$-evaluated in Π. Thus, by applying the induction hypothesis to C, we obtain that C is Λ-evaluated in Π. $\square$

Now, let us consider the calculus:

$$\mathbb{D}_{\mathbf{St}}(\Pi) \;=\; \mathbb{D}(\mathcal{R}_{\mathbf{St}}, \mathrm{Seq}(\Pi)) \ .$$

3.6.5 Lemma *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{St}}$*. For any* $\pi : \Gamma \vdash H \in$ Subst$^*([\Pi])$*, and for every set* Λ *of formulas of the kind* $\neg\neg K \Rightarrow K$*, if* Γ *is* Λ*-evaluated in* $\mathbb{D}_{\mathbf{St}}([\Pi])$*, then* H *is* Λ*-evaluated in* $\mathbb{D}_{\mathbf{St}}([\Pi])$.

Proof: Let us consider an arbitrary set Λ of formulas of the kind $\neg\neg K \Rightarrow K$ and let us suppose Γ to be Λ-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$. First of all we have to prove Point (i) of Definition 3.6.2, i.e. that there exists a proof $\tau : \Delta \vdash H$ with $\Delta \subseteq \Lambda$ in $\mathbb{D}_{\mathbf{St}}([\Pi])$. To this purpose, since $\pi \in$ Subst$^*([\Pi])$, we note that there exist a proof $\pi' : \Gamma' \vdash H' \in [\Pi]$ and a substitution of individual variables θ such that

$\theta\Gamma' \vdash \theta H' \equiv \Gamma \vdash H$. Thus, by definition, $\mathbb{D}_{\mathbf{St}}([\Pi])$ contains a proof of the sequent $\Gamma' \vdash H'$, and since it is SUBST-closed, there exists also a proof

$$\tau' : \Gamma \vdash H \in \mathbb{D}_{\mathbf{St}}([\Pi]) \ . \tag{3.6}$$

Now, since $\Gamma = \{H_1, \ldots, H_n\}$ is Λ-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$, there exist proofs

$$\tau_1 : \Delta_1 \vdash H_1 \ , \ \ldots \ , \ \tau_n : \Delta_n \vdash H_n \ \in \mathbb{D}_{\mathbf{St}}([\Pi]) \tag{3.7}$$

with $\Delta_1 \subseteq \Lambda, \ldots, \Delta_n \subseteq \Lambda$. By the presence in $\mathbb{D}_{\mathbf{St}}([\Pi])$ of the proofs of Points (3.6) and (3.7), and by the fact that $\mathbb{D}_{\mathbf{St}}([\Pi])$ is CUT-closed, the proof

$$\tau^* : \Delta_0 \cup \ldots \cup \Delta_n \vdash H$$

belongs to $\mathbb{D}_{\mathbf{St}}([\Pi])$. Since $\Delta_0 \cup \ldots \cup \Delta_n \subseteq \Lambda$, this proves Point (i).

Now, we prove by induction on depth(π) that H satisfies Point (ii) of Definition 3.6.2.

Basis: If depth(π) = 0, then the only rule occurring in π is an assumption introduction, that is $H \in \Gamma$. Thus, H is trivially Λ-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$.

Step: We assume that our assertion holds for any set Λ' of formulas of the form $\neg\neg K \Rightarrow K$, and for any proof $\pi' : \Gamma' \vdash A' \in \text{SUBST}^*([\Pi])$ such that depth(π') $\leq h$ ($h \geq 0$); let us suppose that depth(π) $= h+1$. The proof proceeds by cases on the last rule applied in π. Here the interesting case is when the last rule applied in π is St; the other cases can be treated as in the analogous Lemma for the Kreisel-Putnam Logic.

- **Rule** (St).

$$\pi : \Gamma \vdash A \equiv \frac{\pi_1 : \Gamma, \neg\neg B \Rightarrow B \vdash B \vee \neg B}{\Gamma \vdash (\neg B \vee \neg\neg B)}\,\mathsf{St}$$

  Since Γ is Λ-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$, by Proposition 3.6.3 we have that Γ is also $\Lambda \cup \{\neg\neg B \Rightarrow B\}$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$. Therefore, by induction hypothesis on the proof π_1, we obtain that $B \vee \neg B$ is $\Lambda \cup \{\neg\neg B \Rightarrow B\}$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$. Here we have two possible cases:

  Case 1: If B is $\Lambda \cup \{\neg\neg B \Rightarrow B\}$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$, then there exists in $\mathbb{D}_{\mathbf{St}}([\Pi])$ a proof

$$\tau' : \Delta' \vdash B$$

  with $\Delta' \subseteq \Lambda \cup \{\neg\neg B \Rightarrow B\}$. Moreover, by Point (i), there exists in $\mathbb{D}_{\mathbf{St}}([\Pi])$ a proof

$$\tau : \Delta \vdash \neg B \vee \neg\neg B$$

  with $\Delta \subseteq \Lambda$. Since $\mathbb{D}_{\mathbf{St}}([\Pi])$ is R-IN$^{\neg}$-closed, by the presence of the proofs τ and τ' we have that there exists a proof

$$\tau'' : \Delta, \Delta'' \vdash \neg\neg B \in \mathbb{D}_{\mathbf{St}}([\Pi])$$

where $\Delta'' = \Delta' \setminus \{\neg\neg B \Rightarrow B\}$. Since $\Delta \cup \Delta'' \subseteq \Lambda$ and $\neg\neg B$ is a negated formula, we immediately have that $\neg\neg B$ is Λ-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$.

Case 2: Similar to the previous one.

$\square$

3.6.6 Corollary *Let Π be any set of proofs of $\mathcal{ND}_{\mathbf{St}}$. For every proof $\tau : \Gamma \vdash A \in \mathbb{D}_{\mathbf{St}}([\Pi])$ and every substitution θ on the individual variables, if $\theta\Gamma$ is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$ (where $\emptyset$ is the empty set), then θA is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$.*

Proof: First of all, we must show that there exists a proof of the sequent $\vdash \theta A$ in $\mathbb{D}_{\mathbf{St}}([\Pi])$. This proof can be obtained using only the closure under Cut and Subst of $\mathbb{D}_{\mathbf{St}}([\Pi])$ in a way quite similar to the one seen for the corresponding point in Corollary 3.2.5. Hence, there exists a proof $\tau' : \vdash \theta A$ in $\mathbb{D}_{\mathbf{St}}([\Pi])$.

To prove Point (ii) of the definition of Λ-evaluation of the formula A in $\mathbb{D}_{\mathbf{St}}([\Pi])$ we proceed by induction on the (Cut, R-In$^{\neg}$)-depth of the proof $\tau : \Gamma \vdash A$.

Basis: If no application of the rules Cut and R-In$^{\neg}$ occur in τ, then, by definition of $\mathbb{D}_{\mathbf{St}}([\Pi])$, $\tau : \Gamma \vdash A$ is obtained by applying a (possibly empty) sequence of Subst to a sequent in $[\Pi]$. Thus, there exists a proof $\tau' : \Gamma' \vdash A' \in [\Pi]$ such that $\theta'\Gamma' \vdash \theta' A \equiv \Gamma \vdash A$ for some substitution θ'. Then also $\theta\Gamma \vdash \theta A \equiv \theta\theta'\Gamma' \vdash \theta\theta' A'$ has a proof in Subst$^*([\Pi])$. Since $\theta\Gamma$ is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$, by Lemma 3.6.5 we have that θA is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$.

Step: Let us suppose that, for any proof $\tau'' : \Gamma'' \vdash A''$ and for any substitution θ'' such that the (Cut, R-In$^{\neg}$)-depth is less than or equal to h ($h \geq 0$), if $\theta''\Gamma''$ is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$ then $\theta'' A$ is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$. Now, supposing that the (Cut, R-In$^{\neg}$)-depth of the proof τ is $h+1$, the proof goes on, as usual, by cases according to the last between the rules Cut, R-In$^{\neg}$ which occurs in τ. The proof of the case where this rule is Cut coincides with the one given in Corollary 3.5.6.

- R-In$^{\neg}$**-rule**: then the proof may have two different forms, but in all the cases the final sequent of the proof is either $\theta'\Gamma' \vdash \neg\neg\theta' A'$ or $\theta'\Gamma' \vdash \neg\theta' A'$. Now, if $\theta\Gamma$ is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$, we deduce by the Point (i) of Definition 3.6.2, that there exists a proof of $\tau' : \vdash \theta\theta'\neg\neg A'$ in $\mathbb{D}_{\mathbf{St}}([\Pi])$ or $\tau' : \vdash \theta\theta'\neg A'$ in $\mathbb{D}_{\mathbf{St}}([\Pi])$. In both cases the formula on the right side of the sequent is negated, and hence it is immediately $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{St}}([\Pi])$.

$\square$

By the previous Lemma, we can prove the analogous of Theorem 3.2.7 and Corollary 3.2.8, i.e. the following facts:

3.6.7 Theorem (SCR-$\mathcal{ND}_{\mathbf{St}}$) *$\mathcal{ND}_{\mathbf{St}}$ is a strongly constructive calculus w.r.t.* (DP$_{\text{open}}$) *and* (ED$_{\text{open}}$). $\square$

3.6.8 Corollary (SCR-St) **St** *is a strongly constructive formal system w.r.t.* (DP$_{\text{open}}$) *and* (ED$_{\text{open}}$). $\square$

3.7 Harrop Theories

In this section we will consider formal systems obtained by adding to the intuitionistic one a set of Harrop formulas as axioms.

A formula A is an *Harrop formula* iff A is inductively defined as follows:

1. A is $\bot$ or A is atomic;
2. $A \equiv B \wedge C$, and both B and C are Harrop formulas;
3. $A \equiv B \Rightarrow C$, and C is an Harrop formula;
4. $A \equiv \forall x B(x)$, and $B(x)$ is an Harrop formula.

We say that a set of Harrop formulas **Hr** is an *Harrop theory* iff **Hr** is closed under substitution on individual variables. We could get the same result by requiring the more usual condition that any formula in **Hr** is closed. Given an Harrop theory **Hr**, we denote with

$$\mathcal{H}_{\mathbf{Hr}} = \mathcal{H}_{\mathbf{INT}} + \mathbf{Hr}$$

the Hilbert-style calculus obtained by adding all the formulas of **Hr** to the axioms of $\mathcal{H}_{\mathbf{INT}}$. Let $\mathbf{S}_{\mathbf{Hr}}$ be the formal system generated by $\mathcal{H}_{\mathbf{Hr}}$.

Now, let us consider the following generalized rules.

- $\mathrm{E}^{\wedge}$ is the generalized rule of $\wedge$-elimination. That is the generalized rule whose domain consists of all the sequents of the kind $\vdash A \wedge B$ and such that:

$$\begin{array}{rcl} \vdash A & \in & \mathrm{E}^{\wedge}(\vdash A \wedge B) \\ \vdash B & \in & \mathrm{E}^{\wedge}(\vdash A \wedge B) \end{array}$$

- $\mathrm{E}^{\forall}$ is the generalized rule of $\forall$-elimination. That is the generalized rule whose domain consists of all the sequents of the kind $\vdash \forall x A(x)$ and such that:

$$\vdash A(x) \quad \in \quad \mathrm{E}^{\forall}(\vdash \forall x A(x)) \ .$$

- $\textsc{Mp}$ is the generalized rule of modus ponens. That is the generalized rule whose domain consists of all the sequences of sequents of the kind $\vdash A \Rightarrow B; \vdash A$ and:

$$\vdash B \in \textsc{Mp}(\vdash A \Rightarrow B; \vdash A) \ .$$

$\mathrm{E}^{\wedge}$ and $\mathrm{E}^{\forall}$ and $\textsc{Mp}$ are non-increasing rules and so is the compound generalized rule

$$\mathcal{R}_{\mathbf{Hr}} = \textsc{Cut} \cup \textsc{Subst} \cup \textsc{Mp} \cup \mathrm{E}^{\wedge} \cup \mathrm{E}^{\forall} \ .$$

Obviously, $\mathcal{ND}_{\mathbf{INT}}$ is uniformly closed with respect to these rules. Now, given an Harrop theory **Hr**, we denote with $\mathcal{ND}_{\mathbf{Hr}}$ the pseudo-natural deduction system obtained by adding to $\mathcal{ND}_{\mathbf{INT}}$ an axiom-rule

$$\frac{}{\vdash A}\,\textsc{Ax}$$

for any $A \in \mathbf{Hr}$.

It is immediate to verify that $\mathcal{ND}_{\mathbf{Hr}}$ is a presentation for the formal system generated by $\mathcal{H}_{\mathbf{Hr}}$. Moreover, as a trivial consequence of Proposition 3.2.3, we have:

3.7.1 Proposition *$\mathcal{ND}_{\mathbf{Hr}}$ is uniformly $\mathcal{R}_{\mathbf{Hr}}$-closed.* □

Now, we define the calculus

$$\mathbb{D}_{\mathbf{Hr}}(\Pi) \;=\; \mathbb{D}(\mathcal{R}_{\mathbf{Hr}}, \mathrm{Seq}(\Pi))$$

We prove that $\mathcal{ND}_{\mathbf{Hr}}$ is a strongly constructive system using the plain notion of formula evaluated in a set of proofs given in Definition 3.2.1.

3.7.2 Lemma *Let Π be any set of proofs of $\mathcal{ND}_{\mathbf{Hr}}$. For every proof $\pi : \Gamma \vdash A \in$ $\textsc{Subst}^*([\Pi])$, if Γ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, then A is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.*

Proof: Since $\mathcal{R}_{\mathbf{Hr}}$ contains Cut and Subst, it immediately follows from Lemma 3.2.2 that there exists a proof

$$\tau : \vdash A \in \mathbb{D}_{\mathbf{Hr}}([\Pi]) \; . \tag{3.8}$$

This proves Point (i) of Definition 3.2.1. To prove Point (ii) of Definition 3.2.1 we proceed by induction on depth(π).

Basis: If depth(π) $= 0$, we have two cases:

Case 1: The only rule applied in π is an assumption introduction. In this case $A \in \Gamma$ and hence A is trivially evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.

Case 2: The only rule applied in π is an axiom-rule. In this case A is an Harrop formula and Γ is empty. We proceed by a secondary induction on the degree of the Harrop formula A to prove that $\tau : \vdash A \in \mathbb{D}_{\mathbf{Hr}}([\Pi])$ implies that A meets Condition (ii) of the definition of formula evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. The basis case, that is A atomic or $A \equiv \bot$ (or A negated), is immediate. Now, let us suppose that, for any $\tau' \vdash A' \in \mathbb{D}_{\mathbf{Hr}}([\Pi])$ with A' an Harrop formula with degree less than or equal to k, A' is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, and let $k+1$ be the degree of A. The proof goes by cases depending on the principal logical constant of A.

- If $A \equiv B \wedge C$, with B and C Harrop formulas, then, since $\mathbb{D}_{\mathbf{Hr}}([\Pi])$ is $\mathrm{E}^{\wedge}$-closed, we get that B and C are provable in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Thus, by the secondary induction hypothesis, we obtain that both B and C are evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$ and so $A \equiv B \wedge C$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.
- If $A \equiv B \Rightarrow C$ with C an Harrop formula, let us suppose that B is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Then there exists a proof

$$\tau' : \vdash B \in \mathbb{D}_{\mathbf{Hr}}([\Pi]) \; .$$

Since $\mathbb{D}_{\mathbf{Hr}}([\Pi])$ is MP-closed, by the existence of the proofs τ and τ' in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, there exists a proof

$$\tau'' : \vdash C \in \mathbb{D}_{\mathbf{Hr}}([\Pi]) \ .$$

Thus, since C is an Harrop formula, by the secondary induction hypothesis we obtain that C is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.

- If $A \equiv \forall x B(x)$ with $B(x)$ an Harrop formula, from the fact that the proof τ of Point (3.8) belongs to $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, and from the fact that $\mathbb{D}_{\mathbf{Hr}}([\Pi])$ is $\mathrm{E}^{\forall}$-closed, we immediately obtain that there is a proof

$$\tau' : \vdash B(x) \in \mathbb{D}_{\mathbf{Hr}}([\Pi]) \ .$$

Moreover, since $\mathbb{D}_{\mathbf{Hr}}([\Pi])$ is also SUBST-closed, for any term t of the language, there exists in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$ a proof

$$\pi'_t : \vdash B(t/x) \ .$$

Therefore, applying the secondary induction hypothesis, we get that $B(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$ for any term t; hence $\forall x B(x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.

Step: Since $\mathcal{ND}_{\mathbf{Hr}}$ has exactly the same non-zero-ary rules as $\mathcal{ND}_{\mathbf{INT}}$, the proof of the Step of the induction coincides with the one given in Lemma 3.2.4 for the calculus $\mathcal{ND}_{\mathbf{INT}}$. □

3.7.3 Corollary *Let Π be a set of proofs of $\mathcal{ND}_{\mathbf{Hr}}$. For every $\tau : \Gamma \vdash A \in \mathbb{D}_{\mathbf{Hr}}([\Pi])$ and every substitution θ, if $\theta\Gamma$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, then θA is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.*

Proof: The proof of Point (i) of Definition 3.2.1 follows from the CUT-closure of $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. The proof of Point (ii) goes by induction on the (CUT, $\mathcal{R}_{\mathbf{Hr}}$)-depth of the proof τ.

Base: If no CUT-rule and no $\mathcal{R}_{\mathbf{Hr}}$-rule is applied in τ, then $\tau : \Gamma \vdash A$ is obtained by applying a (possibly empty) sequence of SUBST-rules to a sequent $\Gamma' \vdash A'$ such that $\pi' : \Gamma' \vdash A' \in [\Pi]$. This implies that there exists $\pi : \theta\Gamma \vdash \theta A \in \mathrm{SUBST}^*([\Pi])$. By applying Lemma 3.7.2, we have that θA is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.

Step: Let us suppose that the assertion holds for any proof $\tau'' : \Gamma'' \vdash A'' \in \mathbb{D}_{\mathbf{Hr}}([\Pi])$ with (CUT, $\mathcal{R}_{\mathbf{Hr}}$)-depth less than or equal to h ($h \geq 0$), and let us suppose that the (CUT, $\mathcal{R}_{\mathbf{Hr}}$)-depth of τ is $h+1$. We proceed by cases according to the last between the rules CUT and $\mathcal{R}_{\mathbf{Hr}}$ applied in τ. The proof of the case corresponding to the CUT rule coincides with the one given in Corollary 3.2.5. To prove the other case we proceed by cases on the form of the proof τ.

- τ has the form

$$\cfrac{\cfrac{\cfrac{\tau_1 : \vdash A' \wedge B'}{\vdash A'}\,\text{E}^{\wedge}}{\vdots}\,\textsc{Subst}}{\vdash \theta' A'}\,\textsc{Subst}$$

with $\theta' A \equiv A$. Since the $(\textsc{Cut}, \mathcal{R}_{\mathbf{Hr}})$-depth of τ_1 is h, by induction hypothesis on the proof

$$\frac{\tau_1 : \vdash A' \wedge B'}{\vdash \theta\theta'(A' \wedge B')}\,\textsc{Subst}$$

we have that $\theta\theta'(A' \wedge B')$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, and this immediately implies $\theta\theta' A' \equiv \theta A$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. The case where we apply the symmetric rule is analogous.

- τ has the form

$$\cfrac{\cfrac{\cfrac{\tau_1 : \vdash B' \Rightarrow A' \qquad \tau_2 : \vdash B'}{\Gamma' \vdash A'}\,\textsc{Mp}}{\vdots}\,\textsc{Subst}}{\vdash \theta' A'}\,\textsc{Subst}$$

with $\theta' A' \equiv A$. Since the $(\textsc{Cut}, \mathcal{R}_{\mathbf{Hr}})$-depth of τ_1 and τ_2 is less than or equal to h, by induction hypothesis on the proofs

$$\frac{\tau_1 : \vdash B' \Rightarrow A'}{\vdash \theta\theta' B' \Rightarrow \theta\theta' A'}\,\textsc{Subst} \qquad \frac{\tau_2 : \vdash B'}{\vdash \theta\theta' B'}\,\textsc{Subst}$$

we get that $\theta\theta' B \Rightarrow \theta\theta' A$ and $\theta\theta' B$ are both evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, and this immediately implies that $\theta\theta' A' \equiv \theta A$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.

- τ has the form

$$\cfrac{\cfrac{\cfrac{\tau_1 : \vdash \forall x B(x)}{\vdash B(x)}\,\text{E}^{\forall}}{\vdots}\,\textsc{Subst}}{\vdash \theta' B(x)}\,\textsc{Subst}$$

with $\theta' B(x) \equiv A$. Since the $(\textsc{Cut}, \mathcal{R}_{\mathbf{Hr}})$-depth of τ_1 is h, by induction hypothesis on the proof

$$\frac{\tau_1 : \vdash \forall x B(x)}{\vdash \theta\theta' \forall x B(x)} \textsc{Subst}$$

we get that $\theta\theta' \forall x B(x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, and this immediately implies that $\theta\theta' B(x) \equiv \theta A$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.

□

3.7.4 Corollary *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{Hr}}$. *Then* $\mathbb{D}_{\mathbf{Hr}}([\Pi])$ *satisfies* $(\mathrm{DP_{open}})$ *and* $(\mathrm{ED_{open}})$. □

Finally, we obtain, in the usual way, the following strong constructiveness results for $\mathcal{ND}_{\mathbf{Hr}}$ and $\mathbf{S}_{\mathbf{Hr}}$.

3.7.5 Theorem (SCR-$\mathcal{ND}_{\mathbf{Hr}}$) *For any Harrop theory* **Hr**, *if* $\mathcal{ND}_{\mathbf{Hr}}$ *is the corresponding natural deduction calculus,* $\mathcal{ND}_{\mathbf{Hr}}$ *is a strongly constructive calculus w.r.t.* $(\mathrm{DP_{open}})$ *and* $(\mathrm{ED_{open}})$. □

3.7.6 Corollary (SCR-Hr) *For any Harrop theory* **Hr**, *the formal system* $\mathbf{S}_{\mathbf{Hr}}$ *is a strongly constructive formal system w.r.t.* $(\mathrm{DP_{open}})$ *and* $(\mathrm{ED_{open}})$. □

This concludes our explanation of formal systems which turn out to be strongly constructive with respect to $(\mathrm{DP_{open}})$ and $(\mathrm{ED_{open}})$. As the reader can see, they involve pure predicate logics (intermediate predicate logics), or extensions of pure logics with the addition of special (weak) axioms. The choice of the examples has been motivated by the attempt of providing a reasonable number of variants of our method, yet giving rise to reasonably simple illustrations. On the other hand, the method seems to be very powerful also when the traditional tools (even to prove simple, non strong, constructiveness) fail. For instance, our method allows to prove strong (simple) constructiveness when syntactical techniques such as normalization cannot be used, and the semantical methods based on the Kripke models cannot be applied, by the simple fact that no Kripke semantics is known (or can be found) for the systems in hand (as already noticed, no Kripke semantics has ever been provided, as far as we know, for Kreisel-Putnam Logic and Scott Logic in a predicative context). To quote rather complex examples which can be handled using our techniques, the formal system (intermediate predicate logic) obtained by simultaneously adding the Kuroda Principle, the Grzegorczyck Principle and the Kreisel-Putnam Principles to $\mathcal{H}_{\mathbf{INT}}$ can be shown to be strongly constructive with respect to $(\mathrm{DP_{open}})$ and $(\mathrm{ED_{open}})$. The same holds if we furtherly enlarge such a system by means of Harrop axioms. Also, we can considerably extend predicate Scott Logic into predicate logics for which we are not even able to imagine a Kripke-frame semantics, yet getting strongly constructive systems with respect to ($\mathrm{DP_{open}}$) and ($\mathrm{ED_{open}}$).

Thus, the use of our method to provide systems which are strongly constructive with respect to ($\mathrm{DP_{open}}$) and ($\mathrm{ED_{open}}$) seems to be interesting especially for those systems which are intermediate (propositional or predicate) logics. We conclude this Chapter by quoting the following open problem:

> *Is there some formal system generating some intermediate predicate (propositional) logic which is constructive but not strongly constructive with respect to* ($\mathrm{DP_{open}}$) *and* ($\mathrm{ED_{open}}$) *?*

We notice that a similar problem, related to the notion of strong constructivity based on (DP) and (ED), will be solved in Chapter 5.

Chapter 4

Exhibiting strongly constructive theories

4.1 Theories with closed evaluation

In this Chapter we will devote our attention to theories involving calculi where the subproofs of $\pi : \Gamma \vdash \Delta$ only allow to evaluate closed instances of the sequent proved by π. Thus we will obtain, for the systems treated in this Chapter, strong constructiveness results with respect to (DP) and (ED), but not with respect to (DP_{open}) and (ED_{open}). As a consequence, the notion of evaluation we will use in this Chapter will depend on the set of closed terms of the language in hand; this is the reason why we will refer to it as *closed evaluation*. Another important point about the results of this section is that in some cases, namely the cases of the *Descending Chain Principle* (Section 4.4) and of *Markov Arithmetic* (Section 4.5), the strong constructiveness proofs will not take into account only proof-theoretical properties of the formal systems in hand, but also some model-theoretic properties; this requires the introduction of some model-theoretic notions.

Let $\mathcal{L}_{\mathcal{A}}$ be a (first-order) language with extra-logical alphabet $\mathcal{A}$; a *model for $\mathcal{L}_{\mathcal{A}}$* (or an *$\mathcal{A}$-structure*), defined according to classical model theory (for an extensive treatment see e.g. [Chang and Keisler, 1973]), is a couple

$$\mathcal{M}_{\mathcal{A}} = \langle \mathbf{D}, \mathbf{i} \rangle$$

where $\mathbf{D}$ is a non-empty set, called the *domain* of the $\mathcal{A}$-structure, and $\mathbf{i}$ is the interpretation of the symbols of $\mathcal{A}$. That is, $\mathbf{i}$ is a map associating, with any constant symbol c of $\mathcal{A}$ an element of $\mathbf{D}$, with any function symbol $f^{(n)}$ with arity n a function from $\mathbf{D}^n$ into $\mathbf{D}$, and with any relation $r^{(n)}$ of arity n a subset of $\mathbf{D}^n$.

In a model $\mathcal{M}_{\mathcal{A}}$ terms and formulas are interpreted in the usual way, and we write

$$\mathcal{M}_{\mathcal{A}} \models A$$

to mean that the formula A is true (is valid, holds) on $\mathcal{M}_{\mathcal{A}}$ in the classical sense (if A is not closed, this notation means that the universal closure of A holds in $\mathcal{M}_{\mathcal{A}}$).

Given a theory $\mathbf{T}$ over the language $\mathcal{L}_{\mathcal{A}}$, we write

$$\mathcal{M}_{\mathcal{A}} \models \mathbf{T}$$

to mean, as usual, that all the formulas of $\mathbf{T}$ hold in $\mathcal{M}_{\mathcal{A}}$. We say that an $\mathcal{A}$-structure $\mathcal{M}_{\mathcal{A}}$ is *reachable* iff any element of $\mathbf{D}$ is denoted by a closed term of $\mathcal{L}_{\mathcal{A}}$.

To prove the strong constructiveness of the calculi related to theories $\mathbf{T}$ over the language $\mathcal{L}_{\mathcal{A}}$, we will consider the following notion of evaluation of a formula in a set of proofs.

4.1.1 Definition *Let Π be a set of proofs and A be a formula of $\mathcal{L}_{\mathcal{A}}$. We say that A is $\mathcal{L}_{\mathcal{A}}$-evaluated in Π iff the following conditions hold:*

(i) There is a proof $\pi : \vdash A \in \Pi$;

(ii) For every closed instance θA of A, one of the following cases holds:

(a) θA is atomic or negated;

(b) $\theta A \equiv B \wedge C$, and both B and C are $\mathcal{L}_{\mathcal{A}}$-evaluated in Π;

(c) $\theta A \equiv B \vee C$, and either B is $\mathcal{L}_{\mathcal{A}}$-evaluated in Π or C is $\mathcal{L}_{\mathcal{A}}$-evaluated in Π;

(d) $\theta A \equiv B \Rightarrow C$, and either B is not $\mathcal{L}_{\mathcal{A}}$-evaluated in Π or C is $\mathcal{L}_{\mathcal{A}}$-evaluated in Π;

(e) $\theta A \equiv \exists x B(x)$, and $B(t/x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in Π for some closed term t;

(f) $\theta A \equiv \forall x B(x)$, and for any closed term t, $B(t/x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in Π.

We begin our survey of strongly constructive theories presenting Heyting Arithmetic.

4.2 Intuitionistic arithmetic

Let us consider the language $\mathcal{L}_{\mathbf{A}}$ of arithmetic which extra-logical alphabet $\mathbf{A}$ consisting of the constant symbol $\mathtt{0}$, the unary function symbol $\mathtt{s}$, the two binary function symbols $+$ and $*$ and the binary relation symbol $=$. Let $\mathcal{H}_{\mathbf{HA}}$ be the Hilbert-style calculus obtained by adding to the Hilbert-style calculus $\mathcal{H}_{\mathbf{INT}}$ for intuitionistic logic (presented in Section 1.2.1) the following axioms and rules:

Equality axioms:

(i). $x{=}x$
(ii). $x{=}y \wedge y{=}z \Rightarrow x{=}z$
(iii). $x = y \Rightarrow \mathsf{s}x = \mathsf{s}y$
(iv). $x{=}x' \wedge y{=}y' \Rightarrow (x{+}y{=}x'{+}y')$
(v). $x{=}x' \wedge y{=}y' \Rightarrow (x{*}y{=}x'{*}y')$

Successor axioms:

(vi). $\mathsf{s}x = \mathsf{s}y \Rightarrow x = y$
(vii). $\mathsf{s}x = \mathsf{0} \Rightarrow \bot$

Sum axioms:

(viii). $x{+}\mathsf{0}{=}x$
(ix). $x{+}\mathsf{s}y{=}\mathsf{s}(x{+}y)$

Product axioms:

(x). $x{*}\mathsf{0}{=}\mathsf{0}$
(xi). $x{*}\mathsf{s}y{=}(x{*}y){+}x$

Induction axiom scheme:

(xii). $A(\mathsf{0}) \wedge \forall x(A(x) \Rightarrow A(\mathsf{s}x)) \Rightarrow \forall y A(y)$

We indicate with **HA** the formal system of *Heyting Arithmetic* generated by the Hilbert-style calculus $\mathcal{H}_{\mathbf{HA}}$.

With $\mathcal{ND}_{\mathbf{HA}}$ we denote the pseudo-natural deduction system, obtained by adding to $\mathcal{ND}_{\mathbf{INT}}$, presented in Section 1.2.2, the rules for identity, successor, sum, product and induction listed below.

Identity:

$$\frac{}{\vdash x{=}x}\,\mathrm{id}_1 \qquad \frac{\pi_1 : \Gamma \vdash a(t) \qquad \pi_2 : \Delta \vdash t{=}t'}{\Gamma, \Delta \vdash a(t')}\,\mathrm{id}_2$$

where, in id_2, $a(t)$ is an atomic formula.

Successor Rules:

$$\frac{\Gamma \vdash \mathsf{0}{=}\mathsf{s}(x)}{\Gamma \vdash \bot}\,\mathrm{s}_1 \qquad \frac{\Gamma \vdash \mathsf{s}(t){=}\mathsf{s}(t')}{\Gamma \vdash t{=}t'}\,\mathrm{s}_2$$

Sum Rules:

$$\frac{}{\vdash t{+}\mathsf{0}{=}t}\,{+}_1 \qquad \frac{}{\vdash t{+}\mathsf{s}(t'){=}\mathsf{s}(t{+}t')}\,{+}_2$$

Product Rule:

$$\frac{}{\vdash t{*}\mathsf{0}{=}\mathsf{0}}\,{*_1} \qquad \frac{}{\vdash t{*}\mathsf{s}(t'){=}t{*}t'{+}t}\,{*_2}$$

Induction Rule:

$$\frac{\Gamma \vdash A(0) \qquad \Delta, A(p) \vdash A(\mathsf{s}(p))}{\Gamma, \Delta \vdash A(x)}\,\mathsf{Ind}$$

where p does not occur free in any formula in Δ. p is the *proper parameter* (or the *eigenvariable*) of the considered application of the induction rule.

It is trivial to verify that $\mathcal{ND}_{\mathbf{HA}}$ is a presentation for the formal system **HA**. We remark that in $\mathcal{ND}_{\mathbf{HA}}$ the following non-restricted version of the id$_2$-rule is derivable:

$$\frac{\pi_1 : \Gamma \vdash A(t) \qquad \pi_2 : \Delta \vdash t{=}t'}{\Gamma, \Delta \vdash A(t')}\,\mathrm{id}_2^*$$

where $A(t)$ is any formula. Two other useful derived rules of $\mathcal{ND}_{\mathbf{HA}}$ are:

$$\frac{t{=}t'}{t'{=}t}\,\mathrm{sym} \qquad \frac{\pi_1 : \Gamma \vdash A(t) \qquad \pi_2 : \Delta \vdash t'{=}t}{\Gamma, \Delta \vdash A(t')}\,\mathrm{id}_2^{**} \; .$$

The set of *canonical terms* of **HA** is the set of closed terms of $\mathcal{L}_{\mathbf{A}}$ defined as follows:

1. The constant 0 is a canonical term of **HA**;
2. If t is a canonical term of **HA** then $\mathsf{s}t$ is a canonical term of **HA**;
3. Nothing else is a canonical term of **HA**.

To denote the canonical terms of $\mathcal{L}_{\mathbf{A}}$ we use the compact notation $\mathsf{s}^n 0$ inductively defined as follows:

$$\mathsf{s}^n 0 = \begin{cases} 0 & \text{if } n = 0 \\ \mathsf{s}\mathsf{s}^{n-1} 0 & \text{if } n > 0 \end{cases}$$

where we simply denote with $\mathsf{s}0$ the canonical term $\mathsf{s}^1 0$.

It is well known that the following result holds:

4.2.1 Proposition *Let t be a closed term of $\mathcal{L}_{\mathbf{A}}$. Then there exists one and only one canonical term $\mathsf{s}^n\mathsf{0}$ such that $\vdash_{\mathcal{H}_{\mathbf{HA}}} t{=}\mathsf{s}^n\mathsf{0}$ and $\vdash_{\mathcal{ND}_{\mathbf{HA}}} t{=}\mathsf{s}^n\mathsf{0}$.* □

According to the previous proposition, given a closed term t of $\mathcal{L}_{\mathbf{A}}$, we call *canonical form of t in* **HA** the only canonical term $\mathsf{s}^n\mathsf{0}$ such that $\vdash_{\mathcal{H}_{\mathbf{HA}}} t{=}\mathsf{s}^n\mathsf{0}$ ($\vdash_{\mathcal{ND}_{\mathbf{HA}}} t{=}\mathsf{s}^n\mathsf{0}$).

Now, we introduce the following generalized rules:

- $\textsc{Id}_1$, $\textsc{Sum}$, $\textsc{Prod}$ are the generalized rules whose domain contains only the empty sequence ϵ and such that:

$$\begin{array}{rcl} \vdash x{=}x & \in & \textsc{Id}_1(\epsilon); \\ \vdash x{+}\mathsf{0}{=}x & \in & \textsc{Sum}(\epsilon); \\ \vdash x{+}\mathsf{s}y{=}\mathsf{s}(x{+}y) & \in & \textsc{Sum}(\epsilon); \\ \vdash x{*}\mathsf{0}{=}\mathsf{0} & \in & \textsc{Prod}(\epsilon); \\ \vdash x{*}\mathsf{s}y{=}x{*}y{+}x & \in & \textsc{Prod}(\epsilon) . \end{array}$$

Since any of these rules introduces a sequent in which only an atomic formula occurs, these are 1-bounded non-increasing rules.

- $\textsc{Id}_2$ is the generalized rule whose domain contains all the sequences of sequents σ^* which have one of the following forms:

$$\begin{array}{rcl} \sigma^* & \equiv & \Gamma \vdash A(t); \Delta \vdash t{=}t' \\ \sigma^* & \equiv & \Gamma \vdash A(t); \Delta \vdash t'{=}t \end{array}$$

  and such that:

$$\begin{array}{rcl} \Gamma, \Delta \vdash A(t') & \in & \textsc{Id}_2(\Gamma \vdash A(t); \Delta \vdash t{=}t') \\ \Gamma, \Delta \vdash A(t') & \in & \textsc{Id}_2(\Gamma \vdash A(t); \Delta \vdash t'{=}t) \end{array}$$

We set

$$\mathcal{R}_{\mathbf{HA}} = \textsc{Cut} \cup \textsc{Subst} \cup \textsc{Id}_1 \cup \textsc{Id}_2 \cup \textsc{Sum} \cup \textsc{Prod} .$$

Since any of the rules used to define $\mathcal{R}_{\mathbf{HA}}$ is a (1-bounded) non-increasing rule, we have that $\mathcal{R}_{\mathbf{HA}}$ is a (1-bounded) non-increasing generalized rule. Moreover, it is immediate to verify that:

4.2.2 Proposition *$\mathcal{ND}_{\mathbf{HA}}$ is uniformly $\mathcal{R}_{\mathbf{HA}}$-closed.* □

Given any set of proofs Π, we define:

$$\mathbb{D}_{\mathbf{HA}}(\Pi) = \mathbb{D}(\mathcal{R}_{\mathbf{HA}}, \mathrm{Seq}(\Pi)) .$$

To prove the strong constructiveness of $\mathcal{ND}_{\mathbf{HA}}$ we need to evaluate terms in their normal forms inside the calculus $\mathbb{D}_{\mathbf{HA}}([\Pi])$ and to prove that if a formula $A(t)$, with t a closed term, is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, then $A(t')$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$ for any closed term t' such that the formula $t{=}t'$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Namely, we need the two following propositions:

4.2.3 Proposition *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{HA}}$. *For any closed term* t *of* $\mathcal{L}_{\mathbf{A}}$, *there exists a proof* $\tau : \vdash t{=}\mathsf{s}^n 0 \in \mathbb{D}_{\mathbf{HA}}([\Pi])$, *where* $\mathsf{s}^n 0$ *is the canonical form of* t *in* **HA**. *Moreover, there is no proof* $\tau' : \vdash t{=}\mathsf{s}^m 0 \in \mathbb{D}_{\mathbf{HA}}([\Pi])$ *with* $m \neq n$.

Proof: The proof of the first part goes by a straightforward induction on the degree of t. To prove that the canonical form of t in $\mathbb{D}_{\mathbf{HA}}([\Pi])$ is unique, let us suppose that there exists a proof $\tau' : \vdash t{=}\mathsf{s}^m 0 \in \mathbb{D}_{\mathbf{HA}}([\Pi])$ with $m \neq n$. Now, by Proposition 2.3.8, $\mathrm{Seq}(\mathbb{D}_{\mathbf{HA}}([\Pi])) \subseteq \mathrm{Seq}(\mathcal{ND}_{\mathbf{HA}})$, and hence $\vdash t{=}\mathsf{s}^n 0, \vdash t{=}\mathsf{s}^m 0 \in \mathrm{Seq}(\mathcal{ND}_{\mathbf{HA}})$. This give rise to a contradiction, since it implies the inconsistency of $\mathcal{ND}_{\mathbf{HA}}$. $\Box$

4.2.4 Proposition *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{HA}}$, *let* $A(x)$ *be a formula of* $\mathcal{L}_{\mathbf{A}}$ *and let* t *and* t' *be closed terms of* $\mathcal{L}_{\mathbf{A}}$. *If* $t{=}t'$ *is* $\mathcal{L}_{\mathbf{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{HA}}([\Pi])$ *then* $A(t)$ *is* $\mathcal{L}_{\mathbf{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{HA}}([\Pi])$ *iff* $A(t')$ *is* $\mathcal{L}_{\mathbf{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{HA}}([\Pi])$.

Proof: Since $\mathbb{D}_{\mathbf{HA}}([\Pi])$ is ID_2-closed, it is immediate to verify that $\vdash A(t)$ is provable in $\mathbb{D}_{\mathbf{HA}}([\Pi])$ iff $\vdash A(t')$ is provable in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. The proof of Point (ii) of Definition 4.1.1 easily follows by induction on the structure of $A(x)$. $\Box$

4.2.5 Lemma *Let* Π *be a set of proofs of* $\mathcal{ND}_{\mathbf{HA}}$. *For any* $\pi : \Gamma \vdash A \in \mathrm{SUBST}^*([\Pi])$, *if* Γ *is* $\mathcal{L}_{\mathbf{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{HA}}([\Pi])$, *then* A *is* $\mathcal{L}_{\mathbf{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{HA}}([\Pi])$.

Proof: Since $\mathcal{R}_{\mathbf{HA}}$ contains the generalized rules CUT and SUBST, the proof of Point (i) of Definition 4.1.1 immediately follows from Lemma 3.2.2. To prove Point (ii) we proceed by induction on $\mathrm{depth}(\pi)$.

Basis: If $\mathrm{depth}(\pi) = 0$ then either the only rule applied in π is an assumption introduction, or it is one between the rules id_1, $+_1$, $+_2$, $*_1$ and $*_2$. In the former case we have $A \in \Gamma$, and the assertion immediately follows. In the latter case A is an atomic formula and hence, by definition, it is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$.

Step: Let us suppose that the assertion holds for any proof $\pi' : \Gamma' \vdash A'$ belonging to $\mathrm{SUBST}^*([\Pi])$ such that $\mathrm{depth}(\pi') \leq h$ and let $\mathrm{depth}(\pi) = h+1$. We proceed by cases, according to the last rule applied in the proof π. The proof for all the rules of $\mathcal{ND}_{\mathbf{INT}}$ essentially coincides with the one developed in Lemma 3.2.4. If the last rule applied in π is one of id_2, s_1, s_2, the assertion immediately follows from the fact that A is an atomic formula. Now, the only rule we have to analyze is Ind.

- **Induction Rule**.

$$\pi : \Gamma \vdash A \equiv \frac{\pi_1 : \Gamma_1 \vdash B(0) \qquad \pi_2 : \Gamma_2, B(p) \vdash B(\mathsf{s}(p))}{\Gamma \vdash B(x)}\,\mathsf{Ind}$$

  where $\Gamma = \Gamma_1 \cup \Gamma_2$. We begin to prove that $B(\mathsf{s}^m 0)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$ for any $m \geq 0$. We proceed by a secondary induction on m. Since $\Gamma_1 \subseteq \Gamma$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, we have, by the principal induction hypothesis on π_1, that $B(0)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Now, let us suppose that $B(\mathsf{s}^h 0)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, with $h \geq 0$. Let $\pi_2[\mathsf{s}^h 0/p]$ be the proof obtained by replacing any occurrence of the proper parameter p in π_2 with $\mathbf{s}^h 0$. By the stipulations on the proper parameters we made in Section 1.2.2,

$$\pi_2[\mathbf{s}^h 0/p] : \Gamma_2, B(\mathbf{s}^h 0/p) \vdash B(\mathbf{s}^{h+1} 0/p)$$

and, by the principal induction hypothesis on π_2, we obtain that $B(\mathbf{s}^{h+1}0)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Now, let us consider the proof π. To prove that $B(x)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, we must show that any closed instance $\theta B(x)$ of $B(x)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Let $t = \theta(x)$ (where t is a closed term) and let $\mathbf{s}^n 0$ be the canonical form of t in **HA**. Since we have already proved that $B(\mathbf{s}^n 0)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated, we have that also the closed instance $\theta B(\mathbf{s}^n 0)$ of $B(\mathbf{s}^n 0)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. But since, by Proposition 4.2.3, $\mathbb{D}_{\mathbf{HA}}([\Pi])$ contains a proof of the sequent $\vdash t{=}\mathbf{s}^n 0$, we obtain, by Proposition 4.2.4, that $\theta B(t)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. This concludes the proof.

□

4.2.6 Corollary *Let Π be a set of proofs of $\mathcal{ND}_{\mathbf{HA}}$. For every $\tau : \Gamma \vdash A \in \mathbb{D}_{\mathbf{HA}}([\Pi])$ and every substitution θ, if $\theta\Gamma$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, then θA is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$.*

Proof: First of all, we must show that there exists a proof of the sequent $\vdash \theta A$ in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Let $\Gamma = \{H_1, \ldots, H_n\}$. Since $\theta\Gamma$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, by Point (i) of Definition 4.1.1, there exist $\tau_1, \ldots, \tau_n$ in $\mathbb{D}_{\mathbf{HA}}([\Pi])$ such that, for any $i : 1 \leq i \leq n$, $\tau_i : \vdash \theta H_i$. Moreover, since $\mathbb{D}_{\mathbf{HA}}([\Pi])$ is SUBST-closed, it contains a proof $\tau' : \theta\Gamma \vdash \theta A$, and since $\mathbb{D}_{\mathbf{HA}}([\Pi])$ is CUT-closed, it also contains a proof $\tau'' : \vdash \theta A$. The proof of Point (ii) goes on by induction on the $(\text{CUT}, \text{ID}_2)$-depth of τ.

Base: If no CUT-rule and no ID_2-rule is applied in τ, then $\tau : \Gamma \vdash A$ is either obtained by applying a possibly empty sequence of SUBST-rules to a sequent in Seq([Π]), or it is obtained by applying a possibly empty sequence of SUBST-rules to the one-step proof consisting of one of the axiom-rules ID_1, SUM, PROD. In the former case there exists a proof $\tau' : \Gamma' \vdash A' \in [\Pi]$ such that $\theta'\Gamma' \vdash \theta' A' \equiv \Gamma \vdash A$ for some substitution θ'. Then, $\text{SUBST}^*([\Pi])$ also contains a proof of the sequent $\theta\Gamma \vdash \theta A \equiv \theta\theta'\Gamma' \vdash \theta\theta' A'$. Since $\theta\Gamma$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, by Lemma 4.2.5, we have that θA is evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. In the latter case $\tau : \Gamma \vdash A$ is:

$$\dfrac{\dfrac{\dfrac{}{\vdash t{=}t'}\,\#}{\vdots}\,\text{SUBST}}{\theta'(t{=}t')}\,\text{SUBST}$$

where # is one of the rules ID_1,SUM, PROD and $A \equiv \theta'(t{=}t')$. Thus, since A is atomic, we immediately have that it is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$.

Step: The step goes by cases according to the last between the rules CUT and ID$_2$ applied in τ. If this rule is CUT the proof coincides with the one given in Lemma 3.2.5. Otherwise, since the conclusion of an ID$_2$-rule is atomic, the fact that it is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$ immediately follows from the provability of A in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. □

4.2.7 Corollary *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{HA}}$. *Then* $\mathbb{D}_{\mathbf{HA}}([\Pi])$ *satisfies* (DP) *and* (ED).

Proof: Let $A \vee B$ be a closed formula belonging to Theo($\mathbb{D}_{\mathbf{HA}}([\Pi])$). Then there exists a proof $\tau : \vdash A \vee B$ in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Since the empty set of premises is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, by Corollary 4.2.6 we immediately deduce that $A \vee B$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. By Definition 4.1.1, it follows that, for any closed instance $\theta(A \vee B)$ of $A \vee B$, either θA or θB is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. But, since $A \vee B$ is a closed formula, its only closed instance is $A \vee B$ itself. Hence, by Point (i) of Definition 4.1.1, we deduce that either $A \in$ Theo($\mathbb{D}_{\mathbf{HA}}([\Pi])$) or $B \in$ Theo($\mathbb{D}_{\mathbf{HA}}([\Pi])$). Therefore, Theo($\mathbb{D}_{\mathbf{HA}}([\Pi])$) satisfies (DP). In a similar way we can prove that Theo($\mathbb{D}_{\mathbf{HA}}([\Pi])$) enjoys the explicit definability property. □

We remark that if $A \vee B$ is an open formula, from its $\mathcal{L}_{\mathbf{A}}$-evaluation in $\mathbb{D}_{\mathbf{HA}}([\Pi])$ we can only deduce that, for any closed-instance $\theta(A \vee B)$ of $A \vee B$, either θA or θB is is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$; nothing can be said about $A \vee B$.

Since $\mathcal{R}_{\mathbf{HA}}$ is a non-increasing generalized rule, $\mathcal{ND}_{\mathbf{HA}}$ is uniformly $\mathcal{R}_{\mathbf{HA}}$-closed (Proposition 4.2.2) and, by means of the previous corollary, it is trivial to verify (along the lines explained in the proof of Theorem 3.2.7), that $\mathcal{ND}_{\mathbf{HA}}$ meets all the conditions needed to be a strongly constructive calculus.

4.2.8 Theorem (SCR-$\mathcal{ND}_{\mathbf{HA}}$) $\mathcal{ND}_{\mathbf{HA}}$ *is a strongly constructive calculus w.r.t.* (DP) *and* (ED). □

Finally, since we already know that $\mathcal{ND}_{\mathbf{HA}}$ is a presentation for **HA**, to deduce that **HA** is a strongly constructive formal system, we only need to show that $\mathcal{ND}_{\mathbf{HA}}$ agrees with **HA**. But, by of Proposition 2.4.11, this amounts to prove that $\mathcal{ND}_{\mathbf{HA}}$ is uniformly embedded in $\mathcal{H}_{\mathbf{HA}}$, this fact immediately following from Theorem 2.4.3 and an inspection of the proper rules of $\mathcal{ND}_{\mathbf{HA}}$.

4.2.9 Corollary (SCR-HA) **HA** *is a strongly constructive formal system w.r.t.* (DP) *and* (ED). □

4.3 Generalized induction

In this section we consider a first-order schema which is close to the induction principle and is based on the following notion of cover set:

4.3.1 Definition *Given an alphabet $\mathcal{A}$ with a non-empty set of constant symbols, we say that a finite set $\mathcal{C}$ of terms of $\mathcal{L}_{\mathcal{A}}$ is a* cover set for $\mathrm{Term}(\mathcal{L}_{\mathcal{A}})$ *if no term of $\mathcal{C}$ is a variable and, for every closed term t of $\mathcal{L}_{\mathcal{A}}$, there is a term $t' \in \mathcal{C}$ such that $t \equiv \theta t'$, for some substitution θ of individual variables.*

Now, we need the following abbreviations: given a finite set of formulas $\Delta = \{B_1, \ldots, B_k\}$, we set:

$$\bigwedge \Delta \equiv B_1 \wedge \ldots \wedge B_k \ ;$$

given a formula A whose free variables are exactly $x_1, \ldots, x_n$, we set

$$\forall(A) \equiv \forall x_1 \ldots \forall x_n A \ .$$

With a cover set $\mathcal{C} = \{t_1, \ldots, t_n\}$ we associate the following *Generalized Induction Principle*:

$$(\mathsf{Gind}) \qquad \forall(\bigwedge \Delta_1 \Rightarrow A(t_1)) \wedge \ldots \wedge \forall(\bigwedge \Delta_n \Rightarrow A(t_n)) \Rightarrow \forall x A(x)$$

where, for $1 \leq i \leq n$, t_i is a term but not a variable, and if t_i contains k_i variables $y^i_1, \ldots, y^i_{k_i}$, then Δ_i is $\{A(y^i_1), \ldots, A(y^i_{k_i})\}$.

We can express this principle in a pseudo-natural deduction style with the following rule:

Generalized Induction Rule : With a cover set $\mathcal{C} = \{t_1, \ldots, t_n\}$ we associate the *Generalized Induction Rule*:

$$\frac{\Gamma, \Delta_1 \vdash A(t_1) \qquad \ldots \qquad \Gamma, \Delta_n \vdash A(t_n)}{\Gamma \vdash A(x)}\ \mathsf{Gind}$$

where, for $1 \leq i \leq n$, t_i is not a variable and if t_i contains k_i variables $y^i_1, \ldots, y^i_{k_i}$, then Δ_i is $\{A(y^i_1), \ldots, A(y^i_{k_i})\}$. If $k_i = 0$, Δ_i is the empty set. $y^i_1, \ldots, y^i_{k_i}$ are the *eigenvariables* of the induction rule and cannot occur free in $\Gamma, A(x)$. The formulas in Δ_i are the induction hypotheses.

Now, let **Hr** be an Harrop-theory. We denote with

$$\mathcal{H}_{\mathbf{Hr}+(\mathsf{Gind})} = \mathcal{H}_{\mathbf{INT}} + \mathbf{Hr} + \mathbf{ID} + (\mathsf{Gind})$$

the Hilbert-style calculus obtained by adding the formulas of **Hr**, the theory of identity **ID** (i.e. the theory consisting of the equality axioms listed in Section 4.2), and all the instances of the axiom schema Gind to the axioms of $\mathcal{H}_{\mathbf{INT}}$. $\mathbf{T}_{\mathbf{Hr}+\mathbf{ID}+(\mathsf{Gind})}$ will be the formal system generated by the Hilbert-style calculus $\mathcal{H}_{\mathbf{Hr}+\mathbf{ID}+(\mathsf{Gind})}$.

Finally, let us denote with $\mathcal{ND}_{\mathbf{Hr}+\mathbf{ID}+\mathsf{Gind}}$ the pseudo-natural deduction calculus obtained by adding to the calculus $\mathcal{ND}_{\mathbf{INT}}$ the rule Gind, rules id_1 and id_2 for identity given in Section 4.2 and, for any formula $A \in \mathbf{Hr}$, the zero-premises rule

$$\frac{}{\vdash A}\ \mathrm{Ax}$$

To prove the strong constructiveness of the formal system $\mathbf{T}_{\mathbf{Hr}+\mathbf{ID}+(\mathsf{Gind})}$, we use the generalized rule $\mathcal{R}_{\mathbf{Hr}}$ and the abstract calculus $\mathbb{D}_{\mathbf{Hr}}([\Pi])$ defined in Section 3.7.

4.3.2 Lemma *Let* Π *be a set of proofs of* $\mathcal{ND}_{\mathbf{Hr}+\mathbf{ID}+\mathsf{Gind}}$. *For any* $\pi : \Gamma \vdash A \in$ $\textsc{Subst}^*([\Pi])$, *if* Γ *is* $\mathcal{L}_{\mathcal{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, *then* A *is* $\mathcal{L}_{\mathcal{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.

Proof: Since $\mathcal{R}_{\mathbf{Hr}}$ contains $\textsc{Cut}$ and $\textsc{Subst}$, the proof of Point (i) of Definition 4.1.1 immediately follows from Lemma 3.2.2. To prove Point (ii) we proceed by induction on depth(π).

Basis: If depth(π) $= 0$, then either the only rule applied in π is an assumption introduction, or A is an Harrop formula. In both cases the proof essentially coincides with the one given in Lemma 3.7.2.

Step: Now, let us suppose that the assertion holds for any proof $\pi' : \Gamma' \vdash A'$ belonging to $\textsc{Subst}^*([\Pi])$ such that depth(π') $\leq h$, and let depth(π) $= h + 1$. We proceed by cases according to the last rule applied in the proof π. The proof for the rules of $\mathcal{ND}_{\mathbf{INT}}$ is quite similar to the one given in Lemma 3.2.4. Now we analyze the rule Gind.

- **Generalized induction rule**:

$$\frac{\pi_1 : \Gamma, \Delta_1 \vdash A(t_1) \qquad \ldots \qquad \pi_n : \Gamma, \Delta_n \vdash A(t_n)}{\Gamma \vdash A(x)}\,\mathsf{Gind}\ .$$

We begin by proving that, for any closed term t of $\mathcal{L}_{\mathcal{A}}$, $A(t/x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. This proof goes on by a secondary induction on the structure of the term t. The basis is the case where $t = c$ and c is a constant symbol. This implies that c belongs to the cover set $\mathcal{C}$; then, by construction of the rule Gind, there exists a subproof $\pi_i : \Gamma \vdash A(c/x)$, for some $i \in \{1, \ldots, n\}$. In this case the assertion immediately follows from the principal induction hypothesis applied to the proof π_i. Now, let us suppose that the assertion holds for any term t with complexity less or equal to h, and let $h + 1$ be the complexity of t. By definition of cover set, there exists a term $t_i \in \mathcal{C}$ such that, for some substitution θ', $\theta' t_i \equiv t$. Let us consider the proof $\pi_i : \Gamma, \Delta_i \vdash A(t_i)$, and let us apply the substitution θ' to this proof. By the convention on the proper parameters, we have that

$$\theta'\pi_i : \theta'\Gamma, \theta'\Delta_i \vdash A(t)\ .$$

Note that $\theta'\Delta_i$ contains formulas of the kind $A(t')$, with t' a term with complexity less than or equal to h; thus, we can apply the secondary induction hypothesis and deduce that $\theta'\Delta_i$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Thus, by applying the principal induction hypothesis to $\theta'\pi_i$, we get that $A(t/x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Now, to prove that $A(x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, is sufficient to notice that any closed instance $\theta A(x)$ of $A(x)$ can be obtained as a closed instance $\theta' A(t/x)$ for some closed term t of $\mathcal{L}_{\mathcal{S}}$.

□

The proof of the following result coincides with the one given for Corollary 3.7.3.

4.3.3 Corollary *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{Hr}+\mathsf{Gind}}$. *For every proof* $\tau : \Gamma \vdash A \in \mathbb{D}_{\mathbf{Hr}}([\Pi])$ *and every substitution* θ, *if* $\theta\Gamma$ *is* $\mathcal{L}_{\mathcal{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, *then* θA *is* $\mathcal{L}_{\mathcal{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. □

Now we obtain, in the usual way, the strong constructiveness results for $\mathcal{ND}_{\mathbf{Hr}+\mathsf{Gind}}$ and **HA**.

4.3.4 Theorem (SCR-$\mathcal{ND}_{\mathbf{Hr}+\mathbf{ID}+\mathsf{Gind}}$) $\mathcal{ND}_{\mathbf{Hr}+\mathbf{ID}+\mathsf{Gind}}$ *is a strongly constructive calculus w.r.t.* (DP) *and* (ED). □

4.3.5 Corollary (SCR-$\mathbf{T}_{\mathbf{Hr}+\mathbf{ID}+(\mathsf{Gind})}$) $\mathbf{T}_{\mathbf{Hr}+\mathbf{ID}+(\mathsf{Gind})}$ *is a strongly constructive formal system w.r.t.* (DP) *and* (ED). □

4.4 Descending chain principle

Now, let us consider an alphabet $\mathcal{A}$ containing a binary relation symbol $<$ and an Hilbert-style calculus $\mathcal{H}_{\mathbf{PO}}$ over $\mathcal{L}_{\mathcal{A}}$ formalizing $<$ as an irreflexive and transitive relation. In this context is meaningful to consider the following principle, known as the *descending chain principle*.

$$(\mathsf{DCP}) \qquad \exists x A(x) \wedge \forall y(A(y) \Rightarrow \exists z(A(z) \wedge z < y) \vee B) \Rightarrow B$$

Here we consider as an illustrative example the case where (DCP) is added to the Hilbert-style calculus for Heyting Arithmetic. In this case we do not need to add further axioms, it is sufficient to consider $<$ as an abbreviation for:

$$\exists z(x+\mathbf{s}z=y) \ .$$

In this frame let $\mathcal{H}_{\mathbf{HA}+\{(\mathsf{DCP})\}}$ denote the Hilbert-style calculus obtained by adding all the instances of the axiom schema (DCP) to the axioms of $\mathcal{H}_{\mathbf{HA}}$, that is

$$\mathcal{H}_{\mathbf{HA}+\{(\mathsf{DCP})\}} = \mathcal{H}_{\mathbf{HA}} + \{(\mathsf{DCP})\} \ .$$

$\mathbf{T}_{\mathbf{HA}+\{(\mathsf{DCP})\}}$ will denote the formal system generated by the Hilbert style calculus $\mathcal{H}_{\mathbf{HA}+\{(\mathsf{DCP})\}}$.

The Descending Chain Principle can be stated as a pseudo-natural deduction rule in the following way:

Descending Chain Principle :

$$\frac{\Gamma \vdash \exists x A(x) \qquad \Gamma, A(y) \vdash \exists z(A(z) \wedge z < y) \vee B}{\Gamma \vdash B}\ \mathsf{DCP}$$

where y is the *eigenvariable* of the DCP rule and does not occur free in Γ, B.

We denote with $\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$ the pseudo-natural deduction calculus obtained by adding the rule DCP to $\mathcal{ND}_{\mathbf{HA}}$. It is easy to verify that $\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$ is a presentation for the formal system $\mathbf{T}_{\mathbf{HA}+\{(\mathsf{DCP})\}}$.

To prove the strong constructiveness of $\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$, we use the generalized rule $\mathcal{R}_{\mathbf{HA}}$ and the calculus $\mathbb{D}_{\mathbf{HA}}([\Pi])$ defined in Section 4.2. It is immediate to verify that:

4.4.1 Proposition *$\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$ is uniformly $\mathcal{R}_{\mathbf{HA}}$-closed.* □

4.4.2 Lemma *Let Π be a set of proofs of $\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$. For any $\pi : \Gamma \vdash A \in$ $\mathrm{SUBST}^*([\Pi])$, if Γ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, then A is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$.*

Proof: Since $\mathcal{R}_{\mathbf{HA}}$ contains CUT and SUBST, the proof of Point (i) of Definition 4.1.1 immediately follows from Lemma 3.2.2. To prove Point (ii), we proceed by induction on depth(π).

Basis: If depth(π) $= 0$, then either the only rule applied in π is an assumption introduction, or it is one of the axiom-rules id_1, $+_1$, $+_2$, $*_1$ and $*_2$. In the former case we have $A \in \Gamma$, and the assertion immediately follows. In the latter case A is an atomic formula and hence there in nothing more to prove.

Step: Now, let us suppose that the assertion holds for any proof $\pi' : \Gamma' \vdash A'$ belonging to $\mathrm{SUBST}^*([\Pi])$ such that depth(π') $\leq h$, and let depth(π) $= h+1$. We proceed by cases, according to the last rule applied in the proof π. The proof for the other rules is analogous to the one explained in Lemma 4.2.5. Now we analyze the rule DCP.

- DCP **rule**:

$$\frac{\pi_1 : \Gamma \vdash \exists x A(x) \qquad \pi_2 : \Gamma, A(y) \vdash \exists x(A(x) \wedge x < y) \vee B}{\Gamma \vdash B}\ \mathsf{DCP}$$

  We assume that B is not $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$ and show that this gives rise to a contradiction. By induction hypothesis, $\exists x A(x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, and hence there exists at least a closed term t_0 such that $A(t_0/x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. By the conventions on the proper parameters, we have that $\pi_2[t_0/y]$ is a proof of the sequent

$$\Gamma, A(t_0/y) \vdash \exists x(A(x) \wedge x < t_0) \vee B \ ;$$

  hence, by induction hypothesis, $\exists x(A(x) \wedge x < t_0) \vee B$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Since B is not $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, this means that there exists at least a closed term t_1 such that both $A(t_1)$ and $t_1 < t_0$ are $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Now, we can iterate the previous reasoning constructing the proof $\pi_2[t_1/y]$ of the sequent

$$\exists x(A(x) \wedge x < t_1) \vee B \ ,$$

and finding a new closed term t_2 such that both $A(t_2)$ and $t_2 < t_1$ are $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, and so on. In this way we can find an infinite sequence $t_0, t_1, \ldots, t_n, \ldots$ of closed terms of $\mathcal{L}_{\mathcal{A}}$ such that

$$t_1 < t_0 \ , \ t_1 < t_2 \ , \ldots , \ t_{n+1} < t_n \ , \ldots$$

are $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. This implies that all these formulas are provable in $\mathbb{D}_{\mathbf{HA}}([\Pi])$ and hence, by Proposition 2.3.8, they are provable in $\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$. But, since the standard structure of the natural numbers $\mathcal{N}$ is a model of $\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$, this implies that in $\mathcal{N}$ the relation $<$ gives rise to an infinite descending chain; a contradiction. Hence, B must be $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$.

This concludes the proof. □

The proof of the following corollary coincides with the proof of Corollary 4.2.7.

4.4.3 Corollary *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$. *Then* $\mathbb{D}_{\mathbf{HA}}([\Pi])$ *satisfies* (Dp) *and* (Ed). □

We obtain, in the usual way, the strong constructiveness results for $\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$ and $\mathbf{T}_{\mathbf{HA}+\{(\mathsf{DCP})\}}$.

4.4.4 Theorem (SCR-$\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$) $\mathcal{ND}_{\mathbf{HA}+\mathsf{DCP}}$ *is a strongly constructive calculus w.r.t.* (Dp) *and* (Ed). □

4.4.5 Corollary (SCR-$\mathbf{T}_{\mathbf{HA}+\{(\mathsf{DCP})\}}$) $\mathbf{T}_{\mathbf{HA}+\{(\mathsf{DCP})\}}$ *is a strongly constructive formal system w.r.t.* (Dp) *and* (Ed). □

Strong constructiveness can be proved also for more general theories containing the descending chain principle. As an example let us consider a signature $\mathcal{A}$ containing the binary relation $<$, and let

$$\mathcal{H}_{\mathbf{Hr}+\mathbf{ID}+\mathbf{PO}+\{(\mathsf{Kur})\}+\{(\mathsf{DCP})\}} = \mathcal{H}_{\mathbf{INT}} + \mathbf{Hr} + \mathbf{ID} + \mathbf{PO} + (\mathsf{Kur}) + (\mathsf{DCP})$$

be the Hilbert-style calculus obtained by adding to intuitionistic first order logic the axioms for identity, for strict partial orders, that is, axioms characterizing $<$ as in irreflexive and transitive relation:

$$\forall x(\neg x < x)$$
$$\forall x \forall y \forall z(x < y \land y < z \Rightarrow x < z) \ ,$$

the Kuroda principle (see Section 3.3), the descending chain principle and an Harrop theory over the signature $\mathcal{A}$. Moreover, let us suppose that the formal system

$$\mathbf{T}_{\mathbf{Hr}+\mathbf{ID}+\mathbf{PO}+\{(\mathsf{Kur})\}+\{(\mathsf{DCP})\}}$$

generated by the Hilbert-style calculus $\mathcal{H}_{\mathbf{Hr}+\mathbf{ID}+\mathbf{PO}+\{(\mathsf{Kur})\}+\{(\mathsf{DCP})\}}$ satisfies the following additional conditions:

H1 : There exists an $\mathcal{A}$-structure $\mathcal{M} = \langle \mathbf{D}, \mathbf{i} \rangle$ such that

$$\mathcal{M} \models \mathbf{T}_{\mathbf{Hr}+\mathbf{ID}+\mathbf{PO}+\{(\mathsf{Kur})\}+\{(\mathsf{DCP})\}}$$

that is $\mathcal{M}$ satisfies any theorem of the formal system;

H2 : The relation $<^{\mathcal{M}}$ (that is, the interpretation in the structure $\mathcal{M}$ of the relation symbol $<$) is *well founded*, that is any descending chain

$$\ldots e_n <^{\mathcal{M}} e_{n-1} <^{\mathcal{M}} \ldots <^{\mathcal{M}} e_1 <^{\mathcal{M}} e_0$$

is finite.

A pseudo-natural deduction presentation of this formal system can be obtained by adding to $\mathcal{ND}_{\mathbf{INT}}$ an axiom rule for any formula in **Hr**, the rules Kur and DCP, the rules id_1 and id_2 and the rules for irreflexivity and transitivity here stated:

Rules for Partial Orders:

$$\frac{\Gamma \vdash t < t}{\Gamma \vdash \bot} <_1 \qquad \frac{\Gamma \vdash x < y \qquad \Delta \vdash y < z}{\Gamma, \Delta \vdash x < z} <_2$$

It is easy to verify, using the closure properties of $\mathcal{R}_{\mathbf{Hr}}$ defined in Section 3.7, that this is a strongly constructive calculus for $\mathbf{T}_{\mathbf{Hr}+\mathbf{ID}+\mathbf{PO}+\{(\mathsf{Kur})\}+\{(\mathsf{DCP})\}}$.

We conclude this section by recalling that the descending chain principle allows to classically derive, in the frame of the theory of strict partial orders, the *transfinite induction principle*

$$(\mathsf{TIND}) \qquad \forall x(\forall y(y < x \Rightarrow A(y)) \Rightarrow A(x)) \Rightarrow \forall z A(z)$$

that is

$$\vdash_{\mathcal{H}_{\mathbf{CL}}+\mathbf{PO}+\{(\mathsf{DCP})\}} \mathsf{TIND}\ .$$

In a theory formalizing $<$ as a strict linear order, that is, satisfying the axiom

$$\forall x \forall y (\neg x{=}y \Rightarrow x < y \vee y < x)\ ,$$

the axiom (TIND) is the usual *transfinite induction principle* (which, in the frame of Peano Arithmetic is classically implied by the usual induction principle (Ind)). We remark that, in a constructive frame (e.g. in an intuitionistic frame) (DCP) and (Ind) in general are independent principles.

4.5 Markov principle

Finally, to conclude our presentation of examples of strongly constructive system with respect to (DP) and (ED), we present the well known *Markov Principle*. Detailed discussions about the relevance of this principle in the area of constructivism and for program synthesis can be found in [Troelstra, 1973a, Miglioli and Ornaghi, 1981, Voronkov, 1987]. Markov principle is:

$$(\mathsf{Mk}) \qquad \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x A(x) \Rightarrow \exists x A(x)$$

Its formulation as a pseudo-natural deduction rule can be given as follows:

Markov Rule :

$$\frac{\Gamma, \neg\neg\exists x A(z) \vdash \forall x(A(x) \vee \neg A(x))}{\Gamma, \neg\neg\exists x A(x) \vdash \exists x A(x)}\ \mathsf{Mk}$$

As we will see, the proof of strong constructiveness for theories including the Markov Principle will rely, as in the case of the Descending Chain Principle, on some semantical property of the class of the classical models of the theory. Namely, the proof of strong constructiveness of theories including the Markov Principle will rely on the existence of at least a reachable model for the theory. Here, we study in full details the case of Markov Arithmetic enlarged with the Kuroda Principle; a reachable model for such a theory is the standard structure of Natural Numbers.

Let $\mathbf{T}_{\mathbf{HA-Mk-Kur}}$ be the formal system generated by the Hilbert system

$$\mathcal{H}_{\mathbf{HA-Mk-Kur}} = \mathcal{H}_{\mathbf{HA}} + \{(\mathsf{Kur}), (\mathsf{Mk})\}$$

where (Kur) is the Kuroda Principle studied in Section 3.3.

Now, let $\mathcal{ND}_{\mathbf{HA-Mk-Kur}}$ be the pseudo-natural deduction calculus obtained by adding to the calculus $\mathcal{ND}_{\mathbf{HA}}$ the rules Mk and Kur. It is easy to verify that $\mathcal{ND}_{\mathbf{HA-Mk-Kur}}$ is a presentation for the formal system $\mathbf{T}_{\mathbf{HA-Mk-Kur}}$.

To prove the strong constructiveness of the formal system $\mathbf{T}_{\mathbf{HA-Mk-Kur}}$, we use the generalized rule $\mathcal{R}_{\mathbf{HA}}$ and the calculus $\mathbb{D}_{\mathbf{HA}}([\Pi])$ defined in Section 4.2. It is trivial to verify:

4.5.1 Proposition *$\mathcal{ND}_{\mathbf{HA-Mk-Kur}}$ is uniformly $\mathcal{R}_{\mathbf{HA}}$-closed.* □

4.5.2 Lemma *Let Π be a set of proofs of $\mathcal{ND}_{\mathbf{HA-Mk-Kur}}$. For any proof $\pi : \Gamma \vdash A \in \textsc{Subst}^*(\mathrm{SubPr}(\Pi))$, if Γ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, then A is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$.*

Proof: Since $\mathcal{R}_{\mathbf{HA}}$ contains Cut and Subst, the proof of Point (i) of Definition 4.1.1 immediately follows from Lemma 3.2.2. To prove Point (ii) we proceed by induction on depth(π).

Basis: If depth(π) = 0, then either the only rule applied in π is an assumption introduction, or it is one of the zero-premises rules id_1, $+_1$, $+_2$, $*_1$ e $*_2$. In the first case the proof is trivial, in the second case A is an atomic formula.

Step: Now, let us assume that the assertion holds for any proof $\pi' : \Gamma' \vdash A'$ belonging to $\textsc{Subst}^*(\mathrm{SubPr}(\Pi))$ such that depth(π') $\leq h$, and let depth(π) = h+1. We proceed by cases according to the last rule applied in the proof π. The proof for the rules of $\mathcal{ND}_{\mathbf{HA}}$ coincides with the one given in Lemmas 3.2.4 and the case of the rule Kur coincides with the one given in 3.3.2. Now we analyze the rule Mk.

- Mk **rule**:

$$\frac{\pi_1 : \Gamma, \neg\neg\exists x A(x) \vdash \forall x(A(x) \vee \neg A(x))}{\Gamma, \neg\neg\exists x A(x) \vdash \exists x A(x)}\ \mathsf{Mk}$$

We must show that, if $\Gamma, \neg\neg\exists x A(x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, then $\exists x A(x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, that is, for any closed instance $\theta\exists x A(x) \equiv \exists x A'(x)$, there exists at least a term t such that $A(t/x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. By induction hypothesis on the proof π_1, we have that

$$\forall x(A(x) \vee \neg A(x))$$

is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Hence, by definition, for any closed instance

$$\theta\forall x(A(x) \vee \neg A(x)) \equiv \forall x(A'(x) \vee \neg A'(x))$$

of $\forall x(A(x) \vee \neg A(x))$ and for any term t,

$$(A'(t/x) \vee \neg A'(t/x))$$

is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Now, let us suppose that, for any term t, $\neg A'(t/x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. This implies that, for any term t, there exists a proof

$$\tau_t : \neg A'(t/x) \in \mathbb{D}_{\mathbf{HA}}([\Pi]) \ .$$

It is easy to verify that this implies that, for any term t,

$$\vdash_{\mathcal{H}_{\mathbf{PA}}} \neg A'(t/x) \ . \tag{4.1}$$

But, since $\neg\neg\exists x A(x)$ is also $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$, we can analogously deduce that

$$\vdash_{\mathcal{H}_{\mathbf{PA}}} \neg\neg\exists x A'(x)$$

and hence, by the fact that classical negation is idempotent,

$$\vdash_{\mathcal{H}_{\mathbf{PA}}} \exists x A'(x) \ . \tag{4.2}$$

Now, since the standard structure of natural numbers $\mathcal{N}$ is reachable in the language of the theory, by Points (4.1) and (4.2) we get:

$$\mathcal{N} \models \neg A(t/x) \text{ for any term } t \text{ of } \mathcal{L}_{\mathcal{A}} \tag{4.3}$$

$$\mathcal{N} \models \exists x A(x) \ . \tag{4.4}$$

But (4.3) and (4.4) are clearly contradictory, hence, it is not possible that, for any term t, $\neg A'(t/x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Indeed, there exists a term t such that $A'(t/x)$ is $\mathcal{L}_{\mathcal{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}([\Pi])$. Since $A'(x)$ is a generic closed instance of $A(x)$, we have that this holds for any closed instance of $A(x)$. This proves the assertion.

□

4.5.3 Corollary *Let* Π *be any set of proofs of* $\mathcal{ND}_{\mathbf{HA-Mk-Kur}}$. *For every proof* $\tau : \Gamma \vdash A \in \mathbb{D}_{\mathbf{HA}}([\Pi])$ *and every substitution* θ, *if* $\theta\Gamma$ *is* $\mathcal{L}_{\mathcal{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{HA}}([\Pi])$, *then* θA *is* $\mathcal{L}_{\mathcal{A}}$*-evaluated in* $\mathbb{D}_{\mathbf{HA}}([\Pi])$. □

Now, we obtain, in the usual way, the strong constructiveness results for the calculus $\mathcal{ND}_{\mathbf{HA-Mk-Kur}}$ and for the formal system $\mathbf{T}_{\mathbf{HA-Mk-Kur}}$.

4.5.4 Theorem (SCR-$\mathcal{ND}_{\mathbf{HA-Mk-Kur}}$) $\mathcal{ND}_{\mathbf{HA-Mk-Kur}}$ *is a strongly constructive calculus w.r.t.* (DP) *and* (ED). □

4.5.5 Corollary (SCR-$\mathbf{T}_{\mathbf{HA-Mk-Kur}}$) $\mathbf{T}_{\mathbf{HA-Mk-Kur}}$ *is a strongly constructive formal system w.r.t.* (DP) *and* (ED). □

We can prove strong constructiveness also for other systems including the Markov Principle, however also these systems must admit at least a reachable model. For example, any formal system $\mathbf{T}_{\mathsf{Mk}}$ generated by an Hilbert-style calculus $\mathcal{H}_{\mathsf{Mk}}$ satisfying the following conditions is strongly constructive:

H1 : $\mathcal{H}_{\mathsf{Mk}} = \mathcal{H}_{\mathbf{INT}} + \mathbf{HA} + \{(\mathsf{Kur})\} + \{(\mathsf{Mk})\}$
where **Hr** is an Harrop-theory over the language $\mathcal{L}_{\mathcal{A}}$ and (Kur) is the Kuroda principle studied in Section 3.3.

H2 : The formal system $\mathcal{H}_{\mathbf{CL}} + \mathbf{Hr}$ has at least a reachable (classical) model.

Since the pseudo-natural deduction calculus $\mathcal{ND}_{\mathsf{Mk}}$ obtained by adding the rules Mk and Kur and the zero-premises rule

$$\frac{}{\vdash A}\,\text{Ax}\ ,$$

for any $A \in \mathbf{Hr}$, to the natural deduction calculus $\mathcal{ND}_{\mathbf{INT}}$, it is easy to prove, using the generalized rule $\mathcal{R}_{\mathbf{Hr}}$ and the abstract calculus $\mathbb{D}_{\mathbf{Hr}}(\Pi)$, that $\mathcal{ND}_{\mathsf{Mk}}$ is a strongly constructive calculus w.r.t. (DP$_{\text{open}}$) and (ED$_{\text{open}}$) and that $\mathbf{T}_{\mathsf{Mk}}$ is a strongly constructive formal system w.r.t. (DP$_{\text{open}}$) and (ED$_{\text{open}}$).

4.5.6 Remark In the proof of the above Lemma 4.5.2 (as well as in the proof of Lemma 4.4.2), we have combined proof theoretic and model theoretic arguments to get a typical proof theoretic result such as a strong constructivity theorem. This is quite in line with our perspective, oriented to single out formal systems of potential interest for Computer Science *without disregarding any combination of tools which can help in reaching this goal*. Of course, in literature is a lot of examples of formal systems proved to be constructive using semantical tools (e.g. Kripke models). But in those examples generally only simple constructivity is involved, not a more sophisticated proof theoretic notion such as our strong constructivity. On the other hand, a very sophisticated constructive Proof Theory has been developed more or less in the frame of what we call, in the introduction to the present Thesis, the extended intuitionistic tradition. Such a Proof Theory conforms to more orthodox paradigms that our proof theory: one of the main paradigms is to *analyze proof theoretic matters* (e.g. Normalization Theorems, Constructivity Proofs, etc.) *using*, so to say, *proof theoretic tools*; hardly a notion such as the one of classical model can be included in such tools.

This completes our presentation of strongly constructive systems with respect to (DP) and (ED). Like we have made for strong constructiveness with respect to (DP_{open}) and (ED_{open}), the choice of the examples has been mainly inspired to the attempt of giving clear illustrations. Other, much more complex (and perhaps, more interesting) examples could have been chosen. For instance, we can extend Heyting Arithmetic with any set of Harrop axioms together with Descending Chain Principles and both Kreisel-Putnam Principles, yet getting a strongly constructive formal system; we can ever take, in this context, principles stronger than the Kreisel-Putnam ones. Likewise, we can extend Markov Arithmetic (obtained by adding Markov Principle to $\mathcal{H}_{\mathbf{HA}}$) with Harrop axioms, Descending Chain Principles and axiom-schemes for intermediate predicate logics stronger than Scott Principle, yet getting strongly constructive systems.

We remark however, that in the frame of strong constructiveness with respect to (DP) and (ED) (where (DP_{open}) and (ED_{open}) cannot be obtained mainly by the presence of Induction and Descending Chain Principles), *constructive incompatibility* plays a role stronger than in the frame of strong constructiveness with respect to (DP_{open}) and (ED_{open}). For instance, any formal system including intuitionistic logic and Grzegorczyck Principle allows to prove Markov Principle; thus, since there are formal systems including intuitionistic logic, Grzegorczyck Principle and the Kreisel-Putnam Principles which are constructive with respect to (DP_{open}) and (ED_{open}) (see the discussion at the end of Chapter 3) there are also systems which are strongly constructive with respect to (DP_{open}) and (ED_{open}) and include intuitionistic logic, Markov Principle and the Kreisel-Putnam Principles. On the other hand, the simultaneous addition of Kreisel-Putnam Principles and Markov Principle to Heyting Arithmetic cannot give rise to a strongly constructive system with respect to (DP) and (ED) by the simple fact that such a system does not ever satisfy (DP) and (ED) (that is it is not simply constructive): as a matter of fact, it collapse to the classical Peano Arithmetic (see e.g. [Troelstra, 1977]).

To give another example, the addition of Grzegorczyck Principle alone to Heyting Arithmetic give rise, in turn, to the classical Peano Arithmetic; but (Grz) is interesting from the point of view of strong constructiveness, since, as we have seen, its addition to intuitionistic predicate calculus (with the possible addition of Harrop axioms, which may be very interesting axiomatizations of mathematical theories) provides a strongly constructive system.

The above remarks suggest that a good development of the theory of strongly constructive formal systems should be made in connection with an extensive analysis of constructive incompatibility with respect to (DP) and (ED) on the one hand, and with respect to (DP_{open}) and (ED_{open}) on the other hand: the first kind of incompatibility mainly turns out to be simultaneous incompatibility of strong mathematical principles (such as induction) and various strong logical extra-intuitionistic axiom-schemes; the second kind of incompatibility mainly concerns the coexistence of various logical extra-intuitionistic principles.

Note that the treatment of the present section has taken into account only systems which are classically consistent, that is, their theories can be satisfied on some classical model. On the other hand, our notion of strong constructiveness can be applied as well to consistent, even if classically inconsistent, formal systems; e.g., we can prove the strong constructiveness of consistent but not classically consistent formal systems obtained by adding to Heyting Arithmetic suitable instances of a principle known as Church's Thesis (see e.g. [Troelstra, 1973a, Troelstra, 1977, Troelstra and van Dalen, 1988a]). We are analyzing from the point of view of strong constructiveness the whole system involving Church's Thesis, that is the system obtained by adding to Heyting Arithmetic all the instances of Church's Thesis.

Chapter 5

A constructive but not strongly constructive formal system

5.1 Basic recursion theory

The aim of this section is to make this thesis self contained. The material here presented can be found in any good text of recursion theory; we essentially follow [Kleene, 1952, Girard, 1987, Odifreddi, 1989]. Hereafter, we will use symbols as $f^{(n)}, g^{(n)}, h^{(n)}$, possibly with indexes, to denote n-ary number-theoretic functions, that is functions from $\mathbf{N}^n$ into $\mathbf{N}$, and symbols as $p^{(n)}, q^{(n)}$, possibly with indexes, to denote n-ary number-theoretic relations, that is subsets of $\mathbf{N}^n$. We will avoid to write the arity any time it is clear from the context. Finally, we will use the notation $p \leftrightarrow q$ to mean that p holds iff q holds.

5.1.1 Definition *A function $f : \mathbf{N}^k \to \mathbf{N}$ ($k \geq 0$) is* recursive *iff it is obtained by means of the following schemes:*

(R1) - *For any $n, i \in \mathbf{N}$ such that $1 \leq i \leq n$, the projection function $\mathsf{I}_i^{(n)}(x_1, \ldots, x_n)$ such that $\mathsf{I}_i^{(n)}(x_1, \ldots, x_n) = x_i$, is recursive;*
- *The binary function $+$ (sum) is recursive;*
- *The binary function $*$ (product) is recursive;*
- *The binary function $\chi_<$ (the characteristic function of the binary relation $<$) is recursive.*

(R2) *If g is an m-ary recursive function, and $h_1, \ldots, h_m$ are n-ary recursive functions, then the function*

$$f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n))$$

is recursive.

(R3) *If g is an $(n+1)$-ary recursive function and for all $a_1, \dots, a_n \in \mathbf{N}$ there exists $b \in \mathbf{N}$ such that $g(a_1, \dots, a_n, b) = 0$, then the function*

$$f(x_1, \dots, x_n) = \mu y(g(x_1, \dots, x_n, y) = 0)$$

is recursive, where $\mu y g(x_1, \dots, x_n, y) = 0$ is the smallest y such that $g(x_1, \dots, x_n, y) = 0$ holds.

The functions of (R1) are called *initial functions*, the scheme (R2) is called *composition scheme*, and (R3) is called *minimalization scheme*.

It is a well known result that schemes for *constant functions* and the scheme of *primitive recursion* give rise to recursive functions:

5.1.2 Proposition

(R4) *Constant functions are recursive;*

(R5) *Let g be an n-ary recursive function and let h be a $(n+2)$-ary recursive function. Then the $(n+1)$-ary function f defined by means of the following scheme is recursive.*

$$\begin{cases} f(x_1, \dots, x_n, 0) = g(x_1, \dots, x_n) \\ f(x_1, \dots, x_n, k+1) = h(x_1, \dots, x_n, k, f(x_1, \dots, x_n, k)) \end{cases}$$

□

A relevant subclass of the one of recursive functions is the class of the primitive recursive functions.

5.1.3 Definition *A function $f : \mathbf{N}^k \to \mathbf{N}$ $(k \geq 0)$ is* primitive recursive *if it is obtained by means of the schemes (R1), (R2), (R4) and (R5).*

An n-ary relation $p(x_1, \dots, x_n)$ is recursive or primitive recursive if its characteristic function

$$\chi_p(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } p(x_1, \dots, x_n) \text{ is true} \\ 1 & \text{otherwise} \end{cases}$$

is recursive or primitive recursive respectively.

The central role played by the notion of recursive function in mathematical logic and computer science comes from the well known *Church's Thesis*, asserting that *every computable function is recursive.* However, how it is well known, the class of recursive functions fails to have nice "algebraic closure properties". Namely, the set of the recursive functions cannot be enumerated by a recursive function. This is a well known consequence of the fact that (R3) asks for an infinitary condition. Hence, given an arbitrary integer e we cannot effectively decide if it is the "index" of a sequence of applications of (R1), (R2) and (R3) which give rise to a recursive function.

The lack of this property is the main reason why one needs to introduce the concept of *partial recursive function*. A partial recursive function is simply a function that may be undefined for some (possibly all) arguments; the set of the arguments for which it is defined is called its *domain*. In this sense a recursive function is a total partial recursive function, that is a partial recursive function whose domain coincides with $\mathbf{N}$. To define partial recursive functions we need an "extended equality" relation, we denote with $\simeq$. Namely, if t and u are expressions involving partial recursive functions, $t \simeq u$ means that either both t and u are defined, and in that case $t = u$, or t and u are both undefined.

5.1.4 Definition *A function $f : \mathbf{N}^k \to \mathbf{N}$ ($k \geq 0$) is* partial recursive *iff it is obtained by means of the following schemes:*

(R1') - *For any $n, i \in \mathbf{N}$ such that $1 \leq i \leq n$, the projection function $\mathsf{I}_i^{(n)}(x_1, \ldots, x_n)$ such that $\mathsf{I}_i^{(n)}(x_1, \ldots, x_n) \simeq x_i$, is partial recursive;*
- *The binary function $+$ (sum) is partial recursive;*
- *The binary function $*$ (product) is partial recursive;*
- *The binary function $\chi_<$ (the characteristic function of the binary relation $<$) is partial recursive.*

(R2') *If g is an m-ary partial recursive function, and $h_1, \ldots, h_m$ are n-ary partial recursive functions, then the function*

$$f(x_1, \ldots, x_n) \simeq g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n))$$

is partial recursive.

(R3') *If g is an $(n+1)$-ary partial recursive function, then the function*

$$f(x_1, \ldots, x_n) \simeq \mu y(g(x_1, \ldots, x_n, y) = 0)$$

is partial recursive, where: $\mu y(g(x_1, \ldots, x_n, y) = 0)$ is the smallest y such that $g(x_1, \ldots, x_n, 0), \ldots, g(x_1, \ldots, x_n, y)$ are defined and $g(x_1, \ldots, x_n, y) \simeq 0$ if such an y exists; $\mu y(g(x_1, \ldots, x_n, y) = 0)$ is undefined otherwise.

An *explicit definition* $\mathrm{D}_{f^{(n)}}$ (see [Kleene, 1952]) of a partial recursive function is a formal derivation which specifies the partial recursive function $f^{(n)}$, starting from the initial functions, by means of the composition scheme and the minimalization scheme. Namely, $\mathrm{D}_{f^{(n)}}$ is a sequence $g_1, \ldots, g_k$ ($k \geq 1$) of occurrences of functions such that $g_k = f^{(n)}$ and each function g_i of the sequence is either an initial function or is obtained by applying the composition scheme or the minimalization scheme to preceding functions of the sequence.

We remark that the main improvement of the definition of partial recursive functions with respect to Definition 5.1.1 is that in (R3') one does not ask for an infinitary condition. The practical consequence is that the partial recursive functions can be enumerated by a function of the same class (see the Enumeration Theorem 5.1.8 below).

The corresponding notion for relations is the one of recursively enumerable relation.

5.1.5 Definition *An n-ary relation is* recursively enumerable *(*r.e. *for short) if it is the domain of an n-ary partial recursive function.*

A result which will be heavily used in Section 5.3 is the Normal Form Theorem proved by Kleene in 1938.

5.1.6 Theorem (Normal form theorem) *There exist an 1-ary primitive recursive function $\mathsf{U}(x)$ and, for any integer $n \geq 0$, an $(n+2)$-ary primitive recursive relation $\mathsf{T}_n(y, x_1, \ldots, x_n, w)$, with the following properties:*

(i). If f is an n-ary partial recursive function, then there exists an integer e (called an index *of f), such that, for all $a_1, \ldots, a_n$:*

$$f(a_1, \ldots, a_n) \simeq \mathsf{U}(\mu w \mathsf{T}_n(e, a_1, \ldots, a_n, w))$$

(ii). If p is an n-ary recursively enumerable relation, then there exists an integer e (an index *for p) such that, for all $a_1, \ldots, a_n$:*

$$p(a_1, \ldots, a_n) \leftrightarrow \exists w \mathsf{T}_n(e, a_1, \ldots, a_n, w) \ .$$

□

The Normal Form Theorem asserts that every partial recursive function (every r.e. relation) has an index. Of course, this result still holds for recursive and primitive recursive functions. The advantage given by the introduction of the partial recursive functions is that the converse holds as well, and then the partial recursive functions can be enumerated by a function of the same class. That is, fixed an integer n, we have that any integer e defines a n-ary partial recursive function, and an n-ary relation. To state this result formally, we need the following definition.

5.1.7 Definition *For any positive integer n, we will denote with $f_e^{(n)}$ the n-ary partial recursive function of index e, that is the partial recursive function $f_e^{(n)}$ such that, for any $a_1, \ldots, a_n \in \mathbf{N}$:*

$$f_e^{(n)}(a_1, \ldots, a_n) \simeq \mathsf{U}(\mu w \mathsf{T}_n(e, a_1, \ldots, a_n, w)) \ .$$

Now, we can state the symmetric version of the Normal Form Theorem.

5.1.8 Theorem (Enumeration Theorem) *For any given integer n, the sequence $\{f_e^{(n)}\}_{e \in \omega}$ is a partial recursive enumeration of the n-ary partial recursive functions. That is, the following properties hold:*

(i). *For any* $e \in \mathbf{N}$, $f_e^{(n)}$ *is an* n*-ary partial recursive function;*

(ii). *If* g *is an* n*-ary partial recursive function, then there exists an index* $e \in \mathbf{N}$ *such that* $g \simeq f_e^{(n)}$;

(iii). *There exists an* $(n+1)$*-ary partial recursive function* h *such that, for any* $e \in \mathbf{N}$, $h(e, x_1, \ldots, x_n) \simeq f_e^{(n)}(x_1, \ldots, x_n)$.

□

Hereafter, we will denote with

$$\mathcal{E}^{(n)} = \{f_e^{(n)}\}_{e\in\omega} \tag{5.1}$$

a fixed enumeration of partial recursive functions, whose existence is guaranteed by the previous theorem.

As it is well known, for primitive recursive functions a weaker version of the Enumeration Theorem holds. Namely, there exists a recursive function enumerating all the primitive recursive functions, but this function is not primitive recursive. (For unary primitive recursive functions it is the well known *Ackermann* function.) Hereafter, we will denote with

$$\mathcal{E}^{(n)}_{\text{prim}} = \{f_e^{(n)}\}_{e\in\omega} \tag{5.2}$$

a fixed enumeration of the n-ary primitive recursive functions.

A weaker version of the Normal Form Theorem is the *Projection Theorem* (also called *existential quantifier theorem*), see e.g. [Rogers, 1967].

5.1.9 Theorem (Projection Theorem) *A relation* $p(x_1, \ldots, x_n)$ *is r.e. iff there exists a primitive recursive relation* $q(x_1, \ldots, x_n, y)$ *such that, for any* $a_1, \ldots, a_n$*:*

$$p(a_1, \ldots, a_n) \leftrightarrow \exists y q(a_1, \ldots, a_n, y) \ .$$

□

Another important theorem of recursion theory is the so called S^m_n-theorem or parameterization theorem.

5.1.10 Theorem (S^m_n-Theorem) *Given integers* m, n*, there exists a primitive recursive, one-to-one function* $\mathsf{S}^m_n(e, x_1, \ldots, x_n)$ *such that:*

$$f_{\mathsf{S}^m_n(e,x_1,\ldots,x_n)}(y_1, \ldots, y_m) \simeq f_e(x_1, \ldots, x_n, y_1, \ldots, y_m) \ .$$

□

Finally, we present a variant of the Recursion Theorem, known as Fixed-Point Theorem.

5.1.11 Theorem (Fixed-Point Theorem) *For any positive integer n, given an 1-ary recursive function $g^{(n)}$, there exists an index e such that*

$$f_e^{(n)} \simeq f_{g(e)}^{(n)}$$

□

5.2 Notions of representability

Now, let $\mathcal{L}_{\mathbf{A}}$ be the language of (first) order arithmetic, that is the language whose extra-logical alphabet consists of the constant symbol $\mathsf{0}$, the unary function symbol s, the two binary function symbols $+$ and $*$ and the binary relation symbol $=$. With **HA** we denote the the formal system of *Heyting arithmetic* (or *intuitionistic arithmetic*) generated by the Hilbert-style calculus $\mathcal{H}_{\mathbf{HA}}$ obtained by adding to $\mathcal{H}_{\mathbf{INT}}$ axioms (i)-(x) below. On the other hand, with **PA** we denote the formal system of *Peano arithmetic* (or *classical arithmetic*) generated by the Hilbert-style calculus $\mathcal{H}_{\mathbf{PA}}$ obtained by adding to $\mathcal{H}_{\mathbf{CL}}$ axioms (i)-(xii) below.

Equality axioms:

(i). $x{=}x$

(ii). $x{=}y \wedge y{=}z \Rightarrow x{=}z$

(iii). $x = y \Rightarrow \mathsf{s}x = \mathsf{s}y$

(iv). $x{=}x' \wedge y{=}y' \Rightarrow (x{+}y{=}x'{+}y')$

(v). $x{=}x' \wedge y{=}y' \Rightarrow (x{*}y{=}x'{*}y')$

Successor axioms:

(vi). $\mathsf{s}x = \mathsf{s}y \Rightarrow x = y$

(vii). $\mathsf{s}x = \mathsf{0} \Rightarrow \bot$

Sum axioms:

(viii). $x{+}\mathsf{0}{=}x$

(ix). $x{+}\mathsf{s}y{=}\mathsf{s}(x{+}y)$

Product axioms:

(x). $x{*}\mathsf{0}{=}\mathsf{0}$

(xi). $x{*}\mathsf{s}y{=}(x{*}y){+}x$

Induction axiom scheme:

(xii). $A(0) \wedge \forall x(A(x) \Rightarrow A(\mathbf{s}x)) \Rightarrow \forall y A(y)$

Henceforth, we will use the following notational conventions. First of all, given an Hilbert-style calculus we will write $\mathcal{H} \vdash A$ to mean $\vdash_{\mathcal{H}} A$. $a, b, \ldots$, possibly with indexes, will denote natural numbers, while $\widetilde{a}, \widetilde{b}, \ldots$, possibly with indexes, will denote the corresponding numerals. We recall that a *numeral* of $\mathcal{L}_{\mathbf{A}}$ is a term which canonically represents a natural number, according to the following convention: if $a = 0$ then $\widetilde{a}$ is the constant symbol 0 of $\mathcal{L}_{\mathbf{A}}$, and if $a = b + 1$ then $\widetilde{a}$ is $\mathbf{s}\widetilde{b}$.

Now, we state some properties of the calculus $\mathcal{H}_{\mathbf{HA}}$ we will use in the sequel.

5.2.1 Proposition *Let t be a closed term of $\mathcal{L}_{\mathbf{A}}$. Then there exists one and only one numeral $\widetilde{n}$ such that $\mathcal{H}_{\mathbf{HA}} \vdash t = \widetilde{n}$.* □

5.2.2 Proposition *If $\mathcal{H}_{\mathbf{HA}} \vdash t = t'$ and $\mathcal{H}_{\mathbf{HA}} \vdash A(t)$ then $\mathcal{H}_{\mathbf{HA}} \vdash A(t')$.* □

5.2.3 Proposition *$\mathcal{H}_{\mathbf{HA}}$ is a constructive calculus.* □

Now, we introduce the notions of representability we will use in the following sections.

5.2.4 Definition *Let $f(x_1, \ldots, x_n)$ be a number-theoretic function. A formula $\Psi_f(x_1, \ldots, x_n, z)$ of $\mathcal{L}_{\mathbf{A}}$* strongly numeralwise represents *$f(x_1, \ldots, x_n)$ in $\mathcal{H}_{\mathbf{HA}}$ iff:*

1. *For any $a_1, \ldots, a_n, b \in \mathbf{N}$, if $b = f(a_1, \ldots, a_n)$ then $\mathcal{H}_{\mathbf{HA}} \vdash \Psi_f(\widetilde{a}_1, \ldots, \widetilde{a}_n, \widetilde{b})$;*
2. *$\mathcal{H}_{\mathbf{HA}} \vdash \forall x_1 \ldots \forall x_n \exists! z \Psi_f(x_1, \ldots, x_n, z)$.*

A function f is *strongly numeralwise representable* in $\mathcal{H}_{\mathbf{HA}}$ if it is strongly numeralwise represented by some formula of $\mathcal{L}_{\mathbf{A}}$. The previous definition is stronger than the usual definition of numeralwise representable function (quoted e.g. in [Kleene, 1952]), which can be obtained considering Point (1) of Definition 5.2.4 and the following Point (2')

2'. For any $a_1, \ldots, a_n \in \mathbf{N}$, $\mathcal{H}_{\mathbf{HA}} \vdash \exists! z \Psi_f(\widetilde{a}_1, \ldots, \widetilde{a}_n, z)$.

An analogous definition can be given for relations (indeed, it is simply obtained by applying the previous definition to the characteristic function of a relation):

5.2.5 Definition *Let $p(x_1, \ldots, x_n)$ be a number-theoretic relation. A formula $\Psi_p(x_1, \ldots, x_n)$ of $\mathcal{L}_{\mathbf{A}}$* strongly numeralwise expresses *$p(x_1, \ldots, x_n)$ in $\mathcal{H}_{\mathbf{HA}}$ iff:*

1. *For any $a_1, \ldots, a_n \in \mathbf{N}$:*
   (a) *If $p(a_1, \ldots, a_n)$ is true then $\mathcal{H}_{\mathbf{HA}} \vdash \Psi_p(\widetilde{a}_1, \ldots, \widetilde{a}_n)$;*
   (b) *If $p(a_1, \ldots, a_n)$ is false then $\mathcal{H}_{\mathbf{HA}} \vdash \neg\Psi_p(\widetilde{a}_1, \ldots, \widetilde{a}_n)$;*

2. $\mathcal{H}_{\mathbf{HA}} \vdash \forall x_1 \ldots \forall x_n(\Psi_p(x_1, \ldots, x_n) \vee \neg\Psi_p(x_1, \ldots, x_n))$.

A relation is *strongly numeralwise expressible* in $\mathcal{H}_{\mathbf{HA}}$ if it is strongly numeralwise expressed in $\mathcal{H}_{\mathbf{HA}}$ by some formula of $\mathcal{L}_{\mathbf{A}}$. The usual weaker definition of *numeralwise expressible relation* ([Kleene, 1952]) can be obtained by ignoring Point (3) of the previous definition.

Now, we recall that any primitive recursive function $f^{(n)}$ can be characterized by an *explicit definition* $\mathrm{D}^{\mathrm{Prim}}_{f^{(n)}}$ (see [Kleene, 1952]). This is a formal derivation (and an algorithm) which specifies the primitive recursive function $f^{(n)}$, starting from the initial functions, by means of the composition scheme and the primitive recursion scheme. Namely, $\mathrm{D}^{\mathrm{Prim}}_{f^{(n)}}$ is a sequence $g_1, \ldots, g_k$ ($k \geq 1$) of occurrences of functions such that $g_k = f^{(n)}$ and each function g_i of the sequence is either an initial function or is obtained by applying the composition scheme or the primitive recursion scheme to preceding functions of the sequence.

The important point about explicit definitions of primitive recursive functions, is that, starting from the explicit definition $\mathrm{D}^{\mathrm{Prim}}_{f^{(n)}}$ of $f^{(n)}$, we can build up *in an effective way* (by means of the Gödel's β-functions) a formula $\Psi_f(x_1, \ldots, x_n, y)$ of $\mathcal{L}_{\mathbf{A}}$ which strongly numeralwise represents the function $f^{(n)}(x_1, \ldots, x_n)$ in $\mathcal{H}_{\mathbf{HA}}$. (Such an effective correspondence between explicit definitions of recursive functions and formulas describing them, is used also in [Kleene, 1952] to show that every primitive recursive function is numeralwise representable in $\mathcal{H}_{\mathbf{PA}}$.) Here effectiveness means that there is a computable function associating, with any explicit definition $\mathrm{D}^{\mathrm{Prim}}_{f^{(n)}}$ of a primitive recursive function $f^{(n)}$, a formula $\Psi_{f^{(n)}}(x_1, \ldots, x_n, y)$ strongly representing it in $\mathcal{H}_{\mathbf{HA}}$.

To formally characterize this computable function as a full number-theoretic one, let us consider a standard arithmetization $\mathcal{G}_{\mathrm{Prim}}$ of the formalism for primitive recursive functions and a standard arithmetization $\mathcal{G}_{\mathcal{L}_{\mathbf{A}}}$ of the language of first-order arithmetic $\mathcal{L}_{\mathbf{A}}$. This allows us to associate, with any description $\mathrm{D}^{\mathrm{Prim}}_{f^{(n)}}$ and with any formula A, two Gödel numbers, we denote with $\ulcorner \mathrm{D}^{\mathrm{Prim}}_{f^{(n)}} \urcorner$ and $\ulcorner A \urcorner$ respectively. Then, according to the above discussion, we can assert that there exists a recursive function $\mathsf{D}^{\mathsf{prim}}_{\mathsf{fun}}$–$\mathsf{Wff}$ associating, with any natural number (considered as the Gödel number of an explicit definition $\mathrm{D}^{\mathrm{Prim}}_{f^{(n)}}$ of a primitive recursive function $f^{(n)}$) the Gödel number of a formula which strongly numeralwise expresses it in $\mathcal{H}_{\mathbf{HA}}$.

We summarize this fact in the following theorem:

5.2.6 Theorem *There is a recursive function (indeed a primitive recursive function)* $\mathsf{D}^{\mathsf{prim}}_{\mathsf{fun}}$–$\mathsf{Wff}(x)$ *satisfying the following property:*

- *For every explicit definition* $\mathrm{D}^{\mathrm{Prim}}_{f^{(n)}}$ *of a primitive recursive function* $f^{(n)}$*, if* $k = \ulcorner \mathrm{D}^{\mathrm{Prim}}_{f^{(n)}} \urcorner$ *is the Gödel number of it and* $h = \mathsf{D}^{\mathsf{prim}}_{\mathsf{fun}}$–$\mathsf{Wff}(k)$*, then* h *is the Gödel number of a formula* $\Psi_f(x_1, \ldots, x_n, y)$ *which strongly numeralwise expresses in* $\mathcal{H}_{\mathbf{HA}}$ *the function* $f^{(n)}$.

□

Now, we associate with every primitive recursive function $f^{(n)}$ the primitive recursive relation $p_f^{(n)}$ defined as follows:

$$p_f^{(n)}(x_1, \ldots, x_n) \leftrightarrow (f^{(n)}(x_1, \ldots, x_n) = 0) \ . \tag{5.3}$$

In particular, if $f(x_1, \ldots, x_n)$ is the characteristic function of a relation $p(x_1, \ldots, x_n)$, then it defines a relation according to the usual conventions, that is, $p_f(x_1, \ldots, x_n)$ is true iff $f(x_1, \ldots, x_n) = 0$ iff $p(x_1, \ldots, x_n)$ true. On the other hand, the present convention avoids the introduction of unnecessary distinctions between functions defining relations and the other ones, allowing us to consider the explicit definition $\mathrm{D}^{\mathrm{Prim}}_{f^{(n)}}$ of the function of $f^{(n)}$ as the explicit definition of a relation $p^{(n)}$.

Moreover, if $\Psi_f(x_1, \ldots, x_n, y)$ is the formula of $\mathcal{L}_{\mathbf{A}}$ which strongly numeralwise represents $f^{(n)}$ in $\mathcal{H}_{\mathbf{HA}}$, it is easy to verify that the formula

$$\Psi_{p_f}(x_1, \ldots, x_n) \equiv \Psi_f(x_1, \ldots, x_n, 0)$$

strongly numeralwise expresses the primitive recursive relation $p_f^{(n)}$ in $\mathcal{H}_{\mathbf{HA}}$.

By means of the correspondence (5.3), starting from the previously fixed enumeration of primitive recursive functions

$$\mathcal{E}^{(n)}_{\mathrm{prim}} = \{f^{(n)}_e\}_{e \in \omega}$$

defined in (5.2), we can build up the following enumeration of primitive recursive relations:

$$\mathcal{E}^{(n)}_{\mathrm{r-prim}} = \{p^{(n)}_e\}_{e \in \omega} = \{p^{(n)}_{f_e}\}_{e \in \omega} \ . \tag{5.4}$$

The above characterization of the primitive recursive relations and Theorem 5.2.6 immediately yield:

5.2.7 Theorem *There is a recursive function (indeed a primitive recursive function)* $\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Wff}(x)$ *satisfying the following property:*

- *For every explicit definition* $\mathrm{D}^{\mathrm{Prim}}_{p^{(n)}}$ *of a primitive recursive relation* $p^{(n)}$*, if* $k = \ulcorner \mathrm{D}^{\mathrm{Prim}}_{p^{(n)}} \urcorner$ *is the Gödel number of it and* $h = \mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Wff}(k)$*, then* h *is the Gödel number of a formula* $\Psi_p(x_1, \ldots, x_n)$ *which strongly numeralwise expresses in* $\mathcal{H}_{\mathbf{HA}}$ *the relation* $p^{(n)}$*.*

□

Now, let us consider the relation $\mathsf{T}_n(y, x_1, \ldots, x_n, w)$ (for any $n > 0$) defined in the Normal Form Theorem 5.1.6. Since $\mathsf{T}_n(y, x_1, \ldots, x_n, w)$ is a primitive recursive relation, it is strongly numeralwise expressible in $\mathcal{H}_{\mathbf{HA}}$ according to the previous Theorem. Similarly, the function $\mathsf{U}(x)$ defined in the Normal Form Theorem 5.1.6 is primitive recursive, and hence it is strongly numeralwise representable in $\mathcal{H}_{\mathbf{HA}}$ according to Theorem 5.2.6. These facts justify the following definition:

5.2.8 Definition *For any positive integer n, let us fix an explicit definition of $\mathsf{T}_n(y, x_1, \ldots, x_n, w)$ and let h be its Gödel number in the fixed arithmetization $\mathcal{G}_{Prim}$. We denote with*

$$T_n(y, x_1, \ldots, x_n, w)$$

the formula of $\mathcal{L}_\mathbf{A}$ whose Gödel number is $\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Wff}(h)$, which strongly numeralwise expresses $\mathsf{T}_n(y, x_1, \ldots, x_n, w)$ in $\mathcal{H}_\mathbf{HA}$. Moreover, let us fix an explicit definition of $\mathsf{U}(x)$ and let k be the its Gödel number in the fixed arithmetization $\mathcal{G}_{Prim}$. We denote with

$$U(x, y)$$

the formula of $\mathcal{L}_\mathbf{A}$ whose Gödel number is $\mathsf{D}^{\mathsf{prim}}_{\mathsf{fun}}\text{–}\mathsf{Wff}(k)$, which strongly numeralwise represent $\mathsf{U}(x)$ in $\mathcal{H}_\mathbf{HA}$.

Finally, we introduce the representability notions relative to partial recursive functions.

5.2.9 Definition *Let $f(x_1, \ldots, x_n)$ be a number-theoretic function. A formula $\Psi_f(x_1, \ldots, x_n, z)$ of $\mathcal{L}_\mathbf{A}$* exhaustively numeralwise represents *$f(x_1, \ldots, x_n)$ in $\mathcal{H}_\mathbf{HA}$, iff:*

1. *For any $a_1, \ldots, a_n \in \mathrm{dom}(f)$,*

$$\mathcal{H}_\mathbf{HA} \vdash \exists! z \Psi_f(\widetilde{a}_1, \ldots, \widetilde{a}_n, z) \ ;$$

2. *For any $a_1, \ldots, a_n \in \mathrm{dom}(f)$, if $b = f(a_1, \ldots, a_n)$ then*

$$\mathcal{H}_\mathbf{HA} \vdash \Psi_f(\widetilde{a}_1, \ldots, \widetilde{a}_n, \widetilde{b}) \ .$$

The related notion for r.e. relations is the following one:

5.2.10 Definition *Let $p(x_1, \ldots, x_n)$ be a number-theoretic relation. A formula $\Psi_p(x_1, \ldots, x_n)$ of $\mathcal{L}_\mathbf{A}$* positively numeralwise expresses *$p(x_1, \ldots, x_n)$ in $\mathcal{H}_\mathbf{HA}$ iff, for any $a_1, \ldots, a_n \in \mathbf{N}$, $p(a_1, \ldots, a_n)$ is true iff $\mathcal{H}_\mathbf{HA} \vdash \Psi_p(\widetilde{a}_1, \ldots, \widetilde{a}_n)$.*

5.2.11 Theorem *Let $q(x_1, \ldots, x_n, y)$ be a relation strongly numeralwise expressed in $\mathcal{H}_\mathbf{HA}$ by the formula $\Psi_q(x_1, \ldots, x_n, y)$ and let $p(x_1, \ldots, x_n)$ be a relation such that, for any $a_1, \ldots, a_n \in \mathbf{N}$, $p(a_1, \ldots, a_n) \leftrightarrow \exists y q(a_1, \ldots, a_n, y)$. Then the formula $\exists y \Psi_q(x_1, \ldots, x_n, y)$ positively numeralwise expresses the relation $p(x_1, \ldots, x_n)$ in $\mathcal{H}_\mathbf{HA}$.*

Proof: Let us consider a relation $p(x_1, \ldots, x_n)$ such that, for every $a_1, \ldots, a_n \in \mathbf{N}$,

$$p(a_1, \ldots, a_n) \leftrightarrow \exists y q(a_1, \ldots, a_n, y) \tag{5.5}$$

and let $\Psi_q(x_1, \ldots, x_n, y)$ a formula which strongly numeralwise express in $\mathcal{H}_\mathbf{HA}$ the relation $q(x_1, \ldots, x_n, y)$. If $p(a_1, \ldots, a_n)$ holds, by the equivalence (5.5), there exists $b \in \mathbf{N}$ such that $q(a_1, \ldots, a_n, b)$ holds; this implies that

$$\mathcal{H}_\mathbf{HA} \vdash \Psi_q(\widetilde{a}_1, \ldots, \widetilde{a}_n, \widetilde{b})$$

which immediately yields

$$\mathcal{H}_{\mathbf{HA}} \vdash \exists y \Psi_q(\widetilde{a}_1, \ldots, \widetilde{a}_n, y) \ .$$

Conversely, if $\mathcal{H}_{\mathbf{HA}} \vdash \exists y \Psi_q(\widetilde{a}_1, \ldots, \widetilde{a}_n, y)$, by the constructivity of $\mathcal{H}_{\mathbf{HA}}$ we deduce that there exists a closed term t such that

$$\mathcal{H}_{\mathbf{HA}} \vdash \Psi_q(\widetilde{a}_1, \ldots, \widetilde{a}_n, t) \ .$$

Hence, by Propositions 5.2.1 and 5.2.2, there exists a canonical term $\widetilde{b}$ such that :

$$\mathcal{H}_{\mathbf{HA}} \vdash \Psi_q(\widetilde{a}_1, \ldots, \widetilde{a}_n, \widetilde{b}) \tag{5.6}$$

Now, let us suppose that $p(a_1, \ldots, a_n)$ is false. By the equivalence (5.5) we deduce that $q(a_1, \ldots, a_n, b)$ is false, and so

$$\mathcal{H}_{\mathbf{HA}} \vdash \neg\Psi_q(\widetilde{a}_1, \ldots, \widetilde{a}_n, \widetilde{b}) \ .$$

But this and (5.6) gives rise to a contradiction. Hence, $p(a_1, \ldots, a_n)$ must hold. □

The following result follows from the previous theorem.

5.2.12 Theorem *(i) Every partial recursive function is exhaustively numeralwise representable in $\mathcal{H}_{\mathbf{HA}}$. (ii) Every r.e. relation is positively numeralwise expressible in $\mathcal{H}_{\mathbf{HA}}$.*

Proof: Point (ii) easily follows from Theorem 5.2.11 by means of the Projection Theorem 5.1.9, while Point (i) comes similarly using the Normal Form Theorem 5.1.6 and using the formulas $T_n(y, x_1, \ldots, x_n, w)$ and $U(x, y)$. □

Now, let $\mathcal{N} = \langle \mathbf{N}, s, +, *, = \rangle$ be the standard structure of the natural numbers (i.e., let $\mathcal{N}$ be the standard model of the classical number theory of Peano Arithmetic **PA**), and let, for every closed formula A of $\mathcal{L}_{\mathbf{A}}$, $\mathcal{N} \models A$ denote the fact that the formula A holds (is satisfied) in the usual classical sense in the structure $\mathcal{N}$. Then, we can almost immediately translate the content of the Normal Form Theorem 5.1.6, into the following proposition involving satisfaction on $\mathcal{N}$:

5.2.13 Proposition *Let $p(x_1, \ldots, x_n) \equiv \exists y q(x_1, \ldots, x_n, y)$ be an r.e. relation, where $q(x_1, \ldots, x_n, y)$ is primitive recursive, and let $\Psi_q(x_1, \ldots, x_n, y)$ be a formula of $\mathcal{L}_{\mathbf{A}}$ which strongly numeralwise expresses $q(x_1, \ldots, x_n, y)$ in $\mathcal{H}_{\mathbf{HA}}$. Then, there exists an index e such that*

(i). $\mathcal{N} \models \forall x_1 \ldots \forall x_n (\exists y \Psi_q(x_1, \ldots, x_n, y) \Rightarrow \exists w T^n(\widetilde{e}, x_1, \ldots, x_n, w))$.

(ii). $\mathcal{N} \models \forall x_1 \ldots \forall x_n (\exists w T^n(\widetilde{e}, x_1, \ldots, x_n, w) \Rightarrow \exists y \Psi_q(x_1, \ldots, x_n, y))$

Proof: Let e be the index of $p(x_1, \ldots, x_n)$, determined by the Normal Form Theorem 5.1.6, such that

$$p(x_1, \ldots, x_n) \leftrightarrow \exists w \mathsf{T}_n(e, x_1, \ldots, x_n, w)$$

(i) Let us suppose that, for any $a_1, \ldots, a_n \in \mathbf{N}$ and some $b \in \mathbf{N}$,

$$\mathcal{N} \models \Psi_q(\widetilde{a}_1, \ldots, \widetilde{a}_n, \widetilde{b}) \ .$$

Then, since Ψ_q strongly numeralwise expresses q in $\mathcal{H}_{\mathbf{HA}}$, we can deduce that $q(a_1, \ldots, a_n, b)$ holds (otherwise, $\mathcal{H}_{\mathbf{HA}} \vdash \neg\Psi_q(\widetilde{a}_1, \ldots, \widetilde{a}_n, \widetilde{b})$ and hence $\mathcal{H}_{\mathbf{PA}} \vdash \neg\Psi_q(\widetilde{a}_1, \ldots, \widetilde{a}_n, \widetilde{b})$, in contrast with the fact that $\mathcal{N}$ is a model of $\mathcal{H}_{\mathbf{PA}}$), then $p(a_1, \ldots, a_n) \equiv \exists y q(a_1, \ldots, a_n, y)$ holds and this implies that $\exists w \mathsf{T}_n(e, x_1, \ldots, x_n, w)$ holds as well. But since T_n strongly numeralwise expresses the relation T_n in $\mathcal{H}_{\mathbf{HA}}$, we have

$$\mathcal{H}_{\mathbf{HA}} \vdash \exists w T_n(\widetilde{e}, \widetilde{a}_1, \ldots, \widetilde{a}_n, w) \ ,$$

which implies

$$\mathcal{N} \models \exists w T_n(\widetilde{e}, \widetilde{a}_1, \ldots, \widetilde{a}_n, w) \ .$$

(ii) The proof of is quite similar to the one of the previous point. □

The above Proposition 5.2.13, involving the semantical notion of satisfiability on $\mathcal{N}$, is not sufficient for our future purposes, which require provability in $\mathcal{H}_{\mathbf{HA}}$ instead of satisfiability on $\mathcal{N}$. However, we are not interested in replacing $\mathcal{N} \models \ldots$ with $\mathcal{H}_{\mathbf{HA}} \vdash \ldots$ in both points of Proposition 5.2.13. All we need is (i) of Proposition 5.2.13 and the following theorem, which considerably strengthens Point(ii) of the same proposition. The proof of this theorem is rather involved and cumbersome, but can be carried out along the lines quoted in [Wilkie, 1975] for a similar result involving classical arithmetic **PA** (see remark 5.2.15 below).

5.2.14 Theorem *The recursive function $\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}$–$\mathsf{Wff}(x)$ of Theorem 5.2.7 can be chosen so to satisfy (in addition to the properties stated in Theorem 5.2.7) also the following property:*

- *Let $p(x_1, \ldots, x_n) \equiv \exists y q(x_1, \ldots, x_n, y)$ be a recursively enumerable relation where $q(x_1, \ldots, x_n, y)$ is a primitive recursive relation, let $\mathrm{D}^{\mathrm{Prim}}_{q^{(n+1)}}$ be an explicit definition of $q(x_1, \ldots, x_n, y)$, and let h be its Gödel number (in the fixed arithmetization $\mathcal{G}_{Prim}$). Let $k = \mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}$–$\mathsf{Wff}(h)$, and let $\Psi_q(x_1, \ldots, x_n, y)$ be the formula of $\mathcal{L}_{\mathbf{A}}$ whose Gödel number is k (in the fixed arithmetization $\mathcal{G}_{\mathcal{L}_{\mathbf{A}}}$). Then, there exists an index $e \in \mathbf{N}$ such that:*

$$\mathcal{H}_{\mathbf{HA}} \vdash \forall x_1 \ldots \forall x_n (\exists w T_n(e, x_1, \ldots, x_n, w) \Rightarrow \exists y \Psi_q(x_1, \ldots, x_n, y)) \ .$$

□

5.2.15 Remark To better explain what is involved in the result quoted in [Wilkie, 1975], we need some definitions. First of all, let $x < y$ be an abbreviation of the formula $\exists z(x + z = y)$ of $\mathcal{L}_{\mathbf{A}}$ and let $\exists x < yA$ and $\forall x < yA$ be abbreviations of $\exists x(x < y \wedge A)$ and $\forall x(x < y \wedge A)$ respectively. Then the set $\mathbf{\Delta}$ of $\mathsf{bounded\ formulas\ of}$ $\mathcal{L}_{\mathbf{A}}$, is the smallest set of formulas satisfying the following conditions:

(i). Every atomic formula of $\mathcal{L}_{\mathbf{A}}$ belongs to $\mathbf{\Delta}$;

(ii). If A, B belong to $\mathbf{\Delta}$, then so are $A \wedge B$, $A \vee B$ and $\neg A$;

(iii). If A belongs to $\mathbf{\Delta}$, then $\forall x < yA \in \mathbf{\Delta}$ and $\exists x < yA \in \mathbf{\Delta}$, where $y \notin \mathrm{FV(A)}$.

Also, the sets $\mathbf{\Sigma}_n$ and $\mathbf{\Pi}_n$ of formulas of $\mathcal{L}_{\mathbf{A}}$ are so defined by induction on n:

1. $\mathbf{\Sigma}_0 = \mathbf{\Pi}_0 = \mathbf{\Delta}$;
2. $\mathbf{\Sigma}_{n+1} = \{\exists x_1 \ldots \exists x_k A : A \in \Pi_n$ where none of the x_i's is bounded in $A\}$;
3. $\mathbf{\Pi}_{n+1} = \{\forall x_1 \ldots \forall x_k A : A \in \mathbf{\Sigma}_n$ where none of the x_i's is bounded in $A\}$.

The result quoted in [Wilkie, 1975] is the following form of Kleene's Enumeration Theorem.

> If $n, m \geq 1$, there exists a formula $\Theta_n^{(m+1)}(e, x_1, \ldots, x_m)$ of $\mathcal{L}_{\mathbf{A}}$, with exactly $m+1$ free variables, such that:
>
> (i). $\Theta_n^{(m+1)}(e, x_1, \ldots, x_m) \in \mathbf{\Sigma}_n$;
>
> (ii). For all formulas $A(x_1, \ldots, x_m) \in \mathbf{\Sigma}_n$, there exists an index $e \in \mathbf{N}$ such that:
>
> $$\mathbf{PA} \vdash \forall x_1 \ldots \forall x_m (\Theta_n^{(m+1)}(e, x_1, \ldots, x_m) \Leftrightarrow A(x_1, \ldots, x_m)) \ .$$

The proof of this theorem is based on the fact that the usual proof of the enumeration theorem for $\mathbf{\Sigma}_n$ relations, quoted e.g. in [Kleene, 1952], can be carried out in the formal system of Peano arithmetic. ■

To conclude this Section, let us remark that, given the explicit definition $\mathrm{D}^{\mathrm{Prim}}_{q^{(n+1)}}$ of a primitive recursive relation $q^{(n+1)}(x_1, \ldots, x_n, y)$, it is possible to *effectively* build the index e such that, for all $a_1, \ldots, a_n$,

$$\exists y q^{(n+1)}(a_1, \ldots, a_n, y) \Leftrightarrow \exists w \mathsf{T}(e, a_1, \ldots, a_n, w)$$

holds according to Point (ii) of Theorem 5.1.6 (indeed, this is the way according to which one can prove the Enumeration Theorem 5.1.8). Also in this case, we can assert the existence of a general recursive function $\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Ix}(x)$ (indeed, primitive recursive) such that:

- $\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Ix}(x)$ is the recursive function associating, with any natural number, considered as the Gödel number of an explicit definition $\mathrm{D}^{\mathrm{Prim}}_{p^{(n)}}$ of a primitive recursive relation $p^{(n+1)}(x_1, \ldots, x_n, y)$, an index of the recursively enumerable relation $\exists y p^{(n+1)}(x_1, \ldots, x_n, y)$ according to Theorem 5.1.6.

The previous results can be summarized in the following theorem:

5.2.16 Theorem *There exist two* 1*-ary primitive recursive functions* $\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Wff}(x)$ *and* $\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Ix}(x)$ *such that, for any* $h \in \mathbf{N}$*, considered as the Gödel number of an explicit definition of a primitive recursive relation* $p(x_1, \ldots, x_n)$*, the following conditions are satisfied:*

(i). $\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}$–$\mathsf{Wff}(h) = k$ *is the Gödel number of a formula* $\Psi_p(x_1, \ldots, x_n)$ *in the fixed arithmetization of* $\mathcal{L}_{\mathbf{A}}$ *(that is* $k = \ulcorner \Psi_p(x_1, \ldots, x_n) \urcorner$*) which strongly numeralwise expresses the relation* $p^{(n)}_h(x_1, \ldots, x_n)$ *in* $\mathcal{H}_{\mathbf{HA}}$*.*

(ii). *If* $n > 1$*, that is* $\Psi_p(x_1, \ldots, x_n)$ *contains at least two variables, then* $\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}$–$\mathsf{Ix}(h) = e$ *is an index of recursively enumerable relation* $\exists y p^{(n)}(x_1, \ldots, x_{n-1}, y)$ *according to Theorem 5.1.6.*

(a) $\mathcal{N} \models \forall x_1 \ldots \forall x_{n-1}(\exists y \Psi_p(x_1, \ldots, x_{n-1}, y) \Rightarrow \exists w T^n(\widetilde{e}, x_1, \ldots, x_{n-1}, w))$*;*

(b) $\mathcal{H}_{\mathbf{HA}} \vdash \forall x_1 \ldots \forall x_{n-1}(\exists w T^n(\widetilde{e}, x_1, \ldots, x_{n-1}, w) \Rightarrow \exists y \Psi_p(x_1, \ldots, x_{n-1}, y))$*;*

(c) $\{\langle a_1, \ldots, a_{n-1}\rangle \mid \mathcal{H}_{\mathbf{HA}} \vdash \exists y \Psi_p(\widetilde{a}_1, \ldots, \widetilde{a}_{n-1}, y)\}$
$= \{\langle a_1, \ldots, a_{n-1}\rangle \mid \mathcal{N} \models \exists y \Psi_p(\widetilde{a}_1, \ldots, \widetilde{a}_{n-1}, y)\}$*;*

(d) $\{\langle a_1, \ldots, a_n\rangle \mid \mathcal{H}_{\mathbf{HA}} \vdash \exists w T^n(\widetilde{e}, \widetilde{a}_1, \ldots, \widetilde{a}_n, w)\}$
$= \{\langle a_1, \ldots, a_n\rangle \mid \mathcal{N} \models \exists w T^n(\widetilde{e}, \widetilde{a}_1, \ldots, \widetilde{a}_n, w)\}$*.*

□

5.3 The formal system HA*

In this section we will present a formal system which is constructive but not strongly constructive. To this aim we introduce a fundamental result of proof-theory, the so called *partial reflection principle*. Detailed discussions on this principle can be found in [Troelstra, 1973a, Girard, 1987].

Let

$$\mathcal{H} = \mathcal{H}_{\mathbf{HA}} + \{A_1, \ldots, A_n\}$$

be the Hilbert-style calculus obtained by adding the axioms $A_1, \ldots, A_n$ to $\mathcal{H}_{\mathbf{HA}}$. Moreover, let $H(v_1, \ldots, v_n)$ be any formula of $\mathcal{L}_{\mathbf{A}}$ whose free variables are exactly $v_1, \ldots, v_n$, with $n \geq 1$, and let $h \in \mathbf{N}$. We denote with

$$\mathsf{Proof}^{\,h}_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$$

the number-theoretic relation such that, for every $b, a_1, \ldots, a_n \in \mathbf{N}$, the following holds:

- $\mathsf{Proof}^{\,h}_{\mathcal{H}/H^{(n)}}(b, a_1, \ldots, a_n)$ is true iff b is the Gödel number of a proof π of the formula $H(\widetilde{a}_1/v_1, \ldots, \widetilde{a}_n/v_n)$ in the calculus $\mathcal{H}$ such that $\mathrm{dg}(\pi) \leq h$.

Since, the relation $\mathsf{Proof}^{\,h}_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$ is primitive recursive, by Theorem 5.2.7 it is strongly numeralwise representable in $\mathcal{H}_{\mathbf{HA}}$. Now, we can choose an appropriate formula of $\mathcal{L}_{\mathbf{A}}$ strongly numeralwise expressing this relation in $\mathcal{H}_{\mathbf{HA}}$ so to satisfy the following theorem.

5.3.1 Theorem (Partial Reflection Principle) *It is possible to choose the recursive function* $\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Wff}(x)$ *so as to satisfy the following condition:*

- *Let* $\mathcal{H} = \mathcal{H}_{\mathbf{HA}} + \{A_1, \ldots, A_n\}$ *be the Hilbert-style calculus obtained by adding the axioms* $A_1, \ldots, A_n$ *to* $\mathcal{H}_{\mathbf{HA}}$. *Let* $H(v_1, \ldots, v_n)$ *be any formula of* $\mathcal{L}_{\mathbf{A}}$ *whose free variables are exactly* $v_1, \ldots, v_n$ $(n \geq 1)$. *Then, there is an explicit definition of the relation* $\mathsf{Proof}^{\,h}_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$ *with Gödel number* k *(in the fixed arithmetization* $\mathcal{G}_{\mathrm{Prim}}$*), such that, if* $m = \mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Wff}(k)$, *and* $\mathit{Proof}^h_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$ *is the formula of* $\mathcal{L}_{\mathbf{A}}$ *strongly numeralwise expressing the relation* $\mathsf{Proof}^{\,h}_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$ *in* $\mathcal{H}_{\mathbf{HA}}$, *whose Gödel number is* m *(in the fixed arithmetization* $\mathcal{G}_{\mathcal{L}_{\mathbf{A}}}$ *of* $\mathcal{L}_{\mathbf{A}}$*), then:*

$$\mathcal{H} \vdash \forall x_1 \ldots \forall x_n(\mathit{Proof}^h_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n) \Rightarrow H(x_1, \ldots, x_n)) \ .$$

□

Henceforth, with

$$\mathit{Proof}^h_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$$

we will denote some fixed formula of $\mathcal{L}_{\mathbf{A}}$ strongly numeralwise expressing the relation $\mathsf{Proof}^{\,h}_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$ in $\mathcal{H}_{\mathbf{HA}}$ and satisfying the previous theorem.

The reader should observe that a full reflection principle does not hold as we discuss in Remark 5.3.2.

5.3.2 Remark Let

$$\mathsf{Proof}_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$$

be the predicate such that, for any $b, a_1, \ldots, a_n \in \mathbf{N}$,

- $\mathsf{Proof}_{\mathcal{H}/H^{(n)}}(b, a_1, \ldots, a_n)$ is true iff b is the Gödel number of a proof π of the formula $H(\widetilde{a}_1/v_1, \ldots, \widetilde{a}_n/v_n)$ in the calculus $\mathcal{H}$ (no bound is given on the logical complexity of π).

It is well known that any relation $\mathsf{Proof}_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$ is primitive recursive and hence it is strongly numeralwise expressible in $\mathcal{H}_{\mathbf{HA}}$.

Now, let us consider the Hilbert-style calculus for Peano arithmetic $\mathcal{H}_{\mathbf{PA}}$ (for which the partial reflection principle holds too) and the formula in one free variable $\exists z(z{+}z{=}x)$. Then, $\mathsf{Proof}_{\mathcal{H}_{\mathbf{PA}}/\exists z(z+z=x)}(y, x)$ is the predicate such that, for any $b, a \in \mathbf{N}$:

- $\mathsf{Proof}_{\mathcal{H}_{\mathbf{PA}}/\exists z(z+z=x)}(b, a)$ is true iff b is the Gödel number of a proof π of the formula $\exists z(z{+}z{=}\widetilde{a})$ in the calculus $\mathcal{H}_{\mathbf{PA}}$.

Let us assume, by contradiction, that there exists a formula of $\mathcal{L}_{\mathbf{A}}$

$$\mathit{Proof}_{\mathcal{H}_{\mathbf{PA}}/\exists z(z+z=x)}(y, x)$$

strongly numeralwise expressing $\mathsf{Proof}_{\mathcal{H}_{\mathbf{PA}}/\exists z(z+z=x)}(y, x)$ in $\mathcal{H}_{\mathbf{HA}}$ and such that the full reflection principle holds, that is:

$$\mathcal{H}_{\mathbf{PA}} \vdash \forall x(\mathit{Proof}_{\mathcal{H}_{\mathbf{PA}}/\exists z(z+z=x)}(y, x) \Rightarrow \exists z(z{+}z{=}x)) \ .$$

This implies that

$$\mathcal{H}_{\mathbf{PA}} \vdash \exists y Proof_{\mathcal{H}_{\mathbf{PA}}/\exists z(z+z=x)}(y, \mathtt{s}0) \Rightarrow \exists z(z+z=\mathtt{s}0)) \ ,$$

and hence

$$\mathcal{H}_{\mathbf{PA}} \vdash \neg\exists z(z+z=\mathtt{s}0) \Rightarrow \neg\exists y Proof_{\mathcal{H}_{\mathbf{PA}}/\exists z(z+z=x)}(y, \mathtt{s}0) \ .$$

Now, since

$$\mathcal{H}_{\mathbf{PA}} \vdash \neg\exists z(z+z = \mathtt{s}0)$$

we can deduce, by modus ponens, that

$$\mathcal{H}_{\mathbf{PA}} \vdash \neg\exists y Proof_{\mathcal{H}_{\mathbf{PA}}/\exists z(z+z=x)}(y, \mathtt{s}0) \ .$$

But this means that there is no proof of $\exists z(z+z=\mathtt{s}0)$ in $\mathcal{H}_{\mathbf{PA}}$, and this amount to a proof of consistency of $\mathcal{H}_{\mathbf{PA}}$ developed inside $\mathcal{H}_{\mathbf{HA}}$, and this contradicts Gödel's Second Incompleteness Theorem. ■

Now, we notice that, given the calculus $\mathcal{H} = \mathcal{H}_{\mathbf{HA}} + \{A_1, \ldots, A_n\}$ and a formula $H(v_1, \ldots, v_n)$, for any $h \in \mathbf{N}$ the explicit definition of the primitive recursive relation $\mathsf{Proof}^{\,h}_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$ considered in Theorem 5.3.1 can be obtained in an *effective* way. Thus, we can assert that:

5.3.3 Proposition *Given a calculus* $\mathcal{H} = \mathcal{H}_{\mathbf{HA}} + \{A_1, \ldots, A_m\}$ *and a formula* $H(v_1, \ldots, v_n) \in \mathcal{L}_{\mathbf{A}}$ *with exactly* n *free variables* ($n \geq 1$), *there exists a recursive function* $\mathsf{Pr}_{\mathcal{H}/H^{(n)}}(x)$ *(indeed a primitive recursive function) associating, with any* $h \in \mathbf{N}$, *the Gödel number of the explicit definition of* $\mathsf{Proof}^{\,h}_{\mathcal{H}/H^{(n)}}(y, x_1, \ldots, x_n)$ *considered in 5.3.1.* □

Now, given an Hilbert-style calculus $\mathcal{H}$, a formula $A \in \mathcal{L}_{\mathbf{A}}$, and an integer h, we recall that (according to Section 2.1) A is h*-provable in* $\mathcal{H}$ iff $\vdash^h_{\mathcal{H}} A$. Namely, there exists a sequence $B_1, \ldots, B_m$ of formulas of $\mathcal{L}_{\mathbf{A}}$ such that: $\mathrm{dg}(\{B_1, \ldots, B_m\}) \leq h$, and, for any i with $i = 1, \ldots, m$, either B_i is an axiom of $\mathcal{H}$ or it is obtained by applying a rule of $\mathcal{H}$ to a set of formulas $\{C_1, \ldots, C_p\} \subseteq \{B_1, \ldots, B_{i-1}\}$.

5.3.4 Theorem *Let* $\mathcal{H} = \mathcal{H}_{\mathbf{HA}} + \{A_1, \ldots, A_m\}$ *be a consistent Hilbert-style calculus. Then, there exists a recursive function* F *such that, for any* $h \in \mathbf{N}$, *if* $\mathsf{F}(h) = k$ *then:*

(i). $\mathcal{H} \vdash \neg\exists w T(\widetilde{k}, \widetilde{k}, w)$;

(ii). $\neg T(\widetilde{k}, \widetilde{k}, w)$ *is not* h*-provable in* $\mathcal{H}$.

Proof: Given a consistent Hilbert-style calculus

$$\mathcal{H} = \mathcal{H}_{\mathbf{HA}} + \{A_1, \ldots, A_m\}$$

($m \geq 0$ i.e. the set of the extra-axioms may be empty), let us consider the formula $\neg\exists w T(v, v, w)$. We recall that, by Proposition 5.3.3, the function $\mathrm{Pr}_{\mathcal{H}/\neg\exists w T(v,v,w)}(x)$

associates, with any $h \in \mathbf{N}$, the Gödel number (in the given arithmetization $\mathcal{G}_{\mathrm{Prim}}$ of $\mathcal{L}_{\mathbf{A}}$) of an explicit definition of the primitive recursive relation

$$\mathsf{Proof}\,^{h}_{\mathcal{H}/\neg\exists wT(v,v,w)}(y,x)\ .$$

considered in Theorem 5.3.1. Hence,

$$\mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Wff}(\mathsf{Pr}_{\mathcal{H}/\neg\exists wT(v,v,w)}(h))$$

is the Gödel number (in the given arithmetization $\mathcal{G}_{\mathcal{L}_{\mathbf{A}}}$ of $\mathcal{L}_{\mathbf{A}}$) of the formula

$$Proof^{h}_{\mathcal{H}/\neg\exists wT(v,v,w)}(y,x)$$

considered in Theorem 5.3.1. Having this in mind, we introduce the function F so defined: for any $i \in \mathbf{N}$:

$$\mathsf{F}(i) \ = \ \mathsf{D}^{\mathsf{prim}}_{\mathsf{rel}}\text{–}\mathsf{Ix}(\mathsf{Pr}_{\mathcal{H}/\neg\exists wT(v,v,w)}(i))\ .$$

Thus, F is the function associating, with any $i \in \mathbf{N}$, an index, according to Theorem 5.1.6, of the r.e. relation

$$\exists y\mathsf{Proof}\,^{i}_{\mathcal{H}/\neg\exists wT(v,v,w)}(y,x)\ ,$$

where $\mathsf{Pr}_{\mathcal{H}/\neg\exists wT(v,v,w)}(i)$ provides an explicit definition of the primitive recursive relation $\mathsf{Proof}\,_{\mathcal{H}/\neg\exists wT(v,v,w)}(y,x)$. Since F is obtained by composition of two recursive functions, it is recursive.

Now, let us consider any $h \in \mathbf{N}$ and let $\mathsf{F}(h) = k$. By Points (ii)a, (ii)b and (ii)c of Theorem 5.2.16, we have:

$$\mathcal{N} \models \forall x(\exists y Proof^{h}_{\mathcal{H}/\neg\exists wT(v,v,w)}(y,x) \Rightarrow \exists wT(\widetilde{k},x,w)) \tag{5.7}$$

$$\mathcal{H}_{\mathbf{HA}} \vdash \forall x(\exists wT(\widetilde{k},x,w) \Rightarrow \exists y Proof^{h}_{\mathcal{H}/\neg\exists wT(v,v,w)}(y,x)) \tag{5.8}$$

$$\{a \in \mathbf{N} \mid \mathcal{H}_{\mathbf{HA}} \vdash \exists wT(\widetilde{k},\widetilde{a},w)\} = \{a \in \mathbf{N} \mid \mathcal{N} \models \exists wT(\widetilde{k},\widetilde{a},w)\} \tag{5.9}$$

Moreover, by the Partial Reflection Principle (Theorem 5.3.1),

$$\mathcal{H} \vdash \forall x(Proof^{h}_{\mathcal{H}/\neg\exists wT(v,v,w)}(y,x) \Rightarrow \neg\exists wT(x,x,w)) \tag{5.10}$$

which implies

$$\mathcal{H} \vdash \forall x(\exists y Proof^{h}_{\mathcal{H}/\neg\exists wT(v,v,w)}(y,x) \Rightarrow \neg\exists wT(x,x,w)) \tag{5.11}$$

Now, from (5.8) and (5.11) we get:

$$\mathcal{H} \vdash \forall x(\exists wT(\widetilde{k},x,w) \Rightarrow \neg\exists wT(x,x,w))$$

which implies, by an elimination of the universal quantifier

$$\mathcal{H} \vdash \exists wT(\widetilde{k},\widetilde{k},w) \Rightarrow \neg\exists wT(\widetilde{k},\widetilde{k},w)\ . \tag{5.12}$$

But, since $(A \Rightarrow \neg A) \Rightarrow \neg A$ is provable in $\mathcal{H}_{\mathbf{INT}}$, this implies

$$\mathcal{H} \vdash \neg\exists w T(\widetilde{k}, \widetilde{k}, w) \ . \tag{5.13}$$

Now, to conclude the proof, we have to show that $\neg\exists w T(\widetilde{k}, \widetilde{k}, w)$ is not h-provable in $\mathcal{H}$. Let us suppose the contrary. Then the relation $\exists y \mathsf{Proof}^{\,h}_{\mathcal{H}/\neg\exists w T(v,v,w)}(y, \widetilde{k})$ holds and hence, since $Proof^{h}_{\mathcal{H}/\neg\exists w T(v,v,w)}(y, x)$ strongly numeralwise expresses in $\mathcal{H}_{\mathbf{HA}}$ the predicate $\mathsf{Proof}^{\,h}_{\mathcal{H}/\neg\exists w T(v,v,w)}(y, x)$,

$$\mathcal{N} \models \exists y Proof^{h}_{\mathcal{H}/\neg\exists w T(v,v,w)}(y, \widetilde{k})$$

which, by (5.7), implies

$$\mathcal{N} \models \exists w T(\widetilde{k}, \widetilde{k}, w) \ .$$

Hence, by (5.9), we obtain

$$\mathcal{H}_{\mathbf{HA}} \vdash \exists w T(\widetilde{k}, \widetilde{k}, w) \ . \tag{5.14}$$

But, $\mathcal{H}_{\mathbf{HA}} \subseteq \mathcal{H}$ and hence (5.14) and (5.13) imply that $\mathcal{H}$ is inconsistent, contradicting our our hypothesis. Thus $\neg\exists w T(\widetilde{k}, \widetilde{k}, w)$ is not h-provable in $\mathcal{H}$. □

Now, let us consider the enumeration $\mathcal{E}^{(1)}$ of all the partial recursive functions in one variable, defined by means of the Enumeration Theorem in Section 5.1. Then, we can define a recursive function $\mathsf{D}_{\mathsf{fun}}\text{–}\mathsf{Wff}(x)$ associating, with any $i \in \mathbf{N}$ (seen as the Gödel number of an explicit definition of a partial recursive function f, in a suitable arithmetization $\mathcal{G}_{\mathrm{PR}}$ of the formalism of partial recursive functions), the Gödel number of a formula of $\mathcal{L}_{\mathbf{A}}$ with exactly two free variables which exhaustively numeralwise expresses the partial recursive function f_i of the enumeration $\mathcal{E}^{(1)}$. The justification of the recursiveness of the function $\mathsf{D}_{\mathsf{fun}}\text{–}\mathsf{Wff}(x)$ is similar to the one given for the function $\mathsf{D}^{\mathsf{prim}}_{\mathsf{fun}}\text{–}\mathsf{Wff}(x)$ in Section 5.2.

Now, $\forall x \forall z H_j(x, z)$ will denote the following formula of $\mathcal{L}_{\mathbf{A}}$:

$$\forall x \forall z(\ (\Psi_j(x,z) \wedge \neg\exists w T(z,z,w)) \ \vee \ \neg(\Psi_j(x,z) \wedge \neg\exists w T(z,z,w)) \ ) \ ,$$

where $\ulcorner \Psi_{f_j}(x, z) \urcorner = \mathsf{D}_{\mathsf{fun}}\text{–}\mathsf{Wff}(j)$ is the Gödel number of the formula $\Psi_{f_j}(x, z)$ exhaustively numeralwise expressing in $\mathcal{H}_{\mathbf{HA}}$ the function f_j, and we indicate with $\Psi_j(x, z)$ the formula $\Psi_{f_j}(x, z)$.

Starting from formulas $\forall x \forall z H_j(x, z)$, we define the sequence of Hilbert-style calculi $\{\mathcal{H}_j\}_{j \in \omega}$ as follows, for any $j \in \mathbf{N}$:

$$\mathcal{H}_j = \mathcal{H}_{\mathbf{HA}} + \{\forall x \forall z H_j(x, z)\} \ .$$

Since, for any $j \in \omega$, the function F_j associated with the calculus $\mathcal{H}_j$ by Theorem 5.3.4 can be *effectively* determined, starting from an explicit definition of this function (which is indeed primitive recursive) we can effectively determine an index i for this function in the enumeration $\mathcal{E}^{(1)}$ such that $f_i \simeq \mathsf{F}_j$. Hence, we can assert that there exists a general recursive function G with the following properties:

- G is a recursive function associating, with any $j \in \mathbf{N}$ (seen as the index of the Hilbert-style calculus $\mathcal{H}_j$ in the enumeration $\{\mathcal{H}_j\}_{j\in\omega}$), the index of F_j in the enumeration $\mathcal{E}^{(1)}$.

Thus we have:

5.3.5 Proposition *For any $j \in \mathbf{N}$, $\mathsf{G}(j)$ is the index in the enumeration $\mathcal{E}^{(1)}$ of the recursive function $f_{\mathsf{G}(j)}$ such that: for any $h \in \mathbf{N}$, if $f_{\mathsf{G}(j)}(h) = k$ then $\mathcal{H}_j \vdash \neg\exists w T(\widetilde{k}, \widetilde{k}, w)$ but $\neg\exists w T(\widetilde{k}, \widetilde{k}, w)$ is not h-provable in $\mathcal{H}_j$.* □

Now, let us consider the function G. Since G is general recursive, we can apply the *Fixed Point Theorem* 5.1.11 to deduce that there exists an index $i^* \in \mathbf{N}$ such that:

$$f_{i^*} \simeq f_{\mathsf{G}(i^*)} \simeq \mathsf{F}_{i^*} \ .$$

Let us denote with $\Psi_{f^*}(x,z)$ the formula $\Psi_{i^*}(x,z)$ of $\mathcal{L}_{\mathbf{A}}$ which exhaustively numeralwise represents the recursive function $f^* \simeq \mathsf{F}_{i^*}$ in $\mathcal{H}_{\mathbf{HA}}$; that is

$$\ulcorner \Psi_{f^*}(x,z) \urcorner = \mathsf{D}_{\mathsf{fun}}\text{–}\mathsf{Wff}(i^*)$$

(where $\ulcorner \Psi_{f^*}(x,z) \urcorner = \mathsf{D}_{\mathsf{fun}}\text{–}\mathsf{Wff}(i^*)$ is Gödel number of the formula $\Psi_{f^*}(x,z)$). Also, let $\forall x \forall z H^*(x,z)$ be the following formula of $\mathcal{L}_{\mathbf{A}}$:

$$\forall x \forall z(\ (\Psi_{f^*}(x,z) \wedge \neg\exists w T(z,z,w)) \ \vee \ \neg(\Psi_{f^*}(x,z) \wedge \neg\exists w T(z,z,w))\ ) \ .$$

Finally, let us denote with $\mathcal{H}^*$ the Hilbert-style calculus

$$\mathcal{H}_{i^*} = \mathcal{H}_{\mathbf{HA}} + \{\forall x \forall z H^*(x,z)\} \ .$$

Since $f^*(x)$ is a recursive function and $\Psi_{f^*}(x,z)$ exhaustively numeralwise represents it in $\mathcal{H}_{\mathbf{HA}}$, by Proposition 5.3.5 we obtain:

5.3.6 Lemma *For any $h \in \mathbf{N}$, if $k = f^*(h)$, then:*

(i). $\mathcal{H}^* \vdash \neg\exists w T(\widetilde{k}, \widetilde{k}, w)$;

(ii). $\neg\exists w T(\widetilde{k}, \widetilde{k}, w)$ is not h-provable in $\mathcal{H}^*$.

□

Now, let $\mathbf{HA}^*$ be the formal system generated by the Hilbert-style calculus $\mathcal{H}^*$. That is,

$$\mathbf{HA}^* = (\mathcal{L}_{\mathbf{A}}, \mid\!\sim_{\mathcal{H}^*}, \mathcal{H}^*) \ .$$

Of course, $\mathbf{HA}^*$ is consistent, since the formula $\forall x \forall z H^*(x,z)$ holds in $\mathcal{N}$. We will prove that $\mathbf{HA}^*$ is a constructive formal system, but it is not strongly constructive.

First of all, we prove that $\mathbf{HA}^*$ is a constructive formal system, that is we show that its set of theorems satisfies the disjunction property and the explicit definability property for closed formulas.

To this aim, let us consider the pseudo-natural deduction calculus $\mathcal{ND}_{\mathbf{HA}^*}$ for $\mathbf{HA}^*$ obtained by adding to $\mathcal{ND}_{\mathbf{HA}}$ the axiom-rule

$$\frac{}{\vdash H^*(x,z)}\,\text{H}^*$$

It is easy to verify that $\mathcal{ND}_{\mathbf{HA}^*}$ is a presentation for the formal system $\mathbf{HA}^*$.

Now, let us consider the generalized rule

$$\mathcal{R}_{\mathbf{HA}} = \textsc{Cut} \cup \textsc{Subst} \cup \textsc{Id}_1 \cup \textsc{Id}_2 \cup \textsc{Sum} \cup \textsc{Prod} \ .$$

and the abstract calculus

$$\mathbb{D}_{\mathbf{HA}}(\Pi) = \mathbb{D}(\mathcal{R}_{\mathbf{HA}}, \text{Seq}(\Pi)) \ .$$

defined and studied in Section 4.2. It is easy to prove the following fact:

5.3.7 Lemma *For any proof $\pi : \Gamma \vdash A$ belonging to $\mathcal{ND}_{\mathbf{HA}}$, if Γ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$ then A is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$.* □

5.3.8 Lemma *For any proof $\pi : \Gamma \vdash A$ belonging to $\mathcal{ND}_{\mathbf{HA}^*}$, if Γ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$ then A is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$.*

Proof: $\mathcal{R}_{\mathbf{HA}}$ contains Cut and Subst, and hence, by Lemma 3.2.2, we immediately have that $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$ contains a proof $\tau : \vdash A$; hence Point (i) of Definition 4.1.1 is satisfied. To prove Point (ii) we proceed by induction on depth(π).

Basis: If depth(π) = 0 then: either the only rule applied in π is an assumption introduction, or it is one of the axiom-rules id_1, Sum and Prod, or it is the axiom-rule H^*. Since we have already discussed the other cases in Lemma 4.2.5, here we treat only the last case. Thus, let

$$\pi : \Gamma \vdash A \equiv \frac{}{\vdash H^*(x,z)}\,\text{H}^* \ .$$

First of all, by Point (i), there exists a proof τ of this sequent in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$, that is:

$$\tau : \vdash H^*(x,z) \in \mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$$

Now, let us consider a closed instance $\theta H^*(x,z)$ of $H^*(x,z)$. Since the only free variables in $H^*(x,z)$ are x and z and $\mathcal{ND}_{\mathbf{HA}^*}$ is Subst-closed, there exists a proof

$$\pi_1 : \vdash H^*(t,t') \in \mathcal{ND}_{\mathbf{HA}^*}$$

where $t = \theta x$ and $t' = \theta z$. Since t, t' are closed terms, by Proposition 4.2.3 there exist canonical terms $\widetilde{h}, \widetilde{k}$ such that $\vdash t = \widetilde{h}$ and $\vdash t' = \widetilde{k}$ are provable in $\mathcal{ND}_{\mathbf{HA}}$. Therefore, by the closure under $\textsc{Id}_2$ of this calculus, we have that there exists a proof:

$$\pi_2 : \vdash H^*(\widetilde{h},\widetilde{k}) \in \mathcal{ND}_{\mathbf{HA}^*} \ .$$

Let us consider the formula $\Psi_{f^*}(\widetilde{h},\widetilde{k})$. By the assumptions made on this formula and on the function f^*, we have that, if $k = f^*(h)$, then

$$\mathcal{H}_{\mathbf{HA}} \vdash \Psi_{f^*}(\widetilde{h},\widetilde{k})$$

Therefore, since $\mathcal{ND}_{\mathbf{HA}}$ is a calculus for the formal system generated by $\mathcal{H}_{\mathbf{HA}}$, we immediately deduce that there exists a proof

$$\pi_3 : \vdash \Psi_{f^*}(\widetilde{h},\widetilde{k}) \in \mathcal{ND}_{\mathbf{HA}}$$

and hence, still by the closure of $\mathcal{ND}_{\mathbf{HA}}$ under the rule $\textsc{Id}_2$,

$$\pi_4 : \vdash \Psi_{f^*}(t,t') \in \mathcal{ND}_{\mathbf{HA}}$$

Since $\vdash \Psi_{f^*}(t,t') \in \mathrm{Seq}(\mathcal{ND}_{\mathbf{HA}})$, Lemma 5.3.7 implies that $\Psi_{f^*}(t,t')$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$. Now, let us consider the formula $\neg\exists w T(\widetilde{k},\widetilde{k},w)$. By Lemma 5.3.6 we immediately deduce that it is provable in $\mathcal{ND}_{\mathbf{HA}^*}$ and hence

$$\vdash \neg\exists w T(\widetilde{k},\widetilde{k},w) \in \mathrm{Seq}(\mathcal{ND}_{\mathbf{HA}^*})\ .$$

Since this is a negated formula, by Definition 4.1.1 we immediately deduce that it is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$. Still by closure under id_2 of $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$, this implies that $\neg\exists w T(t,t,w)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$, too. Since both $\Psi_{f^*}(t,t')$ and $\neg\exists w T(t,t,w)$ are $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$, we deduce that also their conjunction

$$\Psi_{f^*}(t,t') \wedge \neg\exists w T(t,t,w)$$

is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}(\mathcal{ND}_{\mathbf{HA}^*})$. This immediately implies that $H^*(t,t')$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}(\mathcal{ND}_{\mathbf{HA}^*})$. On the other hand, if $k \not= f^*(h)$, then (since f^* is a total function) it is easy to verify that $\mathcal{H}_{\mathbf{HA}} \vdash \neg\Psi_{f^*}(\widetilde{h},\widetilde{k})$, and this immediately implies that

$$\neg(\Psi_{f^*}(t,t') \wedge \neg\exists w T(t,t,w))$$

is provable in $\mathbb{D}(\mathcal{ND}_{\mathbf{HA}^*})$; since this is a negated formula it is immediately $\mathcal{L}_{\mathbf{A}}$-evaluated in this set, and hence $H^*(t,t')$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}(\mathcal{ND}_{\mathbf{HA}^*})$. Since θ is any closed substitution, we have that $H^*(x,z)$ is $\mathcal{L}_{\mathbf{A}}$-evaluated in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$.

This concludes the proof of the basis case of the induction. The proof of the induction step goes by cases on the last rule applied in π, and since H^* is the only rule of $\mathcal{ND}_{\mathbf{HA}^*}$ which does not belong to $\mathcal{ND}_{\mathbf{HA}}$, this proof coincides with the one given for Lemma 4.2.5. □

5.3.9 Theorem **HA*** *is a constructive formal system with respect to* (Dp) *and* (Ed).

Proof: Let $A \vee B$ be a closed formula such that $A \vee B \in \mathrm{Theo}(\mathbf{HA}^*)$. Then $\pi : \vdash A \vee B \in \mathcal{ND}_{\mathbf{HA}^*}$. Since $\mathcal{ND}_{\mathbf{HA}^*}$ is a $\mathcal{R}_{\mathbf{HA}}$-closed, by Theorem 2.3.9, we have that $\mathcal{ND}_{\mathbf{HA}^*} \approx \mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$. Hence, there exists $\tau : \vdash A \vee B \in \mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$. By Lemma 5.3.8, since $A \vee B$ is a closed formula, we have that at least one between the sequents $\vdash A$ and $\vdash B$ is provable in $\mathbb{D}_{\mathbf{HA}}(\mathcal{ND}_{\mathbf{HA}^*})$, that is, either $A \in \mathrm{Theo}(\mathbf{HA}^*)$ or $B \in \mathrm{Theo}(\mathbf{HA}^*)$. The proof of the explicit definability property is similar. □

Now, let us suppose that $\mathbf{HA}^*$ is a strongly constructive formal system. This implies that there exist a strongly constructive calculus $\mathbf{C}$ which is a presentation for $\mathbf{HA}^*$ and agrees with it. In other words, recalling the definitions given in Section 2.5, there exist a non-increasing generalized rule $\mathcal{R}$ and a function $\phi : \mathbf{N} \to \mathbf{N}$ such that

(I). $\mathbf{C}$ is strongly constructive w.r.t. $\mathcal{R}$ and ϕ. That is:

   (a) $\mathbf{C}$ is uniformly $\mathcal{R}$-closed w.r.t ϕ; ;

   (b) For any $\Pi \subseteq \mathbf{C}$, $\mathrm{Theo}(\mathcal{R}^*_{\mathbf{HA}}([Lambda]))$ is constructive;

(II). $\mathbf{C}$ is a presentation for $\mathbf{HA}^*$. That is, $\mathrm{Theo}(\mathbf{C}) = \mathrm{Theo}(\mid\!\sim_{\mathcal{H}^*})$.

(III). $\mathbf{C}$ agrees with $\mathbf{HA}$. That is: for any $\Pi \subseteq \mathbf{C}$ and for any $h \in \mathbf{N}$ such that $\mathrm{dg}(\Pi) \leq h$, there exists a positive integer k such that $\mathrm{Theo}(\Pi) \subseteq \mathrm{Theo}(\mid\!\sim^k_{\mathcal{H}^*})$.

Since, by elimination of the universal quantifier, we get $\mathcal{H}^* \vdash H^*(x, z)$, we have

$$H^*(x, z) \in \mathrm{Theo}(\mid\!\sim_{\mathcal{H}^*}) \ .$$

Thus, by property (II), there must exist a proof

$$\pi : \vdash H^*(x, z) \in \mathbf{C} \ .$$

Let us consider the set of proofs $[\pi]$, and let h_π be the degree of π. Then $\mathrm{dg}([\pi]) \leq h_\pi$. Hence, by Proposition 2.4.8, there exists a positive integer h' such that, for any $A \in \mathrm{Theo}(\mathbb{D}(\mathcal{R}_{\mathbf{HA}}, \mathrm{Seq}([\pi])))$, there exists a proof $\pi' : \vdash A \in \mathbf{C}$ with $\mathrm{dg}(\pi') \leq h'$. Let

$$\Pi = \left\{\pi' : \vdash A \in \mathbf{C} : \ A \in \mathrm{Theo}(\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi]))) \text{ and } \mathrm{dg}(\pi') \leq h'\right\} \ .$$

Now, by Point (i) of Proposition 2.3.8, we have that

$$\mathbb{D}(\mathcal{R}_{\mathbf{HA}}, \mathrm{Seq}([\pi]))) = \mathcal{R}^*_{\mathbf{HA}}([\pi])$$

and hence, by strong constructiveness of $\mathbf{C}$ (namely by Point (I)b), we deduce that the set of proofs $\mathbb{D}(\mathcal{R}^*_{\mathbf{HA}}, \mathrm{Seq}([\pi]))$ is constructive and hence in constructive the set of proofs Π.

Now, since $H^*(x, z)$ belongs to $\mathbb{D}(\mathcal{R}^*_{\mathbf{HA}}, \mathrm{Seq}([\pi]))$ and $\mathcal{R}$ is standard (that is it includes the generalized rule SUBST); it follows that any closed instance of $H^*(x, z)$ has a proof in Π. Let us consider any $a \in \mathbf{N}$, and let $b = f^*(a)$; by Lemma 5.3.6 and by the fact that $\Psi_{f^*}(x, z)$ exhaustively numeralwise expresses the function $f^*(x)$, we have that

$$\Psi_{f^*}(\widetilde{a}, \widetilde{b}) \wedge \neg\exists w T(\widetilde{b}, \widetilde{b}, w)$$

is provable in $\mathcal{H}_{\mathbf{HA}}$. Now, since Π is constructive and $\mathcal{ND}_{\mathbf{HA}^*}$ is consistent, we deduce that

$$\Psi_{f^*}(\widetilde{a}, \widetilde{b}) \wedge \neg\exists w T(\widetilde{b}, \widetilde{b}, w) \in \mathrm{Theo}(\Pi) \ .$$

We remark that this fact holds for any pair of natural numbers a, b such that $b = f^*(a)$.

Now, let us consider the index k such that, according to property (III), Theo(Π) $\subseteq$ Theo($\mid\!\sim^k_{\mathcal{H}^*}$) and $h' \leq k$; moreover, let us suppose that $j = f^*(k)$. Then, by the above discussion, we have that

$$\Psi_{f^*}(\widetilde{k}, \widetilde{j}) \wedge \neg\exists w T(\widetilde{j}, \widetilde{j}, w) \tag{5.15}$$

belongs to Theo($\mid\!\sim^k_{\mathcal{H}^*}$), hence it is an k-provable formula. Now, let us consider the following instance of the axiom schema $(A \wedge B) \Rightarrow B$:

$$(\Psi_{f^*}(\widetilde{k}, \widetilde{j}) \wedge \neg\exists w T(\widetilde{j}, \widetilde{j}, w)) \Rightarrow \neg\exists w T(\widetilde{k}, \widetilde{k}, w) \ . \tag{5.16}$$

Such an instance, has a degree less than k, namely less than the degree of $H^*(x, z)$, the latter formula belonging to Theo(Π). Hence, by applying the modus ponens to the formulas (5.15) and (5.16), we get

$$\neg\exists w T(\widetilde{j}, \widetilde{j}, w) \in \text{Theo}(\vdash^k_{\mathcal{H}^*}) \ .$$

But this contradicts Lemma 5.3.6. Hence, we must conclude that there cannot exist a strongly constructive calculus for **HA***.

5.3.10 Theorem **HA*** *is not a strongly constructive formal system.* □

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