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Duplication-free tableau calculi together with cut-free and contraction-free sequent calculi for the interpolable propositional intermediate logics

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Abstract

We get cut-free and contraction-free sequent calculi for the interpolable propositional intermediate logics by translating suitable duplication-free tableau calculi developed within a semantical framework. From this point of view, the paper aims also to outline a general semantical method to get cut-free sequent calculi for appropriate intermediate logics.

Keywords: propositional intermediate logics, tableau systems, cut-free sequent calculi, duplication, contraction

Contents

1 Introduction

In the area of automated reasoning, a considerable amount of research has been devoted, in the recent years, to non classical logics in order to get efficient decision procedures and/or good proof-strategies for a relevant number of them. In this frame, an important role has been played, on the one hand, by the comparison of various kinds of calculi, where tableau calculi and cut-free sequent calculi have been taken into account together with prefixed tableau systems, or even with systems using hyper-sequents [Avron, 1991; Fitting, 1983]; on the other hand, nice cut-free sequent calculi and good tableau ones have been provided for some of the most prominent (propositional) modal logics [Fitting, 1972; Wallen, 1990]. Rather recently, also Intuitionistic Logic and the related Proof-Theory (based on such tools as Prawitz's normalizable natural calculus [Prawitz, 1965; Prawitz, 1971], or Gentzen's sequent calculus with eliminable cut [Gentzen, 1969], or Dragalin's improved variant of Gentzen's sequent calculus [Dragalin, 1988], or Fitting's tableau calculus with signed formulas [Fitting, 1969]) have been revisited in order to improve the involved proof-strategies.

Thus, in the intuitionistic framework important results allowing to simplify the search for proofs have been obtained by Dyckhoff [Dyckhoff, 1992] and, independently, Hudelmaier [Hudelmaier, 1993], who have exhibited cut-free and *contraction* free sequent calculi for Intuitionistic Propositional Logic, where a sequent calculus is contraction-free if no formula occurring in the lower sequent of an inference rule (i.e., the sequent obtained by applying the rule) can occur in some of the upper sequents (i.e., the sequents to which the rule is applied); in [Dyckhoff, 1992] also a contraction-free natural calculus is given while in [Hudelmaier, 1993] it is shown that the involved calculus (essentially coinciding with the one of [Dyckhoff, 1992]) gives rise to an $O(n \log n)$-SPACE decision procedure for Intuitionistic Propositional Logic. A quite similar problem has been taken into account on the side of the tableau calculi, where the counterpart of the elimination of contractions is the elimination of *duplications*, a duplication occurring in a proof of a tableau calculus whenever a formula already used by an inference rule is used again by the same inference rule (for a comprehensive discussion about duplications, see e.g. [Miglioli et al., 1997]). In this framework, [Miglioli et al., 1994a; Miglioli et al., 1994b; Miglioli et al., 1997] have improved Fitting's tableau calculus for Intuitionistic Predicate Logic [Fitting, 1969] by progressively reducing the amount of duplications involved in its proofs. To do so, [Miglioli et al., 1994a] has introduced the $\mathbf{F_c}$-signed formulas near the $\mathbf{F}$-signed formulas and the $\mathbf{T}$-signed formulas of Fitting's calculus (see the discussion made in [Miglioli et al., 1994a]), and [Miglioli et al., 1994b; Miglioli et al., 1997] have successively refined the rules for implication, taking into account, in the frame of tableau calculi, the ideas of Dyckhoff's paper; this has led to the machinery of [Miglioli et al., 1997], where a calculus for Intuitionistic Predicate Logic, nearly optimal from the point of view of the elimination of duplications is presented (such a calculus completely avoids duplications in the propositional framework, thus providing, for tableau calculi, a result quite comparable with Dyckhoff's and Hudelmaier's one for sequent calculi; on the other hand, at the predicate level, not all duplications, or *mutatis mutandis*, not all contractions, can be cut off).

Now (as we will furtherly discuss in the present paper), it has turned out that our intuitionistic tableau calculus can be straightforwardly translated into a cut-free and contraction-free sequent calculus, which furtherly strengthens the relations between our work and Dyckhoff's one, to the point that the main difference between the two approaches can be seen to lie more in the tools used to prove the main results than in the content of the results themselves: our techniques are essentially model theoretic, Dyckhoff's ones are essentially proof theoretic. From this point of view, we have furtherly extended our methods to other logics with a Kripke semantics. In particular, [Miglioli et al., 1994b; Miglioli et al., 1997] provide a tableau calculus also for the intermediate predicate logic of Kuroda, which is nearly optimal from the point of view of duplications and provides, by translating the tableau rules into sequent rules, a cut-free sequent calculus; moreover, [Avellone et al., 1997] proposes some generalizations (based on suitable filtration techniques) of the tableau rules in order to reasonably handle a family of propositional intermediate logics.

The methods explained in [Avellone et al., 1997] allow to cover infinitely many logics, but do not provide genuine tableau calculi (and cut-free sequent calculi, on the other side) for them. On the other hand, while a calculus of this kind probably does not exists for the whole class of logics taken into account in [Avellone et al., 1997], such a class includes the class of the interpolable propositional intermediate logics, which has been shown by Maksimova [Maksimova, 1977] to consists of exactly seven logics, namely Intuitionistic Logic, Classical Logic, Dummett Logic (characterized by the class of rooted linearly ordered Kripke frames), Peirce Logic (characterized by the rooted frames with at most two elements), the logic characterized by the rooted frames whose depth is at most 2 (we call $\mathbf{DE}_2$), the logic characterized by the rooted frames with depth at most 2 and with at most two final elements (we call **SM**, since it is the greatest semiconstructive propositional intermediate logic in the sense of [Ferrari and Miglioli, 1993]), and Jankov Logic of the weak excluded middle; and one naturally asks whether there are cut-free sequent calculi for these logics.

The aim of the present paper is just to provide such calculi. To do so, we will extend the semantical machinery of [Miglioli et al., 1997], and will exhibit tableau calculi without duplications for all the considered logics (disregarding, of course, the well known Intuitionistic and Classical Logic), providing for each of them the soundness and the completeness theorem with respect to the related Kripke semantics; the tableau calculi will be successively translated into cut-free and contraction-free sequent calculi, along the lines explained in the next section. Section 3 will take into account, in particular, Dummett Logic, while Section 4 will treat Peirce Logic, Section 5 will study $\mathbf{DE}_2$, Section 6 will analyze **SM** and Section 7 will consider Jankov Logic.

Near the concrete explanation of the various calculi, one of the main goals of the paper is *to illustrate a general method to get cut-free calculi by means of semantical tools.* The method, indeed, seems to be rather promising and applicable to a wide family of predicate intermediate logics. But this will be the subject of further papers.

2 Basic definitions

The set of propositional *well formed formulas* (*wff's* for short) is defined as usual, starting from an enumerable set of propositional variables and using the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$. We say that a wff A is *negated* iff $A \equiv \neg B$ for some wff B.

A *substitution* will be any function σ associating a wff with every propositional variable. To denote the result of the application of the substitution σ to the wff A, we will write $\sigma(A)$ or, more simply, σA.

INT (respectively, **CL**) will denote both an arbitrary calculus for intuitionistic propositional logic (respectively, for classical propositional logic) and the set of intuitionistically valid wff's (respectively, the set of classically valid wff's).

As usual, an *intermediate propositional logic* [Chagrov and Zakharyaschev, 1997; Ferrari and Miglioli, 1993; Ferrari and Miglioli, 1995a; Ferrari and Miglioli, 1995b; Miglioli, 1992; Ono, 1970] will be any set **L** of wff's satisfying the following conditions: (i) **L** is consistent; (ii) $\mathbf{INT} \subseteq \mathbf{L}$; (iii) **L** is closed under detachment; (iv) **L** is closed under arbitrary substitutions.

As is well known, for any intermediate logic **L** we have $\mathbf{INT} \subseteq \mathbf{L} \subseteq \mathbf{CL}$. If $\mathcal{A}$ is a set of axiom-schemes and **L** is a logic, the notation $\mathbf{L} + \mathcal{A}$ will indicate both the deductive system closed under detachment and arbitrary substitutions obtained by adding to **L** the axiom-schemes of $\mathcal{A}$, and the set of theorems (which is a logic) of such a deductive system; of course, if **L** is recursively enumerable and $\mathcal{A}$ is a recursive set of axiom-schemes then $\mathbf{L} + \mathcal{A}$ is recursively enumerable, while nothing can be said, in general, about the recursiveness of $\mathbf{L} + \mathcal{A}$, even if **L** is recursive and $\mathcal{A}$ consists of a single axiom-scheme. If $\mathcal{A} = \{(A)\}$ consists of a single axiom-scheme (A) generated by a wff A, the notation $\mathbf{L} + (A)$ will replace $\mathbf{L} + \{(A)\}$.

As usual, if Γ is any set of wff's and **L** is a logic, we say that a wff A is **L**-*provable from* Γ, and we denote it by $\Gamma \vdash_{\mathbf{L}} A$, iff there are $B_1, \ldots, B_n$ such that $\{B_1, \ldots, B_n\} \subseteq \Gamma$ and $B_1 \wedge \ldots \wedge B_n \rightarrow A \in \mathbf{L}$; $\Gamma \nvdash_{\mathbf{L}} A$ will mean that $\Gamma \vdash_{\mathbf{L}} A$ does not hold.

We assume the reader to be familiar with the notion of *(propositional) Kripke model* $\underline{K} = \langle P, \leq, \Vdash \rangle$, where $\underline{P} = \langle P, \leq \rangle$ is a *poset* or *frame* and $\Vdash$ is the *forcing relation*, defined between elements of P and atomic wff's, and extended in the usual way to arbitrary wff's; we say that $\underline{K}$ is *built on the poset* $\underline{P}$, or that $\underline{P}$ is the *underlying poset of* $\underline{K}$. If $\underline{P} = \langle P, \leq \rangle$ and $\alpha, \beta \in P$, then $\alpha < \beta$ will indicate the fact that $\alpha \leq \beta$ and $\alpha \neq \beta$. Moreover, we call *root of* $\underline{P}$ an element r (if it exists) such that, for every $\alpha \in P$, $r \leq \alpha$.

Given a Kripke frame $\underline{P} = \langle P, \leq \rangle$ and $\alpha, \beta \in P$, β is said to be an *immediate successor of* α in $\underline{P}$ iff $\alpha < \beta$ and, for all $\gamma \in P$ such that $\alpha \leq \gamma \leq \beta$, $\gamma = \alpha$ or $\gamma = \beta$. If $\underline{P} = \langle P, \leq \rangle$ and $\alpha \in P$, then α will be called *final in* $\underline{P}$ or, more simply, *final* iff, for every $\beta \in P$, $\alpha \leq \beta$ implies $\alpha = \beta$. Finally, if $\underline{P} = \langle P, \leq \rangle$, $\alpha \in P$ and $h \geq 1$, we say that α *has at least depth* h *in* $\underline{P}$ iff there exist $\beta_1, \ldots, \beta_h \in P$ such that $\beta_1 = \alpha$, β_h is a final element, and, for all $2 \leq i \leq h$, β_i is an immediate successor of β_{i-1}; we say that α *has at most depth* h *in* $\underline{P}$ iff it is not the case that α has at least depth $h+1$ in $\underline{P}$; we say that α *has depth* h *in* $\underline{P}$ iff α has at least depth h in $\underline{P}$ and α has at most depth h in $\underline{P}$.

If $\mathcal{F}$ is a non empty class of posets, the set

$$\{\underline{K} = \langle P, \leq, \Vdash \rangle \mid \langle P, \leq \rangle \in \mathcal{F}\}$$

will be denoted by $\mathcal{K}(\mathcal{F})$, and we will call it the *class of Kripke models built on* $\mathcal{F}$. Also, for every non empty class $\mathcal{F}$ of frames, $\mathcal{L}(\mathcal{F})$ will indicate the set of wff's

$$\{A \mid \text{ for every } \underline{K} = \langle P, \leq, \Vdash \rangle \in \mathcal{K}(\mathcal{F}) \text{ and for every } \alpha \in P,\ \alpha \Vdash A\}\ .$$

It is well known that, for every non empty class $\mathcal{F}$ of frames, $\mathcal{L}(\mathcal{F})$ is a (propositional) intermediate logic [Chagrov and Zakharyaschev, 1997; Ferrari and Miglioli, 1993; Ferrari and Miglioli, 1995a; Ferrari and Miglioli, 1995b; Miglioli, 1992]. In this paper we will treat intermediate logics **L** for which a Kripke frame semantics is known, that is, for any intermediate logic **L** studied in the paper we know at least a class of frames $\mathcal{F}_{\mathbf{L}}$ such that $\mathcal{L}(\mathcal{F}_{\mathbf{L}}) = \mathbf{L}$. According to this fact we will say that a Kripke model $\underline{K}$ is a **L**-*model*, to mean that $\underline{K}$ is built on a frame belonging to $\mathcal{F}_{\mathbf{L}}$.

In the paper we will present tableau calculi for some intermediate logics; these calculi will work on signed formulas: for any intermediate logic **L** we will use a set $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ of signs depending on **L**; given a wff in the propositional language, a *signed formula (swff) of* **L** will be any string $\mathcal{S}_i A$ with $\mathcal{S}_i \in \{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ and A a (propositional) wff.

We define here the main measures on the formulas and the signed formulas we will use in the paper:

Definition 2.1 ($\textsc{deg}(.)$, $\textsc{ic}(.)$ and $\prec$**)**

1. *The* degree of a wff A*, denoted by* $\textsc{deg}(A)$*, is defined as usual.*

2. *The* degree of a swff $\mathcal{S}A$*, denoted by* $\textsc{deg}(\mathcal{S}A)$*, coincides with the degree of* A*.*

3. *The* implicative complexity of A*, denoted by* $\textsc{ic}(A)$*, is defined as follows: if* A *is implication free, then* $\textsc{ic}(A) = 0$*; if* $A \equiv B \to C$*, then* $\textsc{ic}(A)$ *coincides with* $\textsc{deg}(B)$*; if* $A \equiv \neg B$*, then* $\textsc{ic}(A) \equiv \textsc{ic}(B)$*; if* $A \equiv B \wedge C$ *or* $A \equiv B \vee C$*, then* $\textsc{ic}(A) = \textsc{max}(\textsc{ic}(B), \textsc{ic}(C))$*.*

4. *For a logic* **L** *different from the logic* **Dum** *considered in the next section, the well founded relation* $\prec$ *on pairs of swff's is defined as follows:* $\mathcal{S}A \prec \mathcal{S}'A'$ *iff either* $\textsc{deg}(\mathcal{S}A) < \textsc{deg}(\mathcal{S}'A')$*, or* $\textsc{deg}(\mathcal{S}A) = \textsc{deg}(\mathcal{S}'A')$ *and* $\textsc{ic}(A) < \textsc{ic}(A')$*.*

5. *For the logic* **Dum** *considered in the next section, the well founded relation* $\prec$ *on pairs of swff's is defined as follows:* $\mathcal{S}A \prec \mathcal{S}'A'$ *iff either* $\textsc{deg}(\mathcal{S}A) < \textsc{deg}(\mathcal{S}'A')$*, or* $\textsc{deg}(\mathcal{S}A) = \textsc{deg}(\mathcal{S}'A')$ *and* $\mathcal{S} \equiv \mathbf{F}$ *and* $\mathcal{S}' \equiv \mathbf{F_c}$ *(see the signs related to* **Dum***), or* $\textsc{deg}(\mathcal{S}A) = \textsc{deg}(\mathcal{S}'A')$ *and* $\mathcal{S} \equiv \mathcal{S}' \equiv \mathbf{T}$ *(see the signs related to* **Dum***) and* $\textsc{ic}(A) < \textsc{ic}(A')$*.*

3 Dummett Logic

Dummett Logic is the intermediate logic

$$\mathbf{Dum} = \mathbf{INT} + ((p \to q) \vee (q \to p))$$

obtained by adding to intuitionistic logic the single axiom scheme $((p \to q) \vee (q \to p))$. It is well known (see e.g. [Chagrov and Zakharyaschev, 1997]) that $\mathbf{Dum} = \mathcal{L}(\mathcal{F}_{\mathbf{Dum}})$, where $\mathcal{F}_{\mathbf{Dum}}$ is the class of rooted and linearly ordered Kripke frames $\underline{P} = \langle P, \leq, r \rangle$ (that is the class of all the Kripke frames $\underline{P} = \langle P, \leq \rangle$ with least element and such that, for every $\alpha, \beta \in P$, $\alpha \leq \beta$ or $\beta \leq \alpha$).

The tableau calculus for Dummett Logic uses the signs **T**, **F** and $\mathbf{F_c}$ whose meanings are explained in terms of **Dum**-*realizability* as follows: given a **Dum**-model $\underline{K} = \langle P, \leq, \Vdash \rangle$ and a swff H, we say that an element $\alpha \in P$ **Dum**-*realizes* H, and we write $\alpha \rhd H$, if the following conditions hold:

a If $H \equiv \mathbf{T}A$, then $\alpha \Vdash A$;

b If $H \equiv \mathbf{F}A$, then $\alpha \nVdash A$;

c If $H \equiv \mathbf{F_c}A$, then $\alpha \Vdash \neg A$.

We say that α **Dum**-*realizes a set of swff's* S (and we write $\alpha \rhd S$) iff α **Dum**-realizes any swff in S. A set of swff's S is **Dum**-*realizable* iff there is some element α of a **Dum**-model $\underline{K}$ such that $\alpha \rhd S$.

By a *configuration* we mean a finite sequence $S_1/S_2/\ldots/S_n$ (with $n \geq 1$), where every S_j is a set of swff's. A *configuration is* **Dum**-*realizable* iff at least a S_j is **Dum**-realizable. A **Dum**-*proof table* is a finite sequence of applications of the rules of the calculus **Dum**-T (see Table 1, Table 2, and Table 3 below), starting from some configuration. A **Dum**-proof table is *closed* iff all the sets S_j of its final configuration are contradictory, where a set S is *contradictory* if one of the following conditions holds:

1. $\mathbf{T}A \in S$ and $\mathbf{F}A \in S$;

2. $\mathbf{T}A \in S$ and $\mathbf{F_c}A \in S$.

A proof of a wff B in **Dum**-T is a *closed proof-table* in **Dum**-T starting from $\mathbf{F}B$. Finally, we say that a set of swff's S is **Dum**-*consistent* iff no **Dum**-proof table starting from any finite subset of S is closed. We say that a set of swff's of the form $\{\mathbf{T}A, \mathbf{F}A\}$ or of the form $\{\mathbf{T}A, \mathbf{F_c}A\}$ is a complementary pair. It is immediate to verify that:

Proposition 3.1 *If S is a set of swff's and contains a complementary pair, then S is not* **Dum**-*realizable.* □

$\dfrac{S, \mathbf{T}(A \wedge B)}{S, \mathbf{T}A, \mathbf{T}B}\mathbf{T}\wedge$	$\dfrac{S, \mathbf{F}(A \wedge B)}{S, \mathbf{F}A / S, \mathbf{F}B}\mathbf{F}\wedge$	$\dfrac{S, \mathbf{F_c}(A \wedge B)}{S, \mathbf{F_c}A / S, \mathbf{F_c}B}\mathbf{F_c}\wedge$
$\dfrac{S, \mathbf{T}(A \vee B)}{S, \mathbf{T}A / S, \mathbf{T}B}\mathbf{T}\vee$	$\dfrac{S, \mathbf{F}(A \vee B)}{S, \mathbf{F}A, \mathbf{F}B}\mathbf{F}\vee$	$\dfrac{S, \mathbf{F_c}(A \vee B)}{S, \mathbf{F_c}A, \mathbf{F_c}B}\mathbf{F_c}\vee$
See TABLE 2	See TABLE 3	$\dfrac{S, \mathbf{F_c}(A \to B)}{S, \mathbf{F}(A \to B), \mathbf{F_c}B}\mathbf{F_c}{\to}$
$\dfrac{S, \mathbf{T}\neg A}{S, \mathbf{F_c}A}\mathbf{T}\neg$	See TABLE 3	$\dfrac{S, \mathbf{F_c}\neg A}{S, \mathbf{F}\neg A}\mathbf{F_c}\neg$

TABLE 1: TABLEAU CALCULUS FOR **Dum**

In the above TABLE 1, S_c is the *certain part of* S; formally:

$$S_c = \{\mathbf{T}X : \mathbf{T}X \in S\} \cup \{\mathbf{F_c}X : \mathbf{F_c}X \in S\}$$

$$\frac{S, \mathbf{T}(A \to B)}{S, \mathbf{F}A / S, \mathbf{T}B}\mathbf{T}{\to}_{AN} \quad \text{with } A \text{ atomic or negated.}$$

$$\frac{S, \mathbf{T}((A \wedge B) \to C)}{S, \mathbf{T}(A \to (B \to C))}\mathbf{T}{\to}\wedge \qquad \frac{S, \mathbf{T}((A \vee B) \to C)}{S, \mathbf{T}(A \to C), \mathbf{T}(B \to C)}\mathbf{T}{\to}\vee$$

$$\frac{S, \mathbf{T}((A \to B) \to C)}{S, \mathbf{F}(A \to B), \mathbf{T}(B \to C) / S, \mathbf{T}C}\mathbf{T}{\to}{\to}$$

TABLE 2: $\mathbf{T}{\to}$-RULES

Every rule of TABLE 1 and of TABLE 2 is applied to a swff of a set S_i occurring in a configuration $S_1/\ldots/S_i/\ldots$; e.g., the notation $S, \mathbf{T}(A \wedge B)$ points out that the rule $\mathbf{T}\wedge$ is applied to the swff $\mathbf{T}(A \wedge B)$ of the set $S \cup \{\mathbf{T}(A \wedge B)\}$, where S is possibly empty.

$$\frac{S, \mathbf{F}(A_1 \to B_1), \ldots, \mathbf{F}(A_n \to B_n), \mathbf{F}\neg C_1, \ldots, \mathbf{F}\neg C_m}{S_c, S^1_{\mathbf{F}\to}, \mathbf{T}A_1, \mathbf{F}B_1 \;/\ldots/\; S_c, S^n_{\mathbf{F}\to}, \mathbf{T}A_n, \mathbf{F}B_n \;/\; S_c, S^1_{\mathbf{F}\neg}, \mathbf{T}C_1 /\ldots/\; S_c, S^m_{\mathbf{F}\neg}, \mathbf{T}C_m}\mathbf{F}{\to}\neg$$

TABLE 3: $\mathbf{F}{\to}\neg$-RULES

In TABLE 3, $m \geq 0$ and $n \geq 0$ with $n + m \geq 1$, and:

- For $i = 1, \ldots, n$,

$$S^i_{\mathbf{F}\to} \;=\; \{\mathbf{F}(A_1 \to B_1), \ldots, \mathbf{F}(A_n \to B_n), \mathbf{F}\neg C_1, \ldots, \mathbf{F}\neg C_m\} \setminus \{\mathbf{F}(A_i \to B_i)\}\ ;$$

- For $j = 1, \ldots, m$,

$$S^j_{\mathbf{F}\neg} = \{\mathbf{F}(A_1 \to B_1), \ldots, \mathbf{F}(A_n \to B_n), \mathbf{F}\neg C_1, \ldots, \mathbf{F}\neg C_m\} \setminus \{\mathbf{F}\neg C_j\} .$$

We call **Dum**-*regular rule* any rule of the calculus **Dum**-T different from $\mathbf{F}\to\neg$ and we call **Dum**-*regular swff* any swff to which a regular rule is applicable.

We remark that all the rules of the calculus **Dum**-T are duplication-free, in the sense explained in [Avellone et al., 1997; Dyckhoff, 1992; Miglioli et al., 1994b; Miglioli et al., 1994a; Miglioli et al., 1997] and in the Introduction. For instance, the rule $\mathbf{F_c}\to$ replaces the swff $\mathbf{F_c}(A\to B)$ with the two swff's $\mathbf{F}(A\to B)$ and $\mathbf{F_c}B$ and, according to Point (5) of Definition 2.1, we have that both $\mathbf{F}(A\to B) \prec \mathbf{F_c}(A\to B)$ and $\mathbf{F_c}B \prec \mathbf{F_c}(A\to B)$.

The soundness theorem for the system **Dum**-T with respect to the Kripke semantics for the logic **Dum** has the following form:

Theorem 3.2 (Soundness) *If a* **Dum**-*proof table starting from a swff* $\mathbf{F}A$ *is closed, then* $A \in \mathbf{Dum}$.

Proof: Assume the contrary. Then there are a **Dum**-model $\underline{K} = \langle P, \leq, \Vdash\rangle$ and $\alpha \in P$ such that $\alpha \nVdash A$. This implies that $\alpha \rhd \mathbf{F}A$. Since the rules of **Dum**-T preserve **Dum**-realizability, we deduce that the final configuration of the closed **Dum**-proof table for $\{\mathbf{F}A\}$ is **Dum**-realizable. But this means that a complementary pair is **Dum**-realizable, contradicting Proposition 3.1. □

The proof of the above theorem is based on the fact that the rules of the calculus preserve **Dum**-realizability. More precisely, if a configuration is **Dum**-realized in a state of a Kripke model $\underline{K}$ built on the class of Kripke frames $\mathcal{F}_{\mathbf{Dum}}$, then the configuration obtained by applying to the former configuration one of the rules is **Dum**-realized in a (possibly different) element of $\underline{K}$. As a consequence, if a tableau starting from $\mathbf{F}A$ is closed, then $\mathbf{F}A$ is not **Dum**-realizable, and A is forced in all the elements of all Kripke models $\underline{K}$ built on the class of Kripke frames $\mathcal{F}_{\mathbf{Dum}}$. The proof requires a careful analysis of the various rules, but does not present particular difficulties. We only observe that the rules $\mathbf{F_c}\wedge$, $\mathbf{F_c}\to$, $\mathbf{F_c}\neg$, and $\mathbf{F}\to\neg$, which differ from the ones given in [Miglioli et al., 1994b; Miglioli et al., 1994a; Miglioli et al., 1997] for the intuitionistic calculus, are correct by the particular form of the frames for Dummett Logic. For instance:

–*Rule* $\mathbf{F_c}\to$. Let $S, \mathbf{F_c}(A\to B)$ be a set of swff's **Dum**-realized on an element α of a Kripke model $\underline{K}$; hence, $\alpha \Vdash \neg(A\to B)$. This implies that (i) for all the elements β of $\underline{K}$, if $\alpha \leq \beta$, then $\beta \nVdash A\to B$; therefore, for all elements β of $\underline{K}$ such that $\alpha \leq \beta$, there exists an element γ of $\underline{K}$ such that $\beta \leq \gamma$, $\gamma \Vdash A$ and $\gamma \nVdash B$. Hence (ii) for all elements β of $\underline{K}$, if $\alpha \leq \beta$ then $\beta \nVdash B$. Finally, by (i), $\alpha \rhd \mathbf{F}(A\to B)$ and, by (ii), $\alpha \rhd \mathbf{F_c}B$.

–*Rule* $\mathbf{F_c}\neg$. Let $S, \mathbf{F_c}\neg A$ be a set of swff's **Dum**-realized on an element α of a Kripke model $\underline{K}$; hence, $\alpha \Vdash \neg\neg A$. This implies that (i) for any element β of $\underline{K}$, if $\alpha \leq \beta$, then $\beta \nVdash \neg A$; hence we have that, for any element β of $\underline{K}$ such that $\alpha \leq \beta$, there exists an element γ of $\underline{K}$ such that $\beta \leq \gamma$ and $\gamma \Vdash A$. Therefore, $\alpha \rhd \mathbf{F}\neg A$.

–*Rule* $\mathbf{F}\to\neg$. Let S be a set of swff's **Dum**-realized on an element α of a Kripke model $\underline{K}$ such that the set S contains at least a swff of the kind $\mathbf{F}(A\to B)$ or a swff of the kind

$\mathbf{F}\neg A$. Now, let

$$\begin{aligned} S_{\mathbf{F}\wedge,\vee,a} &= \{\mathbf{F}(A\wedge B) \mid \mathbf{F}(A\wedge B)\in S\}\cup\{\mathbf{F}(A\vee B)\mid \mathbf{F}(A\vee B)\in S\}\cup \\ &\quad \cup\{\mathbf{F}a \mid \mathbf{F}a\in S \text{ and } a \text{ is atomic }\} \\ S_{\mathbf{F}\to,\neg} &= \{\mathbf{F}(A\to B)\mid \mathbf{F}(A\to B)\in S\}\cup\{\mathbf{F}\neg A\mid \mathbf{F}\neg A\in S\}\ . \end{aligned}$$

Moreover let $\{\mathbf{F}A_1\to B_1,\ldots,\mathbf{F}A_n\to B_n,\mathbf{F}\neg C_1,\ldots,\mathbf{F}\neg C_m\}$ be any subset of $S_{\mathbf{F}\to,\neg}$ and let $S'_{\mathbf{F}\to,\neg} = S\setminus\{\mathbf{F}A_1\to B_1,\ldots,\mathbf{F}A_n\to B_n,\mathbf{F}\neg C_1,\ldots,\mathbf{F}\neg C_m\}$. We have

$$S = S_c\cup S_{\mathbf{F}\wedge,\vee,a}\cup S'_{\mathbf{F}\to,\neg}\cup\{\mathbf{F}A_1\to B_1,\ldots,\mathbf{F}A_n\to B_n,\mathbf{F}\neg C_1,\ldots,\mathbf{F}\neg C_m\}$$

Since the elements of $\underline{K}$ are linearly ordered, there are a sequence $H_1,\ldots,H_k$ of the swff's of $\{\mathbf{F}A_1\to B_1,\ldots,\mathbf{F}A_n\to B_n,\mathbf{F}\neg C_1,\ldots,\mathbf{F}\neg C_m\}$ and a sequence (with possible repetitions) $\beta_1,\ldots,\beta_k$ of elements of $\underline{K}$ such that, for all $1\le i\le k-1$, $\beta_i\le\beta_{i+1}$ and, for all $1\le i\le k$,

- If $H_i\equiv\mathbf{F}(A\to B)$ then $\beta_i\rhd\mathbf{T}A$ and $\beta_i\rhd\mathbf{F}B$;
- If $H_i\equiv\mathbf{F}\neg A$ then $\beta_i\rhd\mathbf{T}A$.

Therefore, it is easy to prove that β_1 **Dum**-realizes the resulting configuration of the rule $\mathbf{F}_{\to,\neg}$. As a matter of fact, if $H_1\equiv\mathbf{F}(A_h\to B_h)$, for some $1\le h\le n$, β_1 **Dum**-realizes the set of swff $S_c, S^h_{\mathbf{F}_\to}, \mathbf{T}A_h, \mathbf{F}B_h$; otherwise, if $H_1\equiv\mathbf{F}\neg C_h$, for some $1\le h\le m$, β_1 **Dum**-realizes the set of swff $S_c, S^h_{\mathbf{F}_\neg}, \mathbf{T}A_h$.

Now we begin to discuss the problem of the completeness of the calculus **Dum**-T. The completeness theorem has the following form: *If a formula A is valid in any Kripke model built on a frame belonging to the class $\mathcal{F}_{\mathbf{Dum}}$, then there is a closed proof table in* **Dum**-T *starting from* $\mathbf{F}A$. According to the semantical interpretation of the swff's, it suffices to prove the following fact: *If a finite set S of swff's is* **Dum**-*consistent, then there is a Kripke model $\underline{K}$ built on a frame belonging to the class $\mathcal{F}_{\mathbf{Dum}}$ together with an element α of $\underline{K}$ such that S is* **Dum**-*realized on α.* Thus, following [Fitting, 1969; Miglioli et al., 1994b; Miglioli et al., 1994a; Miglioli et al., 1997], our proof is based on a general method allowing to built up a rooted Kripke model $\underline{K}_{\mathbf{Dum}}(S)$ on which a **Dum**-consistent set of swff's S is **Dum**-realized by the root.

Before explaining the construction of the model $\underline{K}_{\mathbf{Dum}}(S)$, we introduce the following notion:

Definition 3.3 *If* **L** *is one of the logics considered in this paper,* **L**-T *is the related tableau calculus, and H is a swff, we call* **L**-extension(s) of H *the set(s) associated with H as follows:*

1. *If H is not a* **L**-*splitting swff (i.e. the application of the rule related to H gives rise to a single configuration), then the only* **L**-*extension of H is the only set $\mathcal{R}_H$ coinciding with the configuration obtained by applying the rule related to H in* **L**-T *to the configuration $\{H\}$.*
2. *If H is a* **L**-*splitting swff then the* **L**-*extensions of H are the sets $\mathcal{R}^1_H,\ldots,\mathcal{R}^n_H$ coinciding with the various sets in the configuration obtained by applying the rule related to H in* **L**-T *to the configuration $\{H\}$.*

Now we turn to the construction of the model $\underline{K}_{\mathbf{Dum}}(S)$; this is done in two main steps. In the first step, starting from a **Dum**-consistent set of swff's S, we construct two sets S^* and $\overline{S}$, called the **Dum**-*saturated set* of S and the **Dum**-*node set of* S respectively. The set $\overline{S}$ will be the root of the model $\underline{K}_{\mathbf{Dum}}(S)$, and the swff's in $\overline{S}$ will determine the forcing relation in $\overline{S}$. In the second step we will construct a set which will be called the **Dum**-successor of $\overline{S}$. The model $\underline{K}_{\mathbf{Dum}}(S)$ will be constructed by iterating the two steps on the new element, and so on.

Step A

Let $A_1, \dots, A_n \dots$ be any listing of the swff's of S (without duplications of swff's); starting from this enumeration we construct the following sequence $\{S_i\}_{i\in\omega}$ $(i \geq 0)$ of set of swff's:

- $S_0 = \emptyset$;
- Let $S_i = \{H_1, \dots, H_k\}$; then

$$S_{i+1} = \bigcup_{H_j \in S_i} \mathcal{U}(H_j, i) \cup \{A_{i+1}\}$$

where, setting

$$S' = \mathcal{U}(H_1, i) \cup \cdots \cup \mathcal{U}(H_{j-1}, i) \cup \{H_j, \dots, H_k, A_{i+1}, A_{i+2}, \dots\} ,$$

we have:

1. If H_j is a **Dum**-regular swff different from $\mathbf{T}(A \to B)$ with A atomic, then $\mathcal{U}(H_j, i)$ is any **Dum**-extension $\mathcal{R}_{H_j}$ of H_j such that $(S' \setminus \{H_j\}) \cup \mathcal{R}_{H_j}$ is **Dum**-consistent.
2. If H_j is of the kind $\mathbf{T}(A \to B)$ with A atomic and $(S' \setminus \{\mathbf{T}(A \to B)\}) \cup \{\mathbf{T}B\}$ is **Dum**-consistent, then $\mathcal{U}(H_j, i) = \{\mathbf{T}B\}$.
3. In all the other cases, $\mathcal{U}(H_j, i) = \{H_j\}$.

It is easy to prove, by induction on $i \geq 0$, that, if S is **Dum**-consistent, then any S_i is **Dum**-consistent. This allows to deduce that also the **Dum**-*saturated set of* S

$$S^* = \bigcup_{i \geq 0} S_i$$

is **Dum**-consistent. Now, we call **Dum**-*final in a set* V (of swff's) any swff $H \in V$ such that

1. No **Dum**-regular rule can be applied to H;
2. $H \equiv \mathbf{T}(A \to B)$ with A atomic, and $\mathbf{T}B \notin V$.

We call **Dum**-*node set of* S the set:

$$\overline{S} = \{H \mid H \text{ is } \mathbf{Dum}\text{-final in } S^*\}$$

Obviously, for any **Dum**-consistent set of swff's there exists at least a **Dum**-consistent **Dum**-node set of S. We remark that if S is a finite set S, then all the sets S^* and $\overline{S}$ are finite too.

Step B

Let $\overline{S}$ be a non empty finite node set of a **Dum**-consistent and finite set of swff's S; we define the **Dum**-*successor set* U *of* $\overline{S}$ as follows:

- U is one of the consistent sets obtained by applying the rule $\mathbf{F}\rightarrow\neg$ to $\overline{S}$, taking as $\{\mathbf{F}(A_1\rightarrow B_1),\ldots,\mathbf{F}(A_n\rightarrow B_n),\mathbf{F}\neg C_1,\ldots,\mathbf{F}\neg C_m\}$ (see Table 3) the set $\{\mathbf{F}(A\rightarrow B) \mid \mathbf{F}(A\rightarrow B)\in\overline{S}\}\cup\{\mathbf{F}\neg C \mid \mathbf{F}\neg C\in\overline{S}\}$.

Now, given a **Dum**-consistent and finite set of swff's S, we define the structure $\underline{K}_{\mathbf{Dum}}(S) = \langle P,\leq,\Vdash\rangle$ as follows:

1. The root of $\underline{K}_{\mathbf{Dum}}(S)$ is a node set $\overline{S}$ of S;
2. For any $\overline{\Gamma}\in P$, let $\overline{U}$ be the **Dum**-node set of the **Dum**-successor set U of $\overline{\Gamma}$; then $\overline{U}$ belongs to P and $\overline{U}$ is an immediate successor of $\overline{\Gamma}$ in $\underline{K}_{\mathbf{Dum}}(S)$;
3. $\leq$ is the transitive and reflexive closure of the immediate successor relation;
4. For any element $\overline{\Gamma}\in P$ and for any propositional variable p, $\overline{\Gamma}\Vdash p$ iff $\mathbf{T}p\in\overline{\Gamma}$

It is easy to verify that, given a **Dum**-consistent and finite set of swff's S, any $\underline{K}_{\mathbf{Dum}}(S) = \langle P,\leq,\Vdash\rangle$ is a finite Kripke model built on a rooted and linearly ordered frame, that is $\underline{K}_{\mathbf{Dum}}(S)$ is a **Dum**-model.

The following lemma is the main step towards the proof of the completeness theorem.

Lemma 3.4 *Let S be a* **Dum**-*consistent and finite set of swff's and let* $\underline{K}_{\mathbf{Dum}}(S) = \langle P,\leq,\Vdash\rangle$ *be one of the* **Dum**-*models defined above. Then, for any* $\overline{\Gamma}\in P$ *and for any swff* $H\in\Gamma^*$, $\overline{\Gamma}\rhd H$ *in* $\underline{K}_{\mathbf{Dum}}(S)$.

Proof: The proof is by induction on the well founded relation $\prec$ of Definition 2.1.

Basis: Let $\mathcal{S}A\in S^*$ with A of degree zero, i.e., A is an atomic wff; then $\mathcal{S}A\in\overline{\Gamma}$. Now, if $\mathcal{S}$ is $\mathbf{T}$, $\mathbf{T}A$ is **Dum**-realized in $\overline{\Gamma}$ by the definition of forcing; if $\mathcal{S}$ is $\mathbf{F}$, then, since $\overline{\Gamma}$ is **Dum**-consistent, $\mathbf{T}A$ cannot belong to $\overline{\Gamma}$, hence $\overline{\Gamma}\nVdash A$ which implies that $\mathbf{F}A$ is **Dum**-realized in $\overline{\Gamma}$. Finally, if $\mathcal{S}$ is $\mathbf{F_c}$, $\mathbf{F_c}A$ belongs to $\overline{\Gamma}$ and to any element $\overline{\Delta}$ in $\underline{K}_{\mathbf{Dum}}(S)$ such that $\overline{\Gamma}\leq\overline{\Delta}$ (if any); since any such $\overline{\Delta}$ is **Dum**-consistent, we have that $\overline{\Delta}\nVdash A$ for any element in $\underline{K}_{\mathbf{Dum}}(S)$ such that $\overline{\Gamma}\leq\overline{\Delta}$, therefore $\overline{\Gamma}\Vdash\neg A$, that is $\overline{\Gamma}\rhd\mathbf{F_c}A$.

Step: Suppose, by induction hypothesis, that the lemma holds for any swff H' such that $H'\prec H$. The proof goes on by cases according to the form of H. Here we only give some illustrative examples.

Case $H\equiv\mathbf{T}((B\rightarrow C)\rightarrow D)$: then $H\in\Gamma^*$ implies $\mathcal{U}(H,i)\subseteq\Gamma^*$ (for some i), where $\mathcal{U}(H,i)$ is either $\{\mathbf{T}D\}$ or $\{\mathbf{F}(B\rightarrow C),\mathbf{T}(C\rightarrow D)\}$. Thus, by induction hypothesis, either $\overline{\Gamma}$ **Dum**-realizes $\mathbf{T}D$ and hence $\mathbf{T}((B\rightarrow C)\rightarrow D)$, or $\overline{\Gamma}$ **Dum**-realizes (i) $\mathbf{F}(B\rightarrow C)$ and (ii) $\mathbf{T}(C\rightarrow D)$. In the latter case, let $\overline{\Delta}\in P$ be such that $\overline{\Gamma}\leq\overline{\Delta}$ and (iii) $\overline{\Delta}\Vdash B\rightarrow C$. We prove that $\overline{\Delta}\Vdash D$. As a matter of fact, let Δ^* be the saturated set related to $\overline{\Delta}$; by construction, since $\mathbf{F}(B\rightarrow C)\notin\Delta^*$ (otherwise, by induction hypothesis, $\overline{\Delta}\nVdash B\rightarrow C$), there is $\overline{\Theta}$ such that $\overline{\Gamma}\leq\overline{\Theta}\leq\overline{\Delta}$ and the swff's $\mathbf{T}B$ and $\mathbf{F}C$ belong to the saturated set related to $\overline{\Theta}$; hence $\mathbf{T}B$ is **Dum**-realized in $\overline{\Theta}$ and *a fortiori* $\mathbf{T}B$ is **Dum**-realized in $\overline{\Delta}$.

By (ii) and (iii), the latter fact implies that $\overline{\Delta} \Vdash D$.

Case $H \equiv \mathbf{T}(B \to C)$, with B an atomic wff: We have to prove that, for every $\overline{\Delta}$ such that $\overline{\Gamma} \leq \overline{\Delta}$, either $\overline{\Delta} \nVdash B$ or $\overline{\Delta} \Vdash C$. We have two cases:

1. $\mathbf{T}(B \to C) \in \overline{\Delta}$; then $\mathbf{T}(B \to C)$ is **Dum**-final in Δ^*, being Δ^* the saturated set related to $\overline{\Delta}$. In this case $\mathbf{T}C \notin \Delta^*$ and $\{\mathbf{T}C\} \cup \Delta^*$ is not **Dum**-consistent (by the construction defined in Step A), while $\{\mathbf{F}B\} \cup \Delta^*$ is **Dum**-consistent; this implies that $\mathbf{T}B \notin \overline{\Delta}$. It follows, by definition of the forcing relation, that $\overline{\Delta} \nVdash B$.

2. $\mathbf{T}(B \to C) \notin \overline{\Delta}$; then, there are $\overline{\Theta}$ and $\overline{\Lambda}$ such that $\overline{\Gamma} \leq \overline{\Theta} \leq \overline{\Lambda} \leq \overline{\Delta}$, $\mathbf{T}(B \to C) \in \overline{\Theta}$, $\mathbf{T}(B \to C) \notin \overline{\Lambda}$ and $\overline{\Lambda}$ is an immediate successor of $\overline{\Theta}$. Since $\mathbf{T}(B \to C) \in \overline{\Theta}$, by construction of $\underline{K}_{\mathbf{Dum}}(S)$ we have that $\mathbf{T}(B \to C) \in \Lambda^*$, where Λ^* is the saturated set related to $\overline{\Lambda}$; on the other hand, since $\mathbf{T}(B \to C) \notin \overline{\Lambda}$, we have that $\mathbf{T}(B \to C)$ is not **Dum**-final in Λ^*, hence (by definition of Λ^*) $\mathbf{T}C \in \Lambda^*$. The latter fact implies, by induction hypothesis, that $\overline{\Lambda} \Vdash C$; *a fortiori*, $\overline{\Delta} \Vdash C$.

Case $H \equiv \mathbf{F_c}(B \to C)$: then H belongs to Γ^* implies $\mathcal{U}(H, i) \subseteq \Gamma^*$ (for some i), where $\mathcal{U}(H, i)$ is $\{\mathbf{F}(B \to C), \mathbf{F_c}C\}$; hence, by induction hypothesis, (i) $\overline{\Gamma} \Vdash \neg C$ and, by construction of $\underline{K}_{\mathbf{Dum}}(S)$, there is $\overline{\Delta}$ such that $\overline{\Gamma} \leq \overline{\Delta}$, and the swff's $\mathbf{T}B$ and $\mathbf{F}C$ belong to the saturated set related to $\overline{\Delta}$; hence, by induction hypothesis, (ii) $\overline{\Delta} \Vdash B$ and $\overline{\Delta} \nVdash C$. Now, let $\overline{\Theta}$ be any element of $\underline{K}_{\mathbf{Dum}}(S)$ such that $\overline{\Gamma} \leq \overline{\Theta}$. We must prove that $\overline{\Theta} \nVdash B \to C$. By the linearity of $\underline{K}_{\mathbf{Dum}}(S)$, we have two cases:

1. If $\overline{\Gamma} \leq \overline{\Theta} \leq \overline{\Delta}$ then the assertion immediately follows, since $\overline{\Delta} \Vdash B$ and $\overline{\Delta} \nVdash C$.

2. If $\overline{\Delta} \leq \overline{\Theta}$ and $\overline{\Theta}$ is different from $\overline{\Delta}$, then, since $\overline{\Gamma} \leq \overline{\Delta} \leq \overline{\Theta}$, by (i), $\overline{\Theta} \Vdash \neg C$ and, by (ii), $\overline{\Theta} \Vdash B$; hence the assertion.

Case $H \equiv \mathbf{F}(B \to C)$: Then, by construction of $\underline{K}_{\mathbf{Dum}}(S)$, there is an element $\overline{\Delta}$ such that $\overline{\Gamma} \leq \overline{\Delta}$ and the swff's $\mathbf{T}B$ and $\mathbf{F}C$ belong to the saturated set related to $\overline{\Delta}$. Therefore, by induction hypothesis, $\overline{\Delta} \Vdash B$ and $\overline{\Delta} \nVdash C$, hence $\overline{\Gamma} \nVdash B \to C$.

Case $H \equiv \mathbf{F}\neg B$: Then, by construction of $\underline{K}_{\mathbf{Dum}}(S)$, there is an element $\overline{\Delta}$ such that $\overline{\Gamma} \leq \overline{\Delta}$ and the swff $\mathbf{T}B$ belongs to the saturated set related to $\overline{\Delta}$. Therefore, by induction hypothesis, $\overline{\Delta} \Vdash B$, hence $\overline{\Gamma} \nVdash \neg B$. □

From the previous Lemma we finally obtain:

Theorem 3.5 (Completeness) *If a wff A is* **Dum***-valid, then there exists a closed* **Dum***-proof table starting from* $\mathbf{F}A$.

Proof: Suppose the contrary; then $\{\mathbf{F}A\}$ is a **Dum**-consistent set of swff's. By Lemma 3.4, this implies that $\mathbf{F}A$ is **Dum**-realizable and this is not possible. □

As anticipated in the Introduction, our tableau calculus for **Dum** can be translated into a multi-succedent cut-free sequent calculus. The translation can be done as follows:

1. One has to reverse the rules, i.e. the configuration above the line is to be put below, and the configuration below the line is to be put above.

2. Each set of swff's in a configuration is translated into a sequent, where:

(a) In the left hand part of the sequent one has to put the **T**-signed formulas (of course without the sign **T**) and the negations of the $\mathbf{F_c}$-signed formulas (of course without the sign $\mathbf{F_c}$);

(b) In the right hand part of the sequent one has to put the **F**-signed formulas (of course without the sign **F**). If the set of signed formulas does not contain any **F**-signed formula, then the right hand part of the sequent is empty (of course the intended meaning of a sequent with an empty right hand part is that the left hand part is inconsistent).

3. The contradictory sets of swff's of the tableau calculus become the starting sequents (axiom-sequents) of the sequent calculus (they are sequents of the form $\Gamma, A \vdash A, \Delta$ or $\Gamma, A, \neg A \vdash \Delta$).

To help the reader we explicitly give the rules of the sequent calculus coming from the above translation.

Axioms:

$$\frac{}{\Gamma, A \vdash A, \Delta} \qquad \frac{}{\Gamma, A, \neg A \vdash \Delta}$$

Rules for $\wedge$:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\text{L}\wedge \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\text{R}\wedge$$

Rules for $\vee$:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\text{L}\vee \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\text{R}\vee$$

Left Rules for $\rightarrow$:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\text{L}\!\rightarrow\!\text{AN} \qquad \text{where } A \text{ is atomic or negated}$$

$$\frac{\Gamma, A \rightarrow (B \rightarrow C) \vdash \Delta}{\Gamma, (A \wedge B) \rightarrow C \vdash \Delta}\text{L}\!\rightarrow\!\wedge \qquad \frac{\Gamma, A \rightarrow C, B \rightarrow C \vdash \Delta}{\Gamma, A \vee B \rightarrow C \vdash \Delta}\text{L}\!\rightarrow\!\vee$$

$$\frac{\Gamma, B \rightarrow C \vdash A \rightarrow B, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, (A \rightarrow B) \rightarrow C \vdash \Delta}\text{L}\!\rightarrow\rightarrow$$

Left Rules for ¬:

$$\frac{\Gamma, \neg A \vdash \Delta \quad \Gamma, \neg B \vdash \Delta}{\Gamma, \neg(A \wedge B) \vdash \Delta}\text{L}\neg\wedge \qquad \frac{\Gamma, \neg A, \neg B \vdash \Delta}{\Gamma, \neg(A \vee B) \vdash \Delta}\text{L}\neg\vee$$

$$\frac{\Gamma, \neg B \vdash A \to B, \Delta}{\Gamma, \neg(A \to B) \vdash \Delta}\text{L}\neg\to \qquad \frac{\Gamma \vdash \neg A, \Delta}{\Gamma, \neg\neg A \vdash \Delta}\text{L}\neg\neg$$

Right Rule for → and ¬:

$$\frac{\Gamma, A_1 \vdash B_1, \Pi^{A_1 \to B_1} \ \ldots \ \Gamma, A_n \vdash B_n, \Pi^{A_n \to B_n} \quad \Gamma, C_1 \vdash \Pi^{\neg C_1} \ \ldots \ \Gamma, C_m \vdash \Pi^{\neg C_m}}{\Gamma \vdash A_1 \to B_1, \ldots, A_n \to B_n, \neg C_1, \ldots, \neg C_m, \Delta}\text{R}\to\neg$$

where $\Pi = \{A_1 \to B_1, \ldots, A_n \to B_n, \neg C_1, \ldots, \neg C_m\}$ and Π^H is $\Pi \setminus \{H\}$.
Remarks:

(**R 3.1**) – As the reader can see, the rules coming from the above translation are correct, i.e. for every rule

$$\frac{\Gamma_1 \vdash \Delta_1 \qquad \ldots \qquad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

of the resulting sequent calculus, one has: if $\Gamma_1 \models \mathbf{Dum}\Delta_1$ and ... and $\Gamma_n \models \mathbf{Dum}\Delta_n$, then $\Gamma \models \mathbf{Dum}\Delta$, where the notation $\Gamma' \models \mathbf{Dum}\Delta'$ means that any state of any **Dum**-model forces some formula of Δ' whenever it forces all the formulas of Γ'. Indeed, assuming that such a rule is not correct, one gets a contradiction, as an immediate consequence of the correctness of the corresponding tableau rule. Thus, also the proof of the soundness theorem for the tableau calculus can be translated into a proof of the soundness theorem for the corresponding sequent calculus. Likewise, one can *directly* look at the proof of the completeness theorem given above for the tableau calculus as a completeness proof, of the usual kind, for the sequent calculus, according to which one gets that $\Gamma' \models \mathbf{Dum}\Delta'$ does not hold if there is no proof of $\Gamma' \vdash \Delta'$ in the considered sequent calculus for **Dum**.

(**R 3.2**) – According to the previous discussion, a proof of the completeness theorem such as the above can be immediately interpreted as a cut-elimination proof carried out by means of semantical tools. This should put into evidence the proof-theoretic content involved in the Kripke semantics for intermediate logics, which allows to capture (yet avoiding a lot of idiosyncratic tricks and combinatorial details) aspects which are usually considered typical of the syntactical realm of Proof Theory.

(**R 3.3**) – Of course, unlike our semantical proofs, a traditional proof of cut-elimination provides a strategy to eliminate cuts in proofs (in our calculi the cut rule is not even included, since it is unnecessary to get the completeness). However, we believe that the semantical approach (producing cut-free calculi via semantically oriented tableau calculi) has a greater heuristic power. We do not see how to start with a massive search for cut-free sequent calculi involving a great number of propositional and predicate logics (in a context far exceeding the few, well known cut-free sequent calculi) following the traditional heuristics of Proof Theory. Even the above calculus for Dummett Logic hardly could have been inspired by that tradition.

(**R 3.4**) – Our semantical approach to the search for cut-free sequent calculi seems to have a great flexibility, which allows to provide different calculi for the same logic, with remarkable differences in the characterization of the logical constants. For instance, the above tableau calculus "interprets" the sign $\mathbf{F_c}$ in terms of the sign $\mathbf{F}$ both in connection with negation and in connection with implication; this corresponds, in the related sequent calculus, to a transfer of formulas from the right hand side of the sequents to the left hand side (with the addition of a negation); also, such a treatment of the sign $\mathbf{F_c}$ is linked with the particular construction of the counter-model explained above in the proof of the Completeness Theorem. On the other hand, one can get a different, complete and duplication-free tableau calculus (still giving rise, with the translation illustrated above, to a cut-free sequent calculus) by "interpreting" (with a transfer of formulas in the opposite direction) the sign $\mathbf{F}$ for negated formulas in terms of $\mathbf{F_c}$. In this frame, one gets the new tableau calculus by replacing the rules $\mathbf{F_c}\neg$ and $\mathbf{F_c}\rightarrow$ respectively with the rules new-$\mathbf{F_c}\neg$ and new-$\mathbf{F_c}\rightarrow$ below

$$\frac{S, \mathbf{F_c}\neg A}{S_c, \mathbf{T}A}\,\text{new-}\mathbf{F_c}\neg \qquad\qquad \frac{S, \mathbf{F_c}(A\rightarrow B)}{S_c, \mathbf{T}A, \mathbf{F_c}B}\,\text{new-}\mathbf{F_c}\!\rightarrow$$

and splitting the rule $F\rightarrow\neg$ into the two rules new-$\mathbf{F}\neg$ and new-$\mathbf{F}\rightarrow$ below

$$\frac{S, \mathbf{F}\neg A}{S, \mathbf{F_c}\neg A}\,\text{new-}\mathbf{F_c}\neg \qquad \frac{S, \mathbf{F}(A_1\rightarrow B_1), \ldots, \mathbf{F}(A_n\rightarrow B_n)}{S_c, S^1_{\mathbf{F}\rightarrow}, \mathbf{T}A_1, \mathbf{F}B_1 \;/\ldots/\; S_c, S^n_{\mathbf{F}\rightarrow}, \mathbf{T}A_n, \mathbf{F}B_n}\,\text{new-}\mathbf{F}\!\rightarrow$$

where, in new-$\mathbf{F}\rightarrow$, $n \geq 1$ and, for any integer $i : 1 \leq i \leq n$, $S^i_{\mathbf{F}\rightarrow} = \{\mathbf{F}(A_1 \rightarrow B_1), \ldots, \mathbf{F}(A_n\rightarrow B_n)\} \setminus \{\mathbf{F}(A_i\rightarrow B_i)\}$.

The completeness proof of the new calculus can be given along the lines described for the former calculus, but requires a special strategy, according to which one has to take into account appropriate sequences of applications of the rules $\mathbf{F_c}\neg$ and $\mathbf{F_c}\rightarrow$, each application, moreover, involving only sets of swff's whose only $\mathbf{F}$-swff's are atomic.

(**R 3.5**) – Still another tableau calculus can be provided for **Dum**, if one prefers to work with a propositional language not containing negation and having $\bot$ (falsehood) as a primitive symbol. To get this calculus one uses only two signs, $\mathbf{T}$ and $\mathbf{F}$, and takes as contradictory sets of swff's having one of the two following forms: $\{S, \mathbf{T}A, \mathbf{F}A\}$, $\{S, , \mathbf{T}\bot\}$.

In this context the $\mathbf{F_c}$-rules disappear, as well as the rules $\mathbf{T}\neg$ and $\mathbf{F}\neg$. On the other hand, the rules $\mathbf{T}\wedge$, $\mathbf{F}\wedge$, $\mathbf{T}\vee$, $\mathbf{F}\vee$, $\mathbf{T}\rightarrow AN$, $\mathbf{T}\rightarrow\vee$, $\mathbf{T}\rightarrow\wedge$ and $\mathbf{T}\rightarrow\rightarrow$ remain unchanged (in absence of negation $\mathbf{T}\rightarrow AN$ is called $\mathbf{T}\rightarrow A$); this set of rules is then completed with the rule new-$\mathbf{F}\rightarrow$ described in the previous remark. The resulting calculus turns out to be complete and duplication free, as the previous ones; also, its completeness proof can be carried out along the same lines; finally, it can be straightforwardly translated into a cut-free sequent calculus (the translations is the one previously discussed, but is simpler, since the sign $\mathbf{F_c}$ is not to be taken into account).

(**R 3.6**) – As it is well known, and has been discussed in the Introduction, **Dum** is one of the seven interpolable intermediate propositional logics; moreover, everyone knows that there are connections between the interpolability property of a formal system and the existence of a cut-free sequent calculus for the system (even if an exact characterization of the state of the affairs in this area has not yet been provided). In this line, it may be interesting to remark that an almost routine proof of the interpolation lemma can be given for **Dum** on the basis of each of the three calculi explained above (the calculus without negation as a primitive symbol is the more suitable). We omit here such a proof, which is rather cumbersome and essentially follows the pattern presented by Fitting in [Fitting, 1969] for Intuitionistic Logic.

(**R 3.7**) – According to a traditional paradigm of Proof Theory, the main interest of the eliminability of cuts lies in the possibility of getting proofs with the subformula property, i.e., proofs where any formula occurring in any sequent which is a premise of any inference rule occurs as a subformula of some formula belonging to the sequent obtained by applying the rule. Indeed, the subformula property is satisfied by the well-known cut-free Gentzen's calculi and their main variants.

On the other hand, in a wider context such as the one we are considering in the present paper, the situation is more intricate; indeed, the absence of the cut-rule does not necessarily yield a literal satisfaction of the subformula property. Form this point of view, the fact that our sequent calculi are duplication-free (or contraction-free, according to the terminology of Dyckhoff [Dyckhoff, 1992]) prevents them from satisfying the subformula property as it is usually meant. But what matters, we believe, is that the "very spirit" of the subformula property is preserved in such calculi, i.e., any formula occurring in any sequent which is a premise of an application of an inference rule can be built (in a simple and direct way) starting from subformulas of formulas occurring in the sequent obtained by the application of the rule. It remains, however, the problem of providing, in a clear and general way, a new condition (different from the old one based on the literal satisfaction of the subformulas property, but capturing the "essential features" of this) allowing to separate the "good" sequent calculi from the "bad" ones, in a framework which is becoming considerably wider than the traditional one.

(**R 3.8**) – Coming back to the discussion made in the previous remark, we believe that it is more important to get duplication-free tableau calculi than to preserve the subformula property, especially in connection with the search for efficient proof strategies. However, we wish to point out that cut-free sequent calculi with the subformula property as it is usually meant can be given for **Dum**. To do so, let us come back to the tableau calculus for **Dum** described in Remark **R 3.5**, whose wff's do not contain negation as a primitive symbol and whose only signs are **T** and **F**. Let us replace in that calculus the rules $\mathbf{T}{\to}A$, $\mathbf{T}{\to}\vee$, $\mathbf{T}{\to}\wedge$ and $\mathbf{T}{\to}{\to}$ with the following single rule

$$\frac{S, \mathbf{T}(A \to B)}{S, \mathbf{T}(A \to B), \mathbf{F}A \;/\; S, \mathbf{T}B}\mathbf{T}{\to}\,.$$

This provides a complete tableau calculus for **Dum** where the rule $\mathbf{T}\to$ gives rise to duplications (the source of duplications is explicitly put into evidence in the left-hand side set of swff's below the line); the corresponding sequent calculus (obtained with the translation previously described) is not contraction-free, but is cut-free and satisfies the subformulas property.

4 Peirce Logic

The logic we consider here is known in literature as *Peirce Logic*. It is the propositional intermediate logic

$$\mathbf{PRC} = \mathbf{INT} + ((\neg q \to p) \to (((p \to q) \to p) \to p))$$

which can be equivalently axiomatized by adding to **INT** the two axiom-schemes $((p \to q) \vee (q \to p))$ and $(p \vee (p \to q \vee \neg q))$. Semantically, it is the logic characterized by the class of posets $\mathcal{F}_{\mathbf{PRC}}$ containing any rooted poset $\langle P, \leq \rangle$ such that $|P| \leq 2$; in other words, $\mathcal{F}_{\mathbf{PRC}}$ is the class of posets consisting of all the rooted frames with at most an element different from the root (see [Chagrov and Zakharyaschev, 1997]).

To give a duplication-free tableau calculus for Peirce Logic, we use the signs **T**, **F**, $\mathbf{F_c}$ and $\mathbf{T_{cl}}$. The meaning of these signs is explained in terms of **PRC**-*realizability* as follows: given a **PRC**-model $\underline{K} = \langle P, \leq, \Vdash \rangle$ and a swff H, we say that an element $\beta \in P$ **PRC**-*realizes* H, and we write $\beta \rhd H$, if (according to the structure of H) the following conditions hold:

a If $H \equiv \mathbf{T}A$, then $\beta \Vdash A$;

b If $H \equiv \mathbf{F}A$, then $\beta \nVdash A$;

c If $H \equiv \mathbf{F_c}A$, then $\beta \Vdash \neg A$;

d If $H \equiv \mathbf{T_{cl}}A$, then $\beta \Vdash \neg\neg A$.

We say that β **PRC**-realizes a set of swff's S iff β **PRC**-realizes any swff in S. A set of swff's S is **PRC**-*realizable* iff there is some element α of a **PRC**-model $\underline{K}$ such that $\alpha \rhd S$.

A *configuration* $S_1/\ldots/S_n$ *(*$n \geq 1$*) is* **PRC**-*realizable* iff at least a S_j $(1 \leq j \leq n)$ is **PRC**-realizable. A **PRC**-*proof table* is a finite sequence of applications of the rules of the calculus **PRC**-T (see Table 4 below) starting from some configuration. A **PRC**-proof table is *closed* iff all the sets S_j of its final configuration are contradictory, where S is *contradictory* if one of the following conditions holds:

1. $\mathbf{T}A \in S$ and $\mathbf{F}A \in S$;

2. $\mathbf{T}A \in S$ and $\mathbf{F_c}A \in S$;

3. $\mathbf{T_{cl}}A \in S$ and $\mathbf{F_c}A \in S$.

$\dfrac{S, \mathbf{T}(A \wedge B)}{S, \mathbf{T}A, \mathbf{T}B}\mathbf{T}\wedge$	$\dfrac{S, \mathbf{T_{cl}}(A \wedge B)}{S, \mathbf{T_{cl}}A, \mathbf{T_{cl}}B}\mathbf{T_{cl}}\wedge$
$\dfrac{S, \mathbf{T}(A \vee B)}{S, \mathbf{T}A / S, \mathbf{T}B}\mathbf{T}\vee$	$\dfrac{S, \mathbf{T_{cl}}(A \vee B)}{S, \mathbf{T_{cl}}A / S, \mathbf{T_{cl}}B}\mathbf{T_{cl}}\vee$
$\dfrac{S, \mathbf{T}(A \to B)}{S, \mathbf{F}A, \mathbf{T_{cl}}B / S, \mathbf{T}B / S, \mathbf{F_c}A}\mathbf{T}\to$	$\dfrac{S, \mathbf{T_{cl}}(A \to B)}{S, \mathbf{T_{cl}}B / S, \mathbf{F_c}A}\mathbf{T_{cl}}\to$
$\dfrac{S, \mathbf{T}\neg A}{S, \mathbf{F_c}A}\mathbf{T}\neg$	$\dfrac{S, \mathbf{T_{cl}}\neg A}{S, \mathbf{F_c}A}\mathbf{T_{cl}}\neg$
$\dfrac{S, \mathbf{F}(A \wedge B)}{S, \mathbf{F}A / S, \mathbf{F}B}\mathbf{F}\wedge$	$\dfrac{S, \mathbf{F_c}(A \wedge B)}{S, \mathbf{F_c}A / S, \mathbf{F_c}B}\mathbf{F_c}\wedge$
$\dfrac{S, \mathbf{F}(A \vee B)}{S, \mathbf{F}A, \mathbf{F}B}\mathbf{F}\vee$	$\dfrac{S, \mathbf{F_c}(A \vee B)}{S, \mathbf{F_c}A, \mathbf{F_c}B}\mathbf{F_c}\vee$
$\dfrac{S, \mathbf{F}(A \to B)}{S, \mathbf{T}A, \mathbf{F}B \ / \ S, \mathbf{T_{cl}}A, \mathbf{F_c}B}\mathbf{F}\to$	$\dfrac{S, \mathbf{F_c}(A \to B)}{S, \mathbf{T_{cl}}A, \mathbf{F_c}B}\mathbf{F_c}\to$
$\dfrac{S, \mathbf{F}\neg A}{S, \mathbf{T_{cl}}A}\mathbf{F}\neg$	$\dfrac{S, \mathbf{F_c}\neg A}{S, \mathbf{T_{cl}}A}\mathbf{F_c}\neg$

Table 4: Tableau calculus for **PRC**

We call *complementary pair* any set of the form $\{\mathbf{T}A, \mathbf{F}A\}$, or of the form $\{\mathbf{T}A, \mathbf{F_c}A\}$, or of the form $\{\mathbf{T_{cl}}A, \mathbf{F_c}A\}$. Finally, we say that a set of swff's S is **PRC**-*consistent* iff no **PRC**-proof table for S is closed. It is immediate to verify that:

Proposition 4.1 *If S is a set of swff's and contains a complementary pair, then S is not* **PRC**-*realizable.* □

In this section we call *splitting rules* the rules $\mathbf{T}\vee$, $\mathbf{T} \to$, $\mathbf{F}\wedge$, $\mathbf{F} \to$, $\mathbf{T_{cl}}\vee$, $\mathbf{T_{cl}} \to$, $\mathbf{F_c}\wedge$; these rules, starting from a configuration consisting of a single set, give rise to a configuration consisting of two or three sets. We call *splitting swff* any swff to which a splitting rule can be applied.

It is easy to check that the rules of the calculus preserve realizability; hence the soundness theorem holds for **PRC**-T.

Theorem 4.2 (Soundness) *If a* **PRC**-*proof table starting from a swff* $\mathbf{F}A$ *is closed, then* $A \in$ **PRC**. □

The completeness proof is based (as usual) on a general method allowing to build up models for **PRC**-consistent sets of swff's. Given a **PRC**-consistent set of swff's S we will construct a **PRC**-model $\underline{K}_{\mathbf{PRC}}(S)$ on whose root the set S will be **PRC**-realized.

Let S be a **PRC**-consistent set of swff's and let $A_1, A_2, \ldots, A_n, \ldots$ be any listing (without duplications) of the swff's of S. Starting from this enumeration, we construct the following sequence $\{S_i\}_{i \geq 0}$ of sets of swff's:

- $S_0 = \emptyset$;
- If $S_i = \{H_1, \ldots, H_k\}$, then

$$S_{i+1} = \bigcup_{H_j \in S_i} \mathcal{U}(H_j, i) \cup \{A_{i+1}\}$$

where: $\mathcal{U}(H_j, i)$ is a **PRC**-extension $\mathcal{R}_{H_j}$ of H_j such that $\mathcal{U}(H_1, i) \cup \cdots \cup \mathcal{U}(H_{j-1}, i) \cup \{H_j, \ldots, H_k, A_{i+1}, A_{i+2}, \ldots\} \cup \mathcal{R}_{H_j}$ is **PRC**-consistent.

We call **PRC**-*saturated set of* S the set

$$S^* = \bigcup_{i \geq 0} S_i$$

It is easy to prove, by induction on i, that any S_i is **PRC**-consistent; hence S^* is **PRC**-consistent.

Now, we say that a swff H is **PRC**-*final* in a set of swff's V iff $H \in V$ and $H \equiv \mathcal{S}A$, where $\mathcal{S}$ is one of the signs $\mathbf{T}, \mathbf{F}, \mathbf{F_c}, \mathbf{T_{cl}}$ and A is an atomic wff. We define

$$\begin{array}{rcl} S^+ & = & \{H \mid H \text{ is } \mathbf{PRC}\text{-final in } S^*\} \\ S_0 & = & \{H \in S^+ \mid H \equiv \mathbf{T}A\} \\ S_1 & = & \{H \in S^+ \mid H \equiv \mathbf{T}A \text{ or } H \equiv \mathbf{T_{cl}}A\} \end{array}$$

Now, given a **PRC**-consistent (and finite) set of swff's S, we define the (finite) structure $\mathcal{K}_{\mathbf{PRC}}(S) = \langle P, \leq, \Vdash \rangle$ as follows:

1. $P = \{S_0, S_1\}$;
2. $S_0 \leq S_1$;
3. For any $\alpha \in P$ and for any propositional variable p, $\alpha \Vdash p$ iff $\mathbf{T}p \in \alpha$ or $\mathbf{T_{cl}}p \in \alpha$.

We remark that, given a consistent set of swff's S, in general we can build up different **PRC**-saturated sets for S and hence we can build different models $\mathcal{K}_{\mathbf{PRC}}(S)$; however, all these structures are equivalent with respect to our purpose. It is obvious that $\mathcal{K}_{\mathbf{PRC}}(S)$ is a **PRC**-model. Now, we prove the Fundamental Lemma.

Lemma 4.3 *Let S be a* **PRC**-*consistent set of swff's and let $\underline{K}_{\mathbf{PRC}}(S) = \langle P, \leq, \Vdash \rangle$ be one of the* **PRC**-*models defined above. Then, for any swff $H \in S^*$, $S_0 \rhd H$ in $\underline{K}_{\mathbf{PRC}}(S)$.*

Proof: The proof goes by induction on the degree of the swff H.

Basis: For $\text{DEG}(H) = 0$ we have that $H \equiv \mathcal{S}p$ with p a propositional variable. Now, if $\mathcal{S} \equiv \mathbf{T}$, $\mathbf{T}p \in S_0$ and is **PRC**-realized in S_0 by our definition of forcing. If $\mathcal{S} \equiv \mathbf{F}$, since S_0 is **PRC**-consistent, $\mathbf{T}p \notin S_0$ and hence $S_0 \rhd \mathbf{F}p$ by our definition of forcing. If $\mathcal{S} \equiv \mathbf{F_c}$, since S^* is **PRC**-consistent neither $\mathbf{T}p$ nor $\mathbf{T_{cl}}p$ belong to S^*, hence $S_0 \nVdash p$ and $S_1 \nVdash p$; this implies $S_0 \rhd \mathbf{F_c}p$. Finally, if $\mathcal{S} \equiv \mathbf{T_{cl}}$, then $\mathbf{T_{cl}}p \in S_1$ and, by definition of the forcing relation, we have that $S_1 \Vdash p$, that is $S_0 \rhd \mathbf{T_{cl}}p$.

Step: Now, let us assume that the assertion holds for any swff $H' \in S^*$ with degree less than or equal to h and let us suppose that $\textsc{deg}(H) = h + 1$. The proof goes by cases according to the form of the swff H. Here we give only some illustrative examples.

Case $H \equiv \mathbf{F}(A \rightarrow B)$: $\mathbf{F}(A \rightarrow B) \in S^*$ implies that either $\mathbf{T}A, \mathbf{F}B$ belong to S^* or $\mathbf{T_{cl}}A$ and $\mathbf{F_c}B$ belong to S^*. In the first case, we get, by the induction hypothesis, that $S_0 \Vdash A$ and $S_0 \not\Vdash B$, hence $S_0 \not\Vdash A \rightarrow B$ and the assertion. In the latter case, we get, by induction hypothesis, $S_1 \Vdash A$ and $S_1 \not\Vdash B$, therefore $S_0 \not\Vdash A \rightarrow B$ which means $S_0 \rhd F(A \rightarrow B)$.

Case $H \equiv \mathbf{F}\neg A$: $\mathbf{F}\neg A \in S^*$ implies that $\mathbf{T_{cl}}A \in S^*$ and hence, by induction hypothesis, $S_0 \Vdash \neg\neg A$. Therefore, we have $S_0 \rhd \mathbf{F}\neg A$. □

From the previous lemma we get, along the lines of Theorem 3.5, the following result:

Theorem 4.4 (Completeness) *If a swff A is* **PRC**-*valid, then there exists a closed* **PRC**-*proof table starting from* $\mathbf{F}A$. □

Example 4.5

1)	$\mathbf{F}((A \rightarrow B) \vee (B \rightarrow A))$	$\mathbf{F}\vee$
2)	$\mathbf{F}(A \rightarrow B), \mathbf{F}(B \rightarrow A))$	$\mathbf{F}\rightarrow$
3)	$\mathbf{T}A, \mathbf{F}B, \mathbf{F}(B \rightarrow A)\ /\ \mathbf{T_{cl}}A, \mathbf{F_c}B, F(B \rightarrow A)$	$\mathbf{F}\rightarrow$
4)	$\underline{\mathbf{T}A}, \mathbf{F}B, \mathbf{T}B, \underline{\mathbf{F}A}\ /\ \underline{\mathbf{T}A}, \mathbf{F}B, \mathbf{T_{cl}}B, \underline{\mathbf{F_c}A}\ /\ \mathbf{T_{cl}}A, \mathbf{F_c}B, F(B \rightarrow A)$	X/X/$\mathbf{F}\rightarrow$
5)	$\ldots/\ldots/\mathbf{T_{cl}}A, \underline{\mathbf{F_c}B}, \underline{\mathbf{T}B}, \mathbf{F}A\ /\ \mathbf{T_{cl}}A, \underline{\mathbf{F_c}B}, \underline{\mathbf{T_{cl}}B}, \mathbf{F_c}A$	X/X/X/X

Example 4.6

1)	$\mathbf{F}(A \vee (A \rightarrow B\ \vee \neg B))$	$\mathbf{F}\vee$
2)	$\mathbf{F}A, \mathbf{F}(A \rightarrow B \vee \neg B))$	$\mathbf{F}\rightarrow$
3)	$\underline{\mathbf{F}A}, \underline{\mathbf{T}A}, \mathbf{F}(B \vee \neg B)\ /\ \mathbf{F}A, \mathbf{T_{cl}}A, \mathbf{F_c}(B \vee \neg B)$	X/$\mathbf{F_c}\vee$
4)	$\ldots/\ \mathbf{F}A, \mathbf{T_{cl}}A, \mathbf{F_c}B, \mathbf{F_c}\neg B$	X/$\mathbf{F_c}\neg$
5)	$\ldots/\ \mathbf{F}A, \mathbf{T_{cl}}A, \underline{\mathbf{F_c}B}, \underline{\mathbf{T_{cl}}B}$	X/X

As a considerably more troublesome example, the reader can check that there exists a closed **PRC**-proof table for $\mathbf{F}((\neg A \rightarrow B) \rightarrow (((B \rightarrow A) \rightarrow B) \rightarrow B))$.

Like the one for **Dum**, also the above tableau calculus for **PRC** can be translated into a cut-free sequent calculus, and essentially the same remarks made at the end of the previous section can be repeated. Here we will only list the rules of such a sequent calculus, where the $\mathbf{T_{cl}}$-swff's are translated into doubly negated wff's put in the left hand sides of the sequents:

Axioms:

$$\frac{}{\Gamma, A \vdash A, \Delta} \qquad\qquad \frac{}{\Gamma, A, \neg A \vdash \Delta}$$

Rules for $\wedge$:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,\mathrm{L}\wedge \qquad\qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\,\mathrm{R}\wedge$$

Rules for $\vee$:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\text{L}\vee \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\text{R}\vee$$

Rules for $\rightarrow$:

$$\frac{\Gamma, \neg\neg B \vdash A, \Delta \quad \Gamma, B \vdash \Delta \quad \Gamma, \neg A \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\text{L}\rightarrow$$

$$\frac{\Gamma, A \vdash B, \Delta \quad \Gamma, \neg\neg A, \neg B \vdash \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\text{R}\rightarrow$$

Left Rules for $\neg$:

$$\frac{\Gamma, \neg\neg A, \neg\neg B \vdash \Delta}{\Gamma, \neg\neg(A \wedge B) \vdash \Delta}\text{L}\neg\neg\wedge \qquad \frac{\Gamma, \neg\neg A \vdash \Delta \quad \Gamma, \neg\neg B \vdash \Delta}{\Gamma, \neg\neg(A \vee B) \vdash \Delta}\text{L}\neg\neg\vee$$

$$\frac{\Gamma, \neg A \vdash \Delta \quad \Gamma, \neg\neg B \vdash \Delta}{\Gamma, \neg\neg(A \rightarrow B) \vdash \Delta}\text{L}\neg\neg\rightarrow \qquad \frac{\Gamma, \neg A \vdash \Delta}{\Gamma, \neg\neg\neg A \vdash \Delta}\text{L}\neg\neg\neg$$

$$\frac{\Gamma, \neg A \vdash \Delta \quad \Gamma, \neg B \vdash \Delta}{\Gamma, \neg(A \wedge B) \vdash \Delta}\text{L}\neg\wedge \qquad \frac{\Gamma, \neg A, \neg B \vdash \Delta}{\Gamma, \neg(A \vee B) \vdash \Delta}\text{L}\neg\vee$$

$$\frac{\Gamma, \neg\neg A, \neg B \vdash \Delta}{\Gamma, \neg(A \rightarrow B) \vdash \Delta}\text{L}\neg\rightarrow$$

Right Rule for $\neg$:

$$\frac{\Gamma, \neg\neg A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\text{R}\neg$$

5 The logic $\mathbf{DE}_2$

The logic we are going to consider is

$$\mathbf{DE}_2 = \mathbf{INT} + (p \vee (p \to q \vee \neg q))$$

From the semantical point of view, $\mathbf{DE}_2$ is characterized by the class of all rooted posets whose depth is at most 2, we denote by $\mathcal{F}_{\mathbf{DE}_2}$ (see [Chagrov and Zakharyaschev, 1997]).

To give a duplication-free tableau calculus for $\mathbf{DE}_2$, we consider the language for swff's used for Peirce Logic in the previous section. The meaning of the signs $\mathbf{T}$, $\mathbf{F}$, $\mathbf{F_c}$ and $\mathbf{T_{cl}}$ is explained in terms of $\mathbf{DE}_2$-*realizability* as follows: given a $\mathbf{DE}_2$-model $\underline{K} = \langle P, \leq, \Vdash \rangle$ and a swff H, we say that an element $\alpha \in P$ $\mathbf{DE}_2$-*realizes* H, and we write $\alpha \rhd H$, if (according to the structure of H) the following conditions hold:

1. If $H \equiv \mathbf{T}A$, then $\alpha \Vdash A$;
2. If $H \equiv \mathbf{F}A$, then $\alpha \nVdash A$;
3. If $H \equiv \mathbf{F_c}A$, then $\alpha \Vdash \neg A$;
4. If $H \equiv \mathbf{T_{cl}}A$, then $\alpha \Vdash \neg\neg A$.

We say that β realizes a set of swff's S iff β realizes any swff in S. A set of swff's S is $\mathbf{DE}_2$-*realizable* iff there is some element α of a $\mathbf{DE}_2$-model $\underline{K}$ such that $\alpha \rhd S$.

A *configuration* $S_1/\ldots/S_n$ *(*$n \geq 1$*) is* $\mathbf{DE}_2$-*realizable* iff at least a S_j $(1 \leq j \leq n)$ is $\mathbf{DE}_2$-realizable. A $\mathbf{DE}_2$-*proof table* is a finite sequence of applications of the rules of the calculus $\mathbf{DE}_2$-T (see TABLE 6 below), starting from some configuration. A $\mathbf{DE}_2$-proof table is *closed* iff all the sets S_j of its final configuration are contradictory, where S_j is *contradictory* if one of the following conditions hold:

1. $\mathbf{T}A \in S$ and $\mathbf{F}A \in S$;
2. $\mathbf{T}A \in S$ and $\mathbf{F_c}A \in S$;
3. $\mathbf{T_{cl}}A \in S$ and $\mathbf{F_c}A \in S$.

Accordingly, we call *complementary pair* any set of the form $\{\mathbf{T}A, \mathbf{F}A\}$, or of the form $\{\mathbf{T}A, \mathbf{F_c}A\}$, or of the form $\{\mathbf{T_{cl}}A, \mathbf{F_c}A\}$. Finally, we say that a set of swff's S is $\mathbf{DE}_2$-*consistent* iff no $\mathbf{DE}_2$-proof table for S is closed. It is immediate to verify that:

Proposition 5.1 *If S is a set of swff's and contains a complementary pair, then S is not* $\mathbf{DE}_2$*-realizable.* □

$\dfrac{S, \mathbf{T}(A \wedge B)}{S, \mathbf{T}A, \mathbf{T}B}\,\mathbf{T}\wedge$	$\dfrac{S, \mathbf{T_{cl}}(A \wedge B)}{S, \mathbf{T_{cl}}A, \mathbf{T_{cl}}B}\,\mathbf{T_{cl}}\wedge$
$\dfrac{S, \mathbf{T}(A \vee B)}{S, \mathbf{T}A / S, \mathbf{T}B}\,\mathbf{T}\vee$	$\dfrac{S, \mathbf{T_{cl}}(A \vee B)}{S_c, \mathbf{T_{cl}}A / S_c, \mathbf{T_{cl}}B}\,\mathbf{T_{cl}}\vee$
See TABLE 2 in Section 3	$\dfrac{S, \mathbf{T_{cl}}(A \to B)}{S_c, \mathbf{T_{cl}}B / S, \mathbf{F_c}A}\,\mathbf{T_{cl}}\to$
$\dfrac{S, \mathbf{T}\neg A}{S, \mathbf{F_c}A}\,\mathbf{T}\neg$	$\dfrac{S, \mathbf{T_{cl}}\neg A}{S, \mathbf{F_c}A}\,\mathbf{T_{cl}}\neg$
$\dfrac{S, \mathbf{F}(A \wedge B)}{S, \mathbf{F}A / S, \mathbf{F}B}\,\mathbf{F}\wedge$	$\dfrac{S, \mathbf{F_c}(A \wedge B)}{S_c, \mathbf{F_c}A / S_c, \mathbf{F_c}B}\,\mathbf{F_c}\wedge$
$\dfrac{S, \mathbf{F}(A \vee B)}{S, \mathbf{F}A, \mathbf{F}B}\,\mathbf{F}\vee$	$\dfrac{S, \mathbf{F_c}(A \vee B)}{S, \mathbf{F_c}A, \mathbf{F_c}B}\,\mathbf{F_c}\vee$
$\dfrac{S, \mathbf{F}(A \to B)}{S, \mathbf{T}A, \mathbf{F}B / S_c, \mathbf{T_{cl}}A, \mathbf{F_c}B}\,\mathbf{F}\to$	$\dfrac{S, \mathbf{F_c}(A \to B)}{S, \mathbf{T_{cl}}A, \mathbf{F_c}B}\,\mathbf{F_c}\to$
$\dfrac{S, \mathbf{F}\neg A}{S_c, \mathbf{T_{cl}}A}\,\mathbf{F}\neg$	$\dfrac{S, \mathbf{F_c}\neg A}{S, \mathbf{T_{cl}}A}\,\mathbf{F_c}\neg$

$$S_c = \{\mathbf{T_{cl}}A \mid \mathbf{T}A \in S\} \cup \{\mathbf{F_c}A \mid \mathbf{F_c}A \in S\} \cup \{\mathbf{T_{cl}}A \mid \mathbf{T_{cl}}A \in S\}$$

TABLE 6: TABLEAU CALCULUS FOR $\mathbf{DE}_2$

We call $\mathbf{DE}_2$-*splitting rules* the rules $\mathbf{T}\vee$, $\mathbf{T} \to AN$, $\mathbf{T} \to\to$, $\mathbf{F}\wedge$, $\mathbf{F} \to$, $\mathbf{T_{cl}}\vee$, $\mathbf{T_{cl}} \to$, $\mathbf{F_c}\wedge$. These rules, starting from a configuration consisting of a single set, give rise to a configuration consisting of two sets. We call $\mathbf{DE}_2$-*splitting swff* any swff to which a splitting rule can be applied.

The rules $\mathbf{F} \to$, $\mathbf{F}\neg$, $\mathbf{T_{cl}}\vee$, $\mathbf{F_c}\wedge$ of $\mathbf{DE}_2$-T narrow the set S to S_c (we call the $\mathbf{DE}_2$-*certain part of* S) in at least one of the sets in the final configuration of the rule. We call $\mathbf{DE}_2$-*c-rules* these rules and we call $\mathbf{DE}_2$-*c-swff* every swff to which a $\mathbf{DE}_2$-*c*-rule can be applied. Finally we call $\mathbf{DE}_2$-*regular rule* any rule of the calculus $\mathbf{DE}_2$-T which is not a $\mathbf{DE}_2$-*c*-rule and we call $\mathbf{DE}_2$-*regular swff* any swff to which a $\mathbf{DE}_2$-regular rule can be applied.

It is easy to check that the rules of the calculus preserve realizability, and hence the soundness theorem can be proved for the calculus $\mathbf{DE}_2$-T along the lines explained for the previous calculi.

Theorem 5.2 (Soundness) *If a $\mathbf{DE}_2$-proof table starting from a swff $\mathbf{F}A$ is closed, then $A \in \mathbf{DE}_2$.* □

The completeness proof is based (as usual) on a general method allowing to build up models for $\mathbf{DE}_2$-consistent sets of swff's. Given a $\mathbf{DE}_2$-consistent set of swff's S we will construct a $\mathbf{DE}_2$-model $\underline{K}_{\mathbf{DE}_2}(S)$ on whose root the set S will be $\mathbf{DE}_2$-realized.

Let S be a $\mathbf{DE}_2$-consistent set of swff's and let $A_1, A_2, \dots, A_n, \dots$ be any listing (without duplications) of the swff's of S. Starting from this enumeration we construct the following sequence $\{S_i\}_{i\geq 0}$ of sets of swff's:

- $S_0 = \emptyset$;
- Let $S_i = \{H_1, \dots, H_k\}$; then

$$S_{i+1} = \bigcup_{H_j \in S_i} \mathcal{U}(H_j, i) \cup \{A_i + 1\}$$

where, setting

$$S' = \mathcal{U}(H_1, i) \cup \dots \cup \mathcal{U}(H_{j-1}, i) \cup \{H_j, \dots, H_k, A_{i+1}, A_{i+2}, \dots\} \ ,$$

we have:

1. If a regular rule different from $\mathbf{T}\!\to\! AN$ and $\mathbf{T}\!\to\!\to\!\!\to$ is applicable to H_j, then $\mathcal{U}(H_j, i)$ is a $\mathbf{DE}_2$-extension $\mathcal{R}_{H_j}$ of H_j such that $(S' \setminus \{H_j\}) \cup \mathcal{R}_{H_j}$ is $\mathbf{DE}_2$-consistent.
2. If $S = S_c$ and either a $\mathbf{DE}_2$-c-rule or $\mathbf{T} \to AN$ or $\mathbf{T} \to\!\!\to$ is applicable to H_j, then $\mathcal{U}(H_j, i)$ is one of the $\mathbf{DE}_2$-extensions $\mathcal{R}_{H_j}$ of H_j such that $(S' \setminus \{H_j\}) \cup \mathcal{R}_{H_j}$ is $\mathbf{DE}_2$-consistent.
3. If $S \neq S_c$ and H_j is $\mathbf{T}(A \to B)$ with A atomic or negated and $(S' \setminus \{\mathbf{T}(A \to B)\}) \cup \{\mathbf{T}B\}$ is $\mathbf{DE}_2$-consistent, then $\mathcal{U}(H_j, i) = \{\mathbf{T}B\}$;
4. If $S \neq S_c$ and H_j is $\mathbf{T}((A \to B) \to C)$ and $(S' \setminus \{\mathbf{T}((A \to B) \to C)\}) \cup \{\mathbf{T}C\}$ is $\mathbf{DE}_2$-consistent, then $\mathcal{U}(H_j, i) = \{\mathbf{T}C\}$.
5. If $S \neq S_c$ and H_j is $\mathbf{T_{cl}}(A \to B)$ and $(S' \setminus \{\mathbf{T_{cl}}(A \to B)\}) \cup \{\mathbf{F_c}A\}$ is $\mathbf{DE}_2$-consistent, then $\mathcal{U}(H_j, i) = \{\mathbf{F_c}A\}$.
6. If $S \neq S_c$ and H_j is $\mathbf{F}(A \to B)$ and $(S' \setminus \{\mathbf{F}(A \to B)\}) \cup \{\mathbf{T}A, \mathbf{F}B\}$ is $\mathbf{DE}_2$-consistent, then $\mathcal{U}(H_j, i) = \{\mathbf{T}A, \mathbf{F}B\}$.
7. In all other cases, $\mathcal{U}(H_j, i) = \{H_j\}$.

It is easy to check that, if S is a $\mathbf{DE}_2$-consistent set of swff's and H is a $\mathbf{DE}_2$-regular swff in S, then a $\mathbf{DE}_2$-extension $\mathcal{R}_H$ of H such that $(S \setminus \{H\}) \cup \mathcal{R}_H$ is $\mathbf{DE}_2$-consistent always exists. Analogously, if S is a $\mathbf{DE}_2$-consistent set of swff's such that $S = S_c$ and H is a $\mathbf{DE}_2$-c-swff of S, then a $\mathbf{DE}_2$-extension $\mathcal{R}_H$ of H such that $(S \setminus \{H\}) \cup \mathcal{R}_H$ is $\mathbf{DE}_2$-consistent always exists. Using this facts, it is easy to prove (by induction) that, for any positive integer i, S_i is $\mathbf{DE}_2$-consistent. This allows to deduce that also the $\mathbf{DE}_2$-*saturated set of* S

$$S^* = \bigcup_{i \geq 0} S_i$$

is $\mathbf{DE}_2$-consistent.

Now, we call $\mathbf{DE}_2$-*final* in V any swff $H \in V$ satisfying one of the following conditions:

1. No $\mathbf{DE}_2$-regular rule can be applied to H;

2. $H \equiv \mathbf{T}(A \to B)$ with A atomic or negated, and $\mathbf{T}B \not\in V$;

3. $H \equiv \mathbf{T}((A \to B) \to C)$ and $\mathbf{T}C \not\in V$ and $\{\mathbf{T}A, \mathbf{F}B\} \not\subseteq V$;

4. $H \equiv \mathbf{T_{cl}}(A \to B)$ and $\mathbf{F_c}A \not\in V$;

5. $H \equiv \mathbf{F}(A \to B)$ and $\{\mathbf{T}A, \mathbf{F}B\} \not\subseteq V$.

We call $\mathbf{DE}_2$-*node set of* S the set:

$$\overline{S} = \begin{cases} \{H \in S^* \mid H \text{ is an atomic swff}\} & \text{if } S = S_c \\ \{H \in S^* \mid H \text{ is } \mathbf{DE}_2\text{-final in } S^*\} & \text{otherwise} \end{cases}$$

Obviously, any (finite) $\mathbf{DE}_2$-consistent set of swff's gives rise to a (finite) $\mathbf{DE}_2$-consistent $\mathbf{DE}_2$-node set.

Now, given a $\mathbf{DE}_2$-node set $\overline{S}$, we define the $\mathbf{DE}_2$-*successor sets of* $\overline{S}$ as follows:

- If a $\mathbf{DE}_2$-c-swff H belongs to $\overline{S}$ and $\mathcal{R}_H$ is a $\mathbf{DE}_2$-extension of H such that $U = (\overline{S}_c \setminus \{H\}) \cup \mathcal{R}_H$ is $\mathbf{DE}_2$-consistent, then U is a $\mathbf{DE}_2$-successor set of $\overline{S}$.
- If $H \equiv \mathbf{T}(\neg A \to B) \in \overline{S}$, then $U = (\overline{S}_c \setminus \{H\}) \cup \{\mathbf{T_{cl}}A\}$ is a $\mathbf{DE}_2$-successor set of $\overline{S}$.
- If $H \equiv \mathbf{T}((A \to B) \to C) \in \overline{S}$, then $U = (\overline{S}_c \setminus \{H\}) \cup \{\mathbf{T_{cl}}A, \mathbf{F_c}B, \mathbf{T_{cl}}(B \to C)\}$ is a $\mathbf{DE}_2$-successor set of $\overline{S}$.
- If $H \equiv \mathbf{T_{cl}}(A \to B) \in \overline{S}$, then $U = (\overline{S}_c \setminus \{H\}) \cup \{\mathbf{T_{cl}}B\}$ is a $\mathbf{DE}_2$-successor set of $\overline{S}$.
- If $H \equiv \mathbf{F}(A \to B) \in \overline{S}$, then $U = (\overline{S}_c \setminus \{H\}) \cup \{\mathbf{T_{cl}}A, \mathbf{F_c}B\}$ is a $\mathbf{DE}_2$-successor set of $\overline{S}$.

It is immediate to check that, if $\overline{S}$ is a $\mathbf{DE}_2$-node set of a $\mathbf{DE}_2$-consistent set of swff's S, then any $\mathbf{DE}_2$-successor set of $\overline{S}$ is $\mathbf{DE}_2$-consistent.

If U is a $\mathbf{DE}_2$-successor set of $\overline{S}$ and H is the swff of $\overline{S}$ used to build U in according to the previous definition, we say that U *is the* $\mathbf{DE}_2$-*successor set of* $\overline{S}$ *related to* H.

Now, given a (finite) $\mathbf{DE}_2$-consistent set of swff's S, we define the (finite) structure $\underline{K}_{\mathbf{DE}_2}(S) = \langle P, \leq, \Vdash \rangle$ as follows:

1. Let $\overline{S}$ be any node set of S; then $\overline{S}$ is the root of $\underline{K}_{\mathbf{DE}_2}(S)$;
2. For any $\mathbf{DE}_2$-successor set U of $\overline{S}$, let $\overline{U}$ be any $\mathbf{DE}_2$-node set of U; then $\overline{U}$ is an element of P and, moreover, $\overline{S} \leq \overline{U}$;
3. For every node set $\overline{\Gamma} \in P$ and for every propositional variable p, we define $\overline{\Gamma} \Vdash p$ iff $\overline{\Gamma} \equiv \overline{S}$ and $\mathbf{T}p \in \overline{\Gamma}$, or $\overline{\Gamma} \not\equiv \overline{S}$ and $\mathbf{T_{cl}}p \in \overline{\Gamma}$.

As an immediate consequence of its definition, we have that, for any $\mathbf{DE}_2$-consistent set of swff's S, $\langle P, \leq \rangle$ is a poset belonging to $\mathcal{F}_{\mathbf{DE}_2}$; moreover, since for any atomic swff $\mathbf{T}p \in \overline{S}$ we have that $\mathbf{T_{cl}}p$ belongs to any $\mathbf{DE}_2$-successor set of $\overline{S}$, $\Vdash$ is a well defined forcing relation. Summarizing, for any $\mathbf{DE}_2$-consistent (and finite) set of swff's, $\underline{K}_{\mathbf{DE}_2}(S)$ is a (finite) $\mathbf{DE}_2$-model.

Lemma 5.3 *Let S be a $\mathbf{DE}_2$-consistent set of swff's and let $\underline{K}_{\mathbf{DE}_2}(S) = \langle P, \leq, \Vdash \rangle$ be one of the $\mathbf{DE}_2$-models defined above. Then, for any $\overline{\Gamma} \in P$ and for any swff $H \in \Gamma^*$, $\overline{\Gamma} \rhd H$ in $\underline{K}_{\mathbf{DE}_2}(S)$.*

Proof: The proof goes on by induction on the well founded relation $\prec$.

Basis: If $\text{DEG}(H) = 0$, we have that $H \equiv \mathcal{S}p$ with p a propositional variable. In this case, if $\mathcal{S}$ is either $\mathbf{T}$, $\mathbf{F}$ or $\mathbf{F_c}$, then, since $\mathcal{S}p$ is $\mathbf{DE}_2$-final in Γ^*, we have that $\mathcal{S}p \in \overline{\Gamma}$. Now, if $\mathcal{S} \equiv \mathbf{T}$, $\mathbf{T}p$ is $\mathbf{DE}_2$-realized in $\overline{\Gamma}(\equiv \overline{S})$ by our definition of forcing. If $\mathcal{S} \equiv \mathbf{F}$, since $\overline{\Gamma}(\equiv \overline{S})$ is $\mathbf{DE}_2$-consistent, $\mathbf{T}p \notin \overline{\Gamma}$ and hence $\overline{\Gamma} \rhd \mathbf{F}p$ by our definition of forcing. If $\mathcal{S} \equiv \mathbf{F_c}$, since, by construction of the model, $\mathbf{F_c}p$ also belongs to any $\mathbf{DE}_2$-successor set of $\overline{\Gamma}$ (if any), we get by consistency that neither $\mathbf{T}p$ nor $\mathbf{T_{cl}}p$ belong to $\overline{\Gamma}$ or to any $\mathbf{DE}_2$-successor set of $\overline{\Gamma}$, hence, by our definition of forcing, we get that, for any element $\overline{\Delta} \in P$ such that $\overline{\Gamma} \leq \overline{\Delta}$, $\overline{\Delta} \Vdash\!\!\!\not\;\, p$, and thus $\overline{\Gamma} \rhd \mathbf{F_c}p$. Finally, if $\mathcal{S} \equiv \mathbf{T_{cl}}$ we get that either $\overline{\Gamma} \equiv \overline{S}$, and hence $\mathbf{T_{cl}}p \in \overline{\Gamma}$ implies $\mathbf{T_{cl}}p \in \overline{\Delta}$ for any $\overline{\Delta}$ in P such that $\overline{\Delta} \not\equiv \overline{S}$, or $\overline{\Gamma}$ is different from $\overline{S}$; in both cases the assertion trivially follows.

Step: Let us suppose that the assertion holds for any $\overline{\Delta} \in P$ and for any swff $H' \in \Delta^*$ such that $H' \prec H$. The proof goes by cases according to the form of the swff H. Here we give only some illustrative examples.

Case $H \equiv \mathbf{F}(A \to B)$: If $\{\mathbf{T}A, \mathbf{F}B\} \subseteq \Gamma^*$, the assertion is an immediate consequence of the induction hypothesis. Otherwise, $\mathbf{F}(A \to B)$ is $\mathbf{DE}_2$-final in Γ^*; this also implies $\overline{\Gamma} \equiv \overline{S}$ and hence $\mathbf{F}(A \to B) \in \overline{S}$. Thus, by construction of $\underline{K}_{\mathbf{DE}_2}(S)$, there exists $\overline{\Delta} \in P$ such that $\overline{S} \leq \overline{\Delta}$ and $\overline{\Delta}$ is the $\mathbf{DE}_2$-successor set of $\overline{S}$ related to $\mathbf{F}(A \to B)$; this implies that $\mathbf{T_{cl}}A, \mathbf{F_c}B \in \Delta^*$ and hence, by induction hypothesis, $\overline{\Delta} \Vdash\!\!\!-\, A$ but $\overline{\Delta} \Vdash\!\!\!\not\;\, B$, which implies $\overline{S} \Vdash\!\!\!\not\;\, A \to B$, that is $\overline{S} \rhd \mathbf{F}(A \to B)$.

Case $H \equiv \mathbf{T_{cl}}(A \to B)$: If $\mathbf{T_{cl}}(A \to B) \in \overline{\Gamma}$, then $\mathbf{T_{cl}}(A \to B)$ is $\mathbf{DE}_2$-final in Γ^* (this also implies $\overline{\Gamma} \equiv \overline{S}$). We must show that, for any final element $\overline{\Delta}$ of $\underline{K}_{\mathbf{DE}_2}(S)$, $\overline{\Delta} \Vdash\!\!\!-\, A \to B$. First of all, we remark that, by construction of the model, there exists a final element $\overline{U}$ which is the node set of the $\mathbf{DE}_2$-successor set U of $\overline{S}$ related to H. This implies that $\mathbf{T_{cl}}B \in U^*$ and therefore, by induction hypothesis, $\overline{U} \Vdash\!\!\!-\, B$, which implies $\overline{U} \Vdash\!\!\!-\, A \to B$. Moreover, for any other element $\overline{V}$ of the model $\underline{K}_{\mathbf{DE}_2}(S)$ different from $\overline{S}$ (if any), since $H \in V^*$ we have that either $\mathbf{F_c}A \in V^*$, and hence $\overline{V} \Vdash\!\!\!-\, \neg A$, or $\mathbf{T_{cl}}B \in V^*$, and hence $\overline{V} \Vdash\!\!\!-\, B$. Summarizing, we have that any final element of $\underline{K}_{\mathbf{DE}_2}(S)$ forces $A \to B$. The case where $\mathbf{T_{cl}}A \to B \notin \overline{\Gamma}$ is trivial. □

From the previous Lemma, along the usual lines, we get:

Theorem 5.4 (Completeness) *If a wff A is $\mathbf{DE}_2$-valid, then there exists a closed $\mathbf{DE}_2$-proof table starting from $\mathbf{F}A$.* □

Example 5.5

1)	$\mathbf{F}((A \vee (A \to B \vee \neg B))$	$\mathbf{F}\vee$
2)	$\mathbf{F}A, \mathbf{F}(A \to B \vee \neg B))$	$\mathbf{F}\to$
3)	$\underline{\mathbf{F}A}, \underline{\mathbf{T}A}, \mathbf{F}(B \vee \neg B)$ / $\mathbf{T_{cl}}A, \mathbf{F_c}(B \vee \neg B)$	$X/\mathbf{F}\vee$
4)	... / $\mathbf{T_{cl}}A, \mathbf{F_c}B, \mathbf{F_c}\neg B$	$X/\mathbf{F_c}\neg$
5)	... / $\mathbf{T_{cl}}A, \underline{\mathbf{F_c}B}, \underline{\mathbf{T_{cl}}B}$	X/X

Also for $\mathbf{DE}_2$ we can translate the tableau calculus into a cut-free sequent calculus and repeat the remarks made at the end of Section 3. We end this section by exhibiting such a sequent calculus:

Axioms:

$$\frac{}{\Gamma, A \vdash A, \Delta} \qquad\qquad \frac{}{\Gamma, A, \neg A \vdash \Delta}$$

Rules for $\wedge$:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,\mathrm{L}\wedge \qquad\qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\,\mathrm{R}\wedge$$

Rules for $\vee$:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\,\mathrm{L}\vee \qquad\qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\,\mathrm{R}\vee$$

Left Rules for $\rightarrow$:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\,\mathrm{L}{\rightarrow}\mathrm{AN} \qquad \text{where } A \text{ is atomic or negated}$$

$$\frac{\Gamma, A \rightarrow (B \rightarrow C) \vdash \Delta}{\Gamma, (A \wedge B) \rightarrow C \vdash \Delta}\,\mathrm{L}{\rightarrow}\wedge \qquad\qquad \frac{\Gamma, A \rightarrow C, B \rightarrow C \vdash \Delta}{\Gamma, A \vee B \rightarrow C \vdash \Delta}\,\mathrm{L}{\rightarrow}\vee$$

$$\frac{\Gamma, B \rightarrow C \vdash A \rightarrow B, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, (A \rightarrow B) \rightarrow C \vdash \Delta}\,\mathrm{L}{\rightarrow}{\rightarrow}$$

Right Rule for $\rightarrow$:

$$\frac{\Sigma, \Pi, A \vdash B, \Delta \qquad \neg\neg\Sigma, \Pi, \neg\neg A, \neg B \vdash}{\Sigma, \Pi \vdash A \rightarrow B, \Delta}\,\mathrm{R}{\rightarrow}$$

where Π is a set of negated wff's and $\neg\neg\Sigma$ is the set of the double negations of the wff's of Σ.

Left Rules for ¬:

$$\frac{\Gamma, \neg\neg A, \neg\neg B \vdash \Delta}{\Gamma, \neg\neg(A \wedge B) \vdash \Delta}\,\mathrm{L}\neg\neg\wedge \qquad \frac{\neg\neg\Sigma, \Pi, \neg\neg A \vdash \quad \neg\neg\Sigma, \Pi, \neg\neg B \vdash}{\Sigma, \Pi, \neg\neg(A \vee B) \vdash \Delta}\,\mathrm{L}\neg\neg\vee$$

$$\frac{\Sigma, \Pi, \neg A \vdash \Delta \quad \neg\neg\Sigma, \Pi, \neg\neg B \vdash}{\Sigma, \Pi, \neg\neg(A \to B) \vdash \Delta}\,\mathrm{L}\neg\neg\to \qquad \frac{\Gamma, \neg A \vdash \Delta}{\Gamma, \neg\neg\neg A \vdash \Delta}\,\mathrm{L}\neg\neg\neg$$

$$\frac{\neg\neg\Sigma, \Pi, \neg A \vdash \quad \neg\neg\Sigma, \Pi, \neg B \vdash}{\Sigma, \Pi, \neg(A \wedge B) \vdash \Delta}\,\mathrm{L}\neg\wedge \qquad \frac{\Gamma, \neg\neg A, \neg\neg B \vdash \Delta}{\Gamma, \neg\neg(A \vee B) \vdash \Delta}\,\mathrm{L}\neg\vee$$

$$\frac{\Gamma, \neg\neg A, \neg B \vdash \Delta}{\Gamma, \neg(A \to B) \vdash \Delta}\,\mathrm{L}\neg\to$$

where Π and $\neg\neg\Sigma$ are as above.

Right Rule for ¬:

$$\frac{\neg\neg\Sigma, \Pi, \neg\neg A \vdash}{\Sigma, \Pi \vdash \neg A, \Delta}\,\mathrm{R}\neg$$

where Π and $\neg\neg\Sigma$ are as above.

6 The Logic SM

The logic **SM** is defined by

$$\mathbf{SM} = \mathbf{INT} + \{(p \lor (p \to q \lor \neg q)) \ , \ ((p \to q) \lor (q \to p) \lor ((p \to \neg q) \land (\neg q \to p)))\}$$

and is characterized on the semantical ground by the class $\mathcal{F}_{\mathbf{SM}}$ of all rooted posets whose depth is at most 2 and with at most two final elements.

To treat **SM**, first of all we will give a very simple tableau calculus based on the possibility of describing, by means of two special signs, the behavior of the wff's on the final elements of the posets of $\mathcal{F}_{\mathbf{SM}}$. Such a calculus can be seen as a special kind of "prefixed tableau calculus" (see [Fitting, 1983]) and cannot be translated into a cut-free sequent calculus. After the explanation of it, however, we will present a second tableau calculus translatable into a cut-free sequent one.

To give our first duplication-free tableau calculus for **SM**, we consider the language for swff's using the signs $\mathbf{T}$, $\mathbf{F}$, ${\mathbf{T_{cl}}}^1$, ${\mathbf{F_{cl}}}^1$, ${\mathbf{T_{cl}}}^2$, ${\mathbf{F_{cl}}}^2$ and the propositional language without $\neg$ but with the propositional constant $\bot$. The meaning of these signs is explained in terms of **SM**-*realizability* as follows: let S be a set of swff's of the kind:

$$\mathbf{T}A_1, \ldots, \mathbf{T}A_m \ , \ \mathbf{F}B_1, \ldots, \mathbf{F}B_n \ ,$$
$${\mathbf{T_{cl}}}^1 C_1, \ldots, {\mathbf{T_{cl}}}^1 C_p \ , \ {\mathbf{F_{cl}}}^1 D_1, \ldots, {\mathbf{F_{cl}}}^1 D_q \ ,$$
$${\mathbf{T_{cl}}}^2 E_1, \ldots, {\mathbf{T_{cl}}}^2 E_r \ , \ {\mathbf{F_{cl}}}^2 F_1, \ldots, {\mathbf{F_{cl}}}^2 F_s \ ;$$

given a **SM**-model $\underline{K} = \langle P, \leq, \Vdash \rangle$, we say that an element $\alpha \in P$ **SM**-*realizes* S, and we write $\alpha \rhd S$, if:

1. For $1 \leq i \leq m$, $\alpha \Vdash A_i$;
2. For $1 \leq i \leq n$, $\alpha \nVdash B_i$;
3. There exist two final elements ϕ_1 and ϕ_2 in P (possibly $\phi_1 = \phi_2$) such that
   - For any i such that $1 \leq i \leq p$, $\phi_1 \Vdash C_i$;
   - For any i such that $1 \leq i \leq q$, $\phi_1 \Vdash \neg D_i$;
   - For any i such that $1 \leq i \leq r$, $\phi_2 \Vdash E_i$;
   - For any i such that $1 \leq i \leq s$, $\phi_2 \Vdash \neg F_i$;

A set of swff's S is **SM**-*realizable* iff there is some element α of a **SM**-model $\underline{K}$ such that $\alpha \rhd S$.

A *configuration* $S_1 / \ldots / S_n$ *($n \geq 1$) is* **SM**-*realizable* iff at least a S_j (with $i \leq j \leq n$) is **SM**-realizable. A **SM**-*proof table* is a finite sequence of applications of the rules of the calculus **SM**-T (see TABLE 7 below), starting from some configuration. A **SM**-proof table is *closed* iff all the sets S_j of its final configuration are contradictory, where S_j is *contradictory* if one of the following conditions holds:

1. $\mathbf{T}A \in S$ and $\mathbf{F}A \in S$;
2. $\mathbf{T}\bot \in S$;
3. ${\mathbf{T_{cl}}}^j A \in S$ and ${\mathbf{F_{cl}}}^j A \in S$ $(j \in \{1, 2\})$.

Accordingly, we define *complementary pair* either the set $\{\mathbf{T}\bot\}$ or a set of the form $\{\mathbf{T}A, \mathbf{F}A\}$, or a set of the form $\{\mathbf{T_{cl}}^1 A, \mathbf{F_{cl}}^1 A\}$, or a set of the form $\{\mathbf{T_{cl}}^2 A, \mathbf{F_{cl}}^2 A\}$. Finally, we say that a set of swff's S is **SM**-*consistent* iff no **SM**-proof table for S is closed. It is immediate to verify that:

Proposition 6.1 *If S is a set of swff's and S contains a complementary pair, then S is not* **SM**-*realizable.* □

$$\frac{S, \mathbf{T}(A \wedge B)}{S, \mathbf{T}A, \mathbf{T}B}\mathbf{T}\wedge \qquad \frac{S, \mathbf{F}(A \wedge B)}{S, \mathbf{F}A / S, \mathbf{F}B}\mathbf{F}\wedge$$

$$\frac{S, \mathbf{T}(A \vee B)}{S, \mathbf{T}A / S, \mathbf{T}B}\mathbf{T}\vee \qquad \frac{S, \mathbf{F}(A \vee B)}{S, \mathbf{F}A, \mathbf{F}B}\mathbf{F}\vee$$

See TABLE 2 in Section 3 $\qquad \dfrac{S, \mathbf{F}(A \to B)}{S, \mathbf{T}A, \mathbf{F}B \ / \ S, \mathbf{T_{cl}}^1 A, \mathbf{F_{cl}}^1 B \ / \ S, \mathbf{T_{cl}}^2 A, \mathbf{F_{cl}}^2 B}\mathbf{F}\to$

$$\frac{S, \mathbf{T_{cl}}^j (A \wedge B)}{S, \mathbf{T_{cl}}^j A, \mathbf{T_{cl}}^j B}\mathbf{T}\wedge \qquad \frac{S, \mathbf{F_{cl}}^j (A \wedge B)}{S, \mathbf{F_{cl}}^j A / \mathbf{F_{cl}}^j B}\mathbf{T}\wedge$$

$$\frac{S, \mathbf{T_{cl}}^j (A \vee B)}{S, \mathbf{T_{cl}}^j A / S, \mathbf{T_{cl}}^j B}\mathbf{T}\vee \qquad \frac{S, \mathbf{F_{cl}}^j (A \vee B)}{S, \mathbf{F_{cl}}^j A, \mathbf{F_{cl}}^j B}\mathbf{T}\vee$$

$$\frac{S, \mathbf{T_{cl}}^j (A \to B)}{S, \mathbf{F_{cl}}^j A / S, \mathbf{T_{cl}}^j B}\mathbf{T}\to \qquad \frac{S, \mathbf{F_{cl}}^j (A \to B)}{S, \mathbf{T_{cl}}^j A, \mathbf{F_{cl}}^j B}\mathbf{T}\to$$

TABLE 7: TABLEAU CALCULUS FOR **SM**

It is easy to verify, along the usual lines, that the rules of the calculus preserve realizability, and hence the soundness theorem holds for **SM**-T.

Theorem 6.2 (Soundness) *If a* **SM**-*proof table starting from a swff* $\mathbf{F}A$ *is closed, then* $A \in \mathbf{SM}$. □

The completeness proof is based (as usual) on a general method allowing to build up models for **SM**-consistent sets of swff's. Given a **SM**-consistent set of swff's S we will build up a **SM**-model $\underline{K}_{\mathbf{SM}}(S)$ on whose root the set S will be **SM**-realized.

Let S be a **SM**-consistent set of swff's and let $A_1, A_2, \ldots, A_n, \ldots$ be any listing (without duplications) of the swff's of S. Starting from this enumeration, we construct the following sequence $\{S_i\}_{i \geq 0}$ of sets of swff's:

- $S_0 = \emptyset$;
- Let $S_i = \{H_1, \ldots, H_k\}$; then

$$S_{i+1} = \bigcup_{H_j \in S_i} \mathcal{U}(H_j, i) \cup \{A_i + 1\}$$

where: $\mathcal{U}(H_j, i)$ is a **SM**-extension $\mathcal{R}_{H_j}$ of H_j such that $\mathcal{U}(H_1, i) \cup \cdots \cup \mathcal{U}(H_{j-1}, i) \cup \{H_{j+1}, \ldots, H_k, A_{i+1}, A_{i+2}, \ldots\} \cup \mathcal{R}_{H_j}$ is **SM**-consistent.

We call **SM**-*saturated set of* S the set

$$S^* = \bigcup_{i \geq 0} S_i \ .$$

It is easy to prove, by induction on i, that any S_i is **SM**-consistent; hence S^* is **SM**-consistent.

Now, we say that a swff H is **SM**-*final* in a set of swff's V iff $H \in V$ and $H = \mathcal{S}A$, where $\mathcal{S}$ is one of the signs $\mathbf{T}$, $\mathbf{F}$, $\mathbf{T_{cl}}^1$, $\mathbf{T_{cl}}^2$, $\mathbf{F_{cl}}^1$, $\mathbf{F_{cl}}^2$ and A is an atomic wff. We define

$$\begin{aligned} S^+ &= \{H \mid H \text{ is } \mathbf{SM}\text{-final in } S^*\} \\ S_0 &= \{H \in S^+ \mid H \equiv \mathbf{T}A\} \\ S_1 &= \{H \in S^+ \mid H \equiv \mathbf{T}A \text{ or } H \equiv \mathbf{T_{cl}}^1 A\} \\ S_2 &= \{H \in S^+ \mid H \equiv \mathbf{T}A \text{ or } H \equiv \mathbf{T_{cl}}^2 A\} \end{aligned}$$

Now, given a **SM**-consistent (and finite) set of swff's S, we define the (finite) structure $\mathcal{K}_{\mathbf{SM}}(S) = \langle P, \leq, \Vdash \rangle$ as follows:

1. $P = \{S_0, S_1, S_2\}$;
2. $S_0 \leq S_1$ and $S_0 \leq S_2$;
3. For any propositional variable p, $S_0 \Vdash p$ iff $\mathbf{T}p \in S_0$. For $j \in \{1, 2\}$ and for any propositional variable p, $S_j \Vdash p$ iff $\mathbf{T_{cl}}^j p \in S_j$ or $\mathbf{T}p \in S_j$.

We immediately have that $\underline{K}_{\mathbf{SM}}(S)$ is a **SM**-model. We remark that, given a consistent set of swff's S, in general we can build up different **SM**-saturated sets for S and hence we can build different models $\mathcal{K}_{\mathbf{SM}}(S)$; however, all these structures are equivalent with respect to our purpose. The proof of the main Lemma can be carried out with a straightforward induction on the degree of the swff's.

Lemma 6.3 *Let S be a* **SM**-*consistent set of swff's and let $\underline{K}_{\mathbf{SM}}(S) = \langle P, \leq, \Vdash \rangle$ be one of the* **SM**-*models defined above. Then, for any swff $H \in S^*$, $S_0 \rhd H$ in $\underline{K}_{\mathbf{SM}}(S)$.* □

From the previous lemma we get, along the usual lines, the following result:

Theorem 6.4 (Completeness) *If a wff A is* **SM**-*valid, then there exists a closed* **SM**-*proof table starting from* $\mathbf{F}A$. □

The above calculus is nice, but we do not see how to translate it into a cut-free sequent one. On the other hand, we can provide a second tableau calculus for **SM**, we call **SM**-T^*, as follows:

(I) The propositional language on which **SM**-T^* works is the usual one, i.e. its primitive logical constants are $\wedge$, $\vee$, $\rightarrow$ and $\neg$; the related set of signs is $\{\mathbf{T}, \mathbf{F}, \mathbf{F_c}\}$, i.e., we have the set of swff's previously used for the logic **Dum**; accordingly, the notions of complementary pair and consistent set of swff's are defined as made in the explanation of the calculus for **Dum**.

(II) In this frame, **SM**-T^* includes the rules $\mathbf{T}\wedge$, $\mathbf{T}\vee$, $\mathbf{T}\rightarrow$, $\mathbf{F}\wedge$, $\mathbf{F}\vee$, $\mathbf{F_c}\vee$, $\mathbf{T}\neg$ given in Table 1, and the rules in Table 8 below.

$$\frac{S, \mathbf{F}(A \to B)}{S, \mathbf{T}A, \mathbf{F}B \ / \ S, \mathbf{F}\neg(A \wedge \neg B)}\mathbf{F}\!\to \qquad \frac{S, \mathbf{F}\neg A}{S_c, \mathbf{T}A}\mathbf{F}\neg$$

$$\frac{S_c, \mathbf{T}(A \to B)}{S_c, \mathbf{F_c}A \ / \ S_c, \mathbf{T}B}\text{special-}\mathbf{T}\!\to \qquad \frac{S_c, \mathbf{F_c}(A \wedge B)}{S_c, \mathbf{F_c}A \ / \ S_c, \mathbf{F_c}B}\mathbf{F_c}\wedge$$

$$\frac{S_c, \mathbf{F_c}(A \to B)}{S_c, \mathbf{T}A, \mathbf{F_c}B}\mathbf{F_c}\!\to \qquad \frac{S_c, \mathbf{F_c}\neg A}{S_c, \mathbf{T}A}\mathbf{F_c}\neg$$

$$\frac{S, \mathbf{F}\neg A, \mathbf{F}\neg B, \mathbf{F}\neg C}{S, \mathbf{F}\neg(A \wedge B), \mathbf{F}\neg C \ / \ S, \mathbf{F}\neg(A \wedge C), \mathbf{F}\neg B \ / \ S, \mathbf{F}\neg(B \wedge C), \mathbf{F}\neg A}\text{special-}\mathbf{F}\neg$$

TABLE 8: SECOND CALCULUS FOR **SM**

The rules special-$\mathbf{T}\to$, $\mathbf{F_c}\wedge$, $\mathbf{F_c}\to$ and $\mathbf{F_c}\neg$ of TABLE 8 can be applied only when the set of swff's above the line does not contain **F**-swff's, i.e., it coincides with its certain part. The special-$\mathbf{F}\neg$-rule is to be applied to get sets of swff's containing at most two swff's of the form $\mathbf{F}\neg H$.

Now, starting from a finite set S of swff's consistent in the calculus **SM**-T^*, using the above rules (except $\mathbf{F}\neg$, special-$\mathbf{T}\to$, $\mathbf{F_c}\wedge$, $\mathbf{F_c}\to$ and $\mathbf{F_c}\neg$), and following the usual lines, one expands S into a saturated set S^* containing a node set $\overline{S}$ to which only atomic swff's or swff's of the kind $\mathbf{F_c}H$ or swff's of the kind $\mathbf{F}\neg H$ belong; moreover, at most two swff's of the kind $\mathbf{F}\neg H$ belong to $\overline{S}$ (to do this, an essential role is played by the rule special-$\mathbf{F}\neg$). Then, applying the rule $\mathbf{F}\neg$, one gets (at most) two successor sets, S_1 and S_2 of S, each of them not containing **F**-swff's. Finally, by means of the rules which act on **T**-swff's and $\mathbf{F_c}$-swff's (i.e. special-$\mathbf{T}\to$, $\mathbf{T}\wedge$, $\mathbf{T}\vee$, $\mathbf{F_c}\wedge$, $\mathbf{F_c}\vee$, $\mathbf{F_c}\to$ and $\mathbf{F_c}\neg$), S_1 and S_2 give rise to S_1^* and S_2^* and to the related node sets $\overline{S}_1$ and $\overline{S}_2$, containing only atomic **T**-swff's and atomic $\mathbf{F_c}$-swff's. This provides the required model and allows to prove, along the usual lines, the completeness theorem for the calculus **SM**-T^*.

At this point we can provide the desired cut-free sequent calculus for **SM** by translating **SM**-T^* according to the translation explained in Section 3; such a calculus is the following:

Axioms:

$$\frac{}{\Gamma, A \vdash A, \Delta} \qquad \frac{}{\Gamma, A, \neg A \vdash \Delta}$$

Rules for $\wedge$:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\text{L}\wedge \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\text{R}\wedge$$

Rules for $\vee$:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\text{L}\vee \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\text{R}\vee$$

Left Rules for $\rightarrow$:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\text{L}{\rightarrow}\text{AN} \qquad \text{where } A \text{ is atomic or negated}$$

$$\frac{\Gamma, A \rightarrow (B \rightarrow C) \vdash \Delta}{\Gamma, (A \wedge B) \rightarrow C \vdash \Delta}\text{L}{\rightarrow}\wedge \qquad \frac{\Gamma, A \rightarrow C, B \rightarrow C \vdash \Delta}{\Gamma, A \vee B \rightarrow C \vdash \Delta}\text{L}{\rightarrow}\vee$$

$$\frac{\Gamma, B \rightarrow C \vdash A \rightarrow B, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, (A \rightarrow B) \rightarrow C \vdash \Delta}\text{L}{\rightarrow}{\rightarrow}$$

$$\frac{\Gamma, \neg A \vdash \quad \Gamma, B \vdash}{\Gamma, (A \rightarrow B) \vdash}\text{special-L}{\rightarrow}$$

Right Rule for $\rightarrow$:

$$\frac{\Gamma, A \vdash B, \Delta \qquad \Gamma \vdash \neg(A \wedge \neg B), \Delta}{\Gamma \vdash (A \rightarrow B), \Delta}\text{R}{\rightarrow}$$

Left Rules for $\neg$:

$$\frac{\Gamma, \neg A \vdash \quad \Gamma, \neg B \vdash}{\Gamma, \neg(A \wedge B) \vdash}\text{L}\neg\wedge \qquad \frac{\Gamma, \neg A, \neg B \vdash \Delta}{\Gamma, \neg(A \vee B) \vdash \Delta}\text{L}\neg\vee$$

$$\frac{\Gamma, A, \neg B \vdash}{\Gamma, \neg(A \rightarrow B) \vdash}\text{l}\neg{\rightarrow} \qquad \frac{\Gamma, A \vdash}{\Gamma, \neg\neg A \vdash}\text{L}\neg\neg$$

Right Rules for $\neg$:

$$\frac{\Gamma, A \vdash}{\Gamma \vdash \neg A, \Delta}\text{R}\neg$$

$$\frac{\Gamma \vdash \neg(A \wedge B), \neg C, \Delta \quad \Gamma \vdash \neg(A \wedge C), \neg B, \Delta \quad \Gamma \vdash \neg(B \wedge C), \neg A, \Delta}{\Gamma \vdash \neg A, \neg B, \neg C, \Delta}\text{special-R}\neg$$

Remarks:

(**R 6.1**) Of course, the above tableau calculus **SM**-T^* can be seen as a duplication-free one. In this sense, one has to set $\mathbf{F_c}\neg(A \wedge \neg B) \prec \mathbf{F}(A \to B)$; also, a set such as $S, \mathbf{F}\neg(A \wedge B)$, $\mathbf{F}\neg C$, must be considered smaller than $S, \mathbf{F}\neg A, \mathbf{F}\neg B, \mathbf{F}\neg C$, the latter containing a greater number of swff's of the form $\mathbf{F}\neg H$.

(**R 6.2**) Consider the calculus obtained by replacing the rule special-$\mathbf{F}\neg$ of **SM**-T^* with the following rule, leaving unchanged the other ones:

$$\frac{S, \mathbf{F}\neg A, \mathbf{F}\neg B}{S, \mathbf{F}\neg(A \wedge B)}\,\mathbf{PRC}\text{-special-}\mathbf{F}\neg$$

Then, one gets another duplication-free tableau calculus for the logic **PRC**; from this calculus, by translating, one gets a cut-free sequent calculus.

(**R 6.3**) The rule special-$\mathbf{F}\neg$ of the calculus **SM**-T^* can be generalized also in another direction; for instance, consider the rule

$$\frac{S, \mathbf{F}\neg A_1, \mathbf{F}\neg A_2, \mathbf{F}\neg A_3, \mathbf{F}\neg A_4}{S, \mathbf{F}\neg(A_1 \wedge A_2), F\neg A_3, \mathbf{F}\neg A_4 \;/\; \ldots \;/\; S, \mathbf{F}\neg(A_3 \wedge A_4), F\neg A_1, \mathbf{F}\neg A_2}\,\text{special}_4\text{-}\mathbf{F}\neg$$

(where the set above the line gives rise to six sets below the line). Then, replacing the rule special-$\mathbf{F}\neg$ of the calculus **SM**-T^* with the rule special$_4$-$\mathbf{F}\neg$ (and leaving unchanged the other rules), one gets a duplication-free tableau calculus for the logic $\mathbf{SM}_3$ characterized by the class $\mathcal{F}_{\mathbf{SM}_3}$ of all the rooted posets whose depth is at most two and with at most three final elements. Also, such a tableau calculus for $\mathbf{SM}_3$ can be translated, along the usual lines, into a cut-free sequent calculus.

Of course, one can treat in a similar way the logics $\mathbf{SM}_4$ (characterized by rooted posets whose depth is at most two and with at most four final elements), $\mathbf{SM}_5$, and so on.

(**R 6.4**) Differently from **SM**, the logics $\mathbf{SM}_3$, $\mathbf{SM}_4$, ... do not satisfy the interpolation lemma. Yet, such logics can be characterized by cut-free sequent calculi whose main features seem to coincide with the ones of the sequent calculus previously explained for **SM**.

(**R 6.5**) In line with the above discussion, one can easily extend to the logics $\mathbf{SM}_3$, $\mathbf{SM}_4$, ... the first tableau calculus presented for **SM**: for $\mathbf{SM}_3$ one has to introduce the additional signs $\mathbf{T_{cl}}^3$, $\mathbf{F_{cl}}^3$; for $\mathbf{SM}_4$ the additional signs $\mathbf{T_{cl}}^3$, $\mathbf{F_{cl}}^3$, $\mathbf{T_{cl}}^4$, $\mathbf{F_{cl}}^4$ and so on. We leave the details to the reader.

7 Jankov logic

Jankov Logic is the intermediate logic

$$\mathbf{Jn} = \mathbf{INT} + (\neg p \vee \neg\neg p) \ ,$$

where the axiom schema $(\neg p \vee \neg\neg p)$ characterizing it is also called the *weak law of excluded middle* [Jankov, 1968]. As it is known [Chagrov and Zakharyaschev, 1997; Jankov, 1968], $\mathbf{Jn} = \mathcal{L}(\mathcal{F}_{\mathbf{Jn}})$, where $\mathcal{F}_{\mathbf{Jn}}$ is the class of all the finite and rooted Kripke frames $\underline{P} = \langle P, \leq, r\rangle$ with a single final element.

As a first goal, we improve the calculus for **Jn** given in [Avellone et al., 1997] and based on the notion of generalized tableau calculus. As in [Avellone et al., 1997], our generalized tableau calculus uses the propositional language with $\wedge$, $\vee$, $\rightarrow$, $\neg$ as primitive symbols, and the three signs **T**, **F** and $\mathbf{F_c}$; moreover, it is still based on the tableau system **INT**-T of [Miglioli et al., 1994a] (see also [Miglioli et al., 1994b], where a variant of this calculus is considered) described in TABLE 9.

$\dfrac{S, \mathbf{T}(A\wedge B)}{S, \mathbf{T}A, \mathbf{T}B}\mathbf{T}\wedge$	$\dfrac{S, \mathbf{F}(A\wedge B)}{S, \mathbf{F}A / S, \mathbf{F}B}\mathbf{F}\wedge$	$\dfrac{S, \mathbf{F_c}(A\wedge B)}{S_c, \mathbf{F_c}A / S_c, \mathbf{F_c}B}\mathbf{F_c}\wedge$
$\dfrac{S, \mathbf{T}(A\vee B)}{S, \mathbf{T}A / S, \mathbf{T}B}\mathbf{T}\vee$	$\dfrac{S, \mathbf{F}(A\vee B)}{S, \mathbf{F}A, \mathbf{F}B}\mathbf{F}\vee$	$\dfrac{S, \mathbf{F_c}(A\vee B)}{S, \mathbf{F_c}A, \mathbf{F_c}B}\mathbf{F_c}\vee$
See TABLE 2 in Section 3	$\dfrac{S, \mathbf{F}(A\rightarrow B)}{S_c, \mathbf{T}A, \mathbf{F}B}\mathbf{F}\rightarrow$	$\dfrac{S, \mathbf{F_c}(A\rightarrow B)}{S_c, \mathbf{T}A, \mathbf{F_c}B}\mathbf{F_c}\rightarrow$
$\dfrac{S, \mathbf{T}\neg A}{S, \mathbf{F_c}A}\mathbf{T}\neg$	$\dfrac{S, \mathbf{F}\neg A}{S_c, \mathbf{T}A}\mathbf{F}\neg$	$\dfrac{S, \mathbf{F_c}\neg A}{S_c, \mathbf{T}A}\mathbf{F_c}\neg$

TABLE 9: TABLEAU CALCULUS FOR **INT**

In TABLE 9, S_c is the *certain part of* S; formally:

$$S_c = \{\mathbf{T}X \mid \mathbf{T}X \in S\} \cup \{\mathbf{F_c}X \mid \mathbf{F_c}X \in S\}$$

Given a logic $\mathbf{L} = \mathbf{INT} + (A)$, we call **L**-*selection function* any function $\Sigma_{\mathbf{L}}$ so defined:

1. The domain of $\Sigma_{\mathbf{L}}$ is the set of all the wff's;
2. For every wff H, $\Sigma_{\mathbf{L}}(H)$ is a finite subset of the set of instances of the axiom schema (A).

Let $\Sigma_{\mathbf{L}}$ be a **L**-selection function for the logic $\mathbf{L} = \mathbf{INT} + (A)$; then the $\Sigma_{\mathbf{L}}(H)$-rule is the following rule:

$$\frac{S}{S, \mathbf{T}B_1, \ldots, \mathbf{T}B_n}\Sigma_{\mathbf{L}}\text{(H)-rule}$$

TABLE 10: GENERALIZED RULE

where $\Sigma_{\mathbf{L}}(H) \supseteq \{B_1, \ldots, B_n\}$.

Given a **L**-selection function $\Sigma_{\mathbf{L}}$, the $\Sigma_{\mathbf{L}}(H)$-*generalized tableau calculus for* **L**, we call $\Sigma_{\mathbf{L}}(H)$-GCL, is obtained by adding the $\Sigma_{\mathbf{L}}(H)$-rule to **INT**-T. The notions of configuration, proof-table, etc., for $\Sigma_{\mathbf{L}}(H)$-GCL are defined in the obvious way, where any proof- table of the calculus must start with the configuration $\mathbf{F}H$.

Of course, given any logic $\mathbf{L} = \mathbf{INT}+(A)$ and any **L**-selection function $\Sigma_{\mathbf{L}}$, the calculus $\Sigma_{\mathbf{L}}(H)$-GCL is sound, that is, if one can build a closed proof-table in the calculus starting from $\mathbf{F}H$ then $H \in L$. The problem is to single out (if any) **L**-selection functions $\Sigma_{\mathbf{L}}$ guaranteeing the completeness of $\Sigma_{\mathbf{L}}(H)$-GCL for any $H \in \mathbf{L}$. In this frame, let us denote with $\textsc{IstP}_{\mathbf{L}}(H)$ the set of all the wff's obtained by correctly instantiating the wff A (generating the axiom-scheme (A)) with the wff's of some well defined set $\mathcal{V}(H)$ of subformulas of H (in other words, if A is a generating wff of (A) then $\textsc{IstP}_{\mathbf{L}}(H)$ is the set of all the wff's obtained from A by means of substitutions replacing the variables of A with elements of $\mathcal{V}(H)$). Of course, $\textsc{IstP}_{\mathbf{L}}(H)$ is a finite set. Finally, let un consider the following **L**-selection function:

- $\Sigma_{\mathbf{L}}$ is the function associating, with every wff H, the finite set of wff's $\textsc{IstP}_{\mathbf{L}}(H)$.

Let **L** be any logic; we introduce the following notions:

- A **L**-*saturated set* Γ is any consistent set of wff's closed under **L**-provability (i.e., $\Gamma \vdash_L A$ implies $A \in \Gamma$) and under the disjunction property (i.e., $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$).
- If Γ is a **L**-saturated set, by the *canonical model generated by* Γ, denoted by $\mathcal{C}_L(\Gamma)$, we mean the Kripke model $\underline{K} = \langle P, \leq, r, \Vdash \rangle$ satisfying the following properties:
  1. $P = \{\Gamma' \mid \Gamma \subseteq \Gamma' \text{ and } \Gamma' \text{ is } \mathbf{L}\text{-saturated}\}$;
  2. For any two $\Gamma', \Gamma'' \in P$, $\Gamma' \leq \Gamma''$ iff $\Gamma' \subseteq \Gamma''$;
  3. $r = \Gamma$;
  4. For any $\Gamma' \in P$ and any propositional variable p, $\Gamma' \Vdash p$ iff $p \in \Gamma'$.

The two following propositions are well known [Chagrov and Zakharyaschev, 1997; Ferrari and Miglioli, 1993; Ferrari and Miglioli, 1995a; Ferrari and Miglioli, 1995b; Miglioli, 1992; Smorynski, 1973]:

Proposition 7.1 *If* **L** *is a logic, A is any wff, and Δ is any set of wff's such that* $\Delta \nvdash_L A$, *then there exists a* **L**-*saturated set* Γ *such that* $\Delta \subseteq \Gamma$ *and* $A \notin \Gamma$. □

Proposition 7.2 *If* **L** *is any logic,* Γ *is any* **L**-*saturated set,* Γ' *is any element of* $\mathcal{C}_L(\Gamma)$, *and B is any wff, then* $\Gamma' \Vdash B$ *holds in* $\mathcal{C}_L(\Gamma)$ *iff* $B \in \Gamma'$. □

Now, given a wff H, we denote with $\textsc{Sf}(H)$ the set of subformulas of H; moreover, we let $\mathcal{V}(H)$ to be the set of all propositional variables of H and we set

$$\textsc{RSf}(H) = \{A \mid A \in \mathcal{V}(H), \text{ or } A \equiv K \rightarrow Q \in \textsc{Sf}(H), \text{ or } A \equiv \neg K \in \textsc{Sf}(H)\} .$$

Following [Avellone et al., 1997; Chagrov and Zakharyaschev, 1997; Ferrari and Miglioli, 1993; Ferrari and Miglioli, 1995a; Ferrari and Miglioli, 1995b; Gabbay, 1970; Gabbay, 1981; Miglioli, 1992], given $\underline{K} = \langle P, \leq, \Vdash\rangle$ and $\alpha, \beta \in P$, we set $\alpha \subseteq_H \beta$ iff, for every $A \in \mathrm{RSF}(H)$, if $\alpha \Vdash A$ then $\beta \Vdash A$. Moreover, $\alpha \equiv_H \beta$ iff $\alpha \subseteq_H \beta$ and $\beta \subseteq_H \alpha$. It is easy to see that $\equiv_H$ is an equivalence relation and that:

Proposition 7.3 *The set of equivalence classes of $\equiv_H$ on the set of elements of $\underline{K}$ is finite.* □

As in [Avellone et al., 1997; Chagrov and Zakharyaschev, 1997; Ferrari and Miglioli, 1993; Ferrari and Miglioli, 1995a; Ferrari and Miglioli, 1995b; Gabbay, 1970; Gabbay, 1981; Miglioli, 1992], given $\underline{K} = \langle P, \leq, \Vdash\rangle$, we define the *quotient model* $\underline{K}_{/\equiv_H}$ as the Kripke model $\langle P', \leq', \Vdash'\rangle$ with the following properties:

1. P' is the set of equivalence classes generated by $\equiv_H$ on P;
2. if $[\alpha]$ and $[\beta]$ are two elements of P' (where $[\gamma]$ is the equivalence class of γ), then $[\alpha] \leq' [\beta]$ iff $\alpha \subseteq_H \beta$;
3. for every variable p such that $p \in \mathrm{RSF}(H)$ and for every $[\alpha] \in P'$, $[\alpha] \Vdash' p$ in $\underline{K}_{/\equiv_H}$ iff $\alpha \Vdash p$ in $\underline{K}$;
4. for every variable q such that $q \notin \mathrm{RSF}(H)$ and for every $[\alpha] \in P'$, $[\alpha] \Vdash' q$ in $\underline{K}_{/\equiv_H}$.

The main property of $\underline{K}_{/\equiv_H}$ is stated by the following proposition that can be proved by induction on the wff B as in [Avellone et al., 1997; Chagrov and Zakharyaschev, 1997; Gabbay, 1970; Gabbay, 1981]:

Proposition 7.4 *If $B \in \mathrm{SF}(H)$ then, for every element α of $\underline{K}$, $\alpha \Vdash B$ in $\underline{K}$ iff $[\alpha] \Vdash' B$ in $\underline{K}_{/\equiv_H}$.* □

Let $\mathbf{L} = \mathbf{INT} + (A)$ be any logic, let us suppose that $\mathbf{L} = \mathcal{L}(\mathcal{F})$ for some non empty class of Kripke frames $\mathcal{F}$, and, for any wff H, let $H_{neg} \equiv H \wedge \bigwedge \Xi(H)$, where $\Xi(H)$ is the set of all wff's of the kind $\neg\neg p$, being p a propositional variable belonging to H, and $\bigwedge \Xi(H)$ is the conjunction of all the wff's in $\Xi(H)$. Then:

Theorem 7.5 *Let H be any wff such that $H \in \mathbf{L}$. If, for every $\mathbf{INT}$-saturated set Γ such that $\mathrm{ISTP}_{\mathbf{L}}(H) \subseteq \Gamma$, $\mathcal{C}_{\mathbf{INT}}(\Gamma)_{/\equiv_{H_{neg}}}$ is built on a frame of $\mathcal{F}$, then $\mathrm{ISTP}_{\mathbf{L}}(H) \vdash_{\mathbf{INT}} H$.*

Proof: The proof is a consequence of Propositions 7.1, 7.2 and 7.4. Let us suppose that there exists a wff $H \in \mathbf{L}$ such that, for every $\mathbf{INT}$-saturated set Γ containing $\mathrm{ISTP}_{\mathbf{L}}(H)$, $\mathcal{C}_{\mathbf{INT}}(\Gamma)_{/\equiv_{H_{neg}}}$ is built on a frame of $\mathcal{F}$, but $\mathrm{ISTP}_{\mathbf{L}}(H) \nvdash_{\mathbf{INT}} H$. Then, by Proposition 7.1, there exists an $\mathbf{INT}$-saturated set $\overline{\Gamma}$ such that $\mathrm{ISTP}_{\mathbf{L}}(H) \subseteq \overline{\Gamma}$ and $H \notin \overline{\Gamma}$. Now, let us consider the Kripke model $\mathcal{C}_{\mathbf{INT}}(\overline{\Gamma})_{/\equiv_{H_{neg}}}$; by hypothesis, it is built on a frame of $\mathcal{F}$; thus, $[\overline{\Gamma}] \Vdash' H$ in $\mathcal{C}_{\mathbf{INT}}(\overline{\Gamma})_{/\equiv_{H_{neg}}}$ and, by Proposition 7.4, since $H \in \mathrm{SF}(H_{neg})$, $\overline{\Gamma} \Vdash H$ in $\mathcal{C}_{\mathbf{INT}}(\overline{\Gamma})$. Therefore, by Proposition 7.2, $H \in \overline{\Gamma}$, absurd. □

Given a logic $\mathbf{L} = \mathcal{L}(\mathcal{F}) = \mathbf{INT} + (A)$ for some non empty class of Kripke frames $\mathcal{F}$, we say that $\mathbf{L}$ has the *property of the canonical quotient* ((p.c.q.) for short) iff, for every wff H and every $\mathbf{INT}$-saturated set Γ such that $\mathrm{ISTP}_{\mathbf{L}}(H) \subseteq \Gamma$, $\mathcal{C}_{\mathbf{INT}}(\Gamma)_{/\equiv_{H_{neg}}}$ is built on a frame belonging to $\mathcal{F}$.

Proposition 7.6 **Jn** *satisfies (p.c.q.).*

Proof: Let us suppose the contrary. Then, there exist an **INT**-saturated set Γ and a wff H such that $\mathrm{IstP}_{\mathbf{Jn}}(H) \subseteq \Gamma$, but the underlying frame of $\mathcal{C}_{\mathbf{INT}}(\Gamma)_{/\equiv_{H_{neg}}}$ does not belong to $\mathcal{F}_{\mathbf{Jn}}$. Therefore, by definition of $\mathcal{F}_{\mathbf{Jn}}$, there exist two different final elements $[\Gamma']$ and $[\Gamma'']$ of $\mathcal{C}_{\mathbf{INT}}(\Gamma)_{/\equiv_{H_{neg}}}$. Thus, by construction of $\mathcal{C}_{\mathbf{INT}}(\Gamma)_{/\equiv_{H_{neg}}}$ and the properties of this Kripke model, there is a propositional variable p of $\mathcal{V}(H)$ such that $[\Gamma']\Vdash' p$ and $[\Gamma'']\Vdash' \neg p$ (in $\mathcal{C}_{\mathbf{INT}}(\Gamma)_{/\equiv_{H_{neg}}}$); hence, by Proposition 7.4, $\Gamma'\Vdash \neg p$ and $\Gamma''\Vdash \neg\neg p$ (in $\mathcal{C}_{\mathbf{INT}}(\Gamma)$). It is easy to see that $\Gamma \nVdash \neg p \vee \neg\neg p$ (in $\mathcal{C}_{\mathbf{INT}}(\Gamma)$): as a matter of fact, if $\Gamma\Vdash \neg p \vee \neg\neg p$, then $\Gamma\Vdash \neg p$ or $\Gamma\Vdash \neg\neg p$ in $\mathcal{C}_{\mathbf{INT}}(\Gamma)$; in both cases one gets a contradiction, since $\Gamma''\Vdash \neg p$ contradicts the fact that $\Gamma\Vdash \neg\neg p$, and $\Gamma'\Vdash \neg\neg p$ contradicts the fact that $\Gamma\Vdash \neg p$. But since Γ is **Jn**-saturated, we must have $\neg p \vee \neg\neg p \in \Gamma$, hence $\Gamma\Vdash \neg p \vee \neg\neg p$, absurd. □

Thus, we have proved:

Corollary 7.7 *For every wff H, $H \in$ **Jn** iff* $\mathrm{IstP}_{\mathbf{Jn}}(H)\vdash_{\overline{\mathbf{INT}}} H$. □

Hence:

Theorem 7.8 *For every wff H, $H \in$ **Jn** iff there is a closed proof-table in the calculus* $\Sigma_{\mathbf{Jn}}(H)$-GCL *starting from* $\mathbf{F}H$. □

The above generalized tableau calculi (depending on the wff H to be proved) for **Jn** can be summarized in the following single tableau calculus, where the various $\Sigma_{\mathbf{L}}(H)$-rules are expressed by the the following special rule, to be applied only to sets of swff's consisting of a single **F**-swff:

$$\frac{\mathbf{F}A}{\mathbf{T}(\neg p_1 \vee \neg\neg p_1), \ldots, \mathbf{T}(\neg p_n \vee \neg\neg p_n), \mathbf{F}A}\,\mathbf{Jn}\text{-tab-rule}\ ,$$

where $\{p_1, \ldots, p_n\}$ coincides with the set of propositional variables occurring in A.

To get the whole calculus for **Jn**, one has to add the **Jn**-tab-rule to the rules of any tableau calculus for Intuitionistic Logic, for instance to the duplication-free calculus of Table 9. It is to be remarked, however, that the latter rules can be improved, at least in order to get from them a better translation into a cut-free sequent calculus. In this line, as made in [Miglioli et al., 1994b], the rule $\mathbf{F}(A \vee B)$ of Table 9 can be replaced by the two following separate rules:

$$\frac{S, \mathbf{F}(A \vee B)}{S, \mathbf{F}A}\,\mathbf{F}\vee_1 \qquad \frac{S, \mathbf{F}(A \vee B)}{S, \mathbf{F}B}\,\mathbf{F}\vee_2$$

Moreover, without any change in the completeness proof (more than this, following the idea involved in the completeness proof given in [Miglioli et al., 1994b]), one can replace the rules $\mathbf{T}\!\to\! AN$ and $\mathbf{T}\!\to\!\to$ of Table 2 (in Section 3) respectively with the following ones:

$$\frac{S, \mathbf{T}(A \to B)}{S_c, \mathbf{F}A \ /\ S, \mathbf{T}B}\,\mathbf{T}^{*}\!\to\! AN \quad \text{with A atomic or negated}$$

$$\frac{S, \mathbf{T}((A \to B) \to C)}{S_c, \mathbf{F}(A \to B), \mathbf{T}(B \to C)\ /\ S, \mathbf{T}C}\,\mathbf{T}^{*}\!\to\to$$

Now, due to the presence of the rules $\mathbf{F}\vee_1$, $\mathbf{F}\vee_2$, $\mathbf{T}^* \to AN$ and $\mathbf{T}^* \to\to$ in place of the corresponding ones of TABLE 9, the new tableau calculus for **INT**, we call **INT**-T^*, has the following remarkable property: any set of swff's generated by the calculus starting from any set of swff's containing at most one **F**-swff, *contains at most one* **F**-*swff.* Thus, we can translate (along the usual lines) **INT**-T^* into the following cut-free and duplication-free sequent calculus *whose sequents contain at most one formula in the right hand sides.* In our notation, the set Δ occurring in the right hand parts of the sequents will denote either a set containing a single wff or the empty set of wff's:

Axioms:

$$\frac{}{\Gamma, A \vdash A} \qquad \frac{}{\Gamma, A, \neg A \vdash \Delta}$$

Rules for $\wedge$:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,\mathrm{L}\wedge \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\mathrm{R}\wedge$$

Rules for $\vee$:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\,\mathrm{L}\vee \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\mathrm{R}\vee_1 \quad \text{and} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\mathrm{R}\vee_2$$

Left Rules for $\to$:

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,\mathrm{L}{\to}\mathrm{AN} \qquad \text{where } A \text{ is atomic or negated}$$

$$\frac{\Gamma, A \to (B \to C) \vdash \Delta}{\Gamma, (A \wedge B) \to C \vdash \Delta}\,\mathrm{L}{\to}\wedge \qquad \frac{\Gamma, A \to C, B \to C \vdash \Delta}{\Gamma, A \vee B \to C \vdash \Delta}\,\mathrm{L}{\to}\vee$$

$$\frac{\Gamma, B \to C \vdash A \to B \quad \Gamma, C \vdash \Delta}{\Gamma, (A \to B) \to C \vdash \Delta}\,\mathrm{L}{\to}{\to}$$

Right Rule for $\to$:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,\mathrm{R}{\to}$$

Left Rules for ¬:

$$\frac{\Gamma, \neg A \vdash \quad \Gamma, \neg B \vdash}{\Gamma, \neg(A \wedge B) \vdash \Delta}\text{L}\neg\wedge \qquad \frac{\Gamma, \neg A, \neg B \vdash \Delta}{\Gamma, \neg(A \vee B) \vdash \Delta}\text{L}\neg\vee$$

$$\frac{\Gamma, A, \neg B \vdash}{\Gamma, \neg(A \to B) \vdash \Delta}\text{L}\neg\to \qquad \frac{\Gamma, A \vdash}{\Gamma, \neg\neg A \vdash \Delta}\text{L}\neg\neg$$

Right Rule for ¬:

$$\frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\text{R}\neg$$

We observe that in rules $L\neg\wedge$, $L\neg\to$, $L\neg\neg$ and $R\neg$ the right hand parts of the sequents above the line are empty.

This provides a good cut-free and contraction-free sequent calculus for Intuitionistic Logic; it differs from the contraction-free calculus of Dyckhoff [Dyckhoff, 1992] essentially for having ¬ as a primitive symbol instead of ⊥ (the calculus of Dyckhoff with ⊥ has a smaller number of rules, but it is not necessarily more efficient in treating the negated formulas).

And now, adding to the above calculus the translation of the **Jn**-tab-rule, we get a sequent calculus for **Jn**, where the translation of the **Jn**-tab-rule is the following:

$$\frac{\neg p_1 \vee \neg\neg p_1, \ldots, \neg p_n \vee \neg\neg p_n \vdash A}{\vdash A}\textbf{Jn}\text{-seq-rule}$$

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables of A, and where the left hand part of the sequent below the line is empty.

The resulting sequent calculus for **Jn** has an inessential contraction (working just once) in the **Jn**-seq-rule; so, it can be seen as essentially contraction-free. Also, the cut rule is not involved in it and the formulas $\neg p_i \vee \neg\neg p_i$ occurring in the **Jn**-seq-rule above the line are very simple wff's built up starting from very simple subformulas (the propositional variables) of the formula below the line. So, it should be defended as a genuine cut-free calculus.

We can, however, provide a second tableau calculus for **Jn** more in line with the calculi treated in the previous sections. To do so, we take the propositional language with ∧, ∨, →, ¬ as primitive symbols, and use the four signs **T**, **F**, $\mathbf{T_{cl}}$ and $\mathbf{F_c}$. Then (even if in a new context) we preserve the rules **T**∧, **F**∧, **T**∨, **F**∨, **T**$\to AN$, **T**$\to\wedge$, **T**$\to\vee$, **T**$\to\to$, **F**$\to$, **T**¬ and $\mathbf{F_c}\neg$ of TABLE 9 (in particular, we take the rules **F**∨, **T**$\to AN$ and **T**$\to\to$ of TABLE 9 instead of the corresponding rules of [Miglioli et al., 1997] previously discussed) and add to them the following rules:

$$\frac{S, \mathbf{F}\neg A}{S, \mathbf{F_c}\neg A}\mathbf{F}\neg \qquad \frac{S, \mathbf{F}(A \to B)}{S, \mathbf{T_{cl}}A, \mathbf{T_{cl}}B, \mathbf{F}(A \to B) \ / \ S, \mathbf{T_{cl}}A, \mathbf{F_c}B, \mathbf{F}(A \to B)}\text{special-}\mathbf{F}\to$$

where special-**F**→ is the *special rule for* **F**→, to be applied only once to the swff $\mathbf{F}(A \to B)$, and hence introducing an inessential duplication;

$$\frac{S, \mathbf{T_{cl}}(A \wedge B)}{S, \mathbf{T_{cl}}A, \mathbf{T_{cl}}B}\mathbf{T}_{cl}\wedge \qquad \frac{S, \mathbf{T_{cl}}(A \vee B)}{S, \mathbf{T_{cl}}A, \mathbf{T_{cl}}B \ / \ S, \mathbf{T_{cl}}A, \mathbf{F_c}B \ / \ S, \mathbf{F_c}A, \mathbf{T_{cl}}B}\mathbf{T}_{cl}\vee$$

$$\frac{S, \mathbf{T_{cl}}\neg A}{S, \mathbf{F_c}A}\mathbf{T}_{cl}\neg \qquad \frac{S, \mathbf{T_{cl}}(A \to B)}{S, \mathbf{F_c}A, \mathbf{T_{cl}}B \ / \ S, \mathbf{F_c}A, \mathbf{F_c}B \ / \ S, \mathbf{T_{cl}}A, \mathbf{T_{cl}}B}\mathbf{T}_{cl}\to$$

$$\frac{S, \mathbf{F_c}(A \wedge B)}{S, \mathbf{F_c}A, \mathbf{T_{cl}}B \ / \ S, \mathbf{F_c}A, \mathbf{F_c}B \ / \ S, \mathbf{T_{cl}}A, \mathbf{F_c}B}\mathbf{F}_c\wedge \qquad \frac{S, \mathbf{F_c}(A \vee B)}{S, \mathbf{F_c}A, \mathbf{F_c}B}\mathbf{F}_c\vee$$

$$\frac{S, \mathbf{F_c}(A \to B)}{S, \mathbf{T_{cl}}A, \mathbf{F_c}B}\mathbf{F}_c\to \qquad \frac{S, \mathbf{F_c}\neg A}{S, \mathbf{T_{cl}}A, \mathbf{F_c}\neg A}\text{special-}\mathbf{F}_c\neg$$

where special-$\mathbf{F_c}\neg$ is the *special rule for* $\mathbf{F_c}\neg$, to be applied only once to the swff $\mathbf{F_c}\neg A$, and hence introducing an inessential duplication.

The above calculus, we call **Jn**-T^*, is immediately seen to be sound. Moreover, it can be shown to be complete, along the following lines. Given a consistent (and finite) set S of swff's, one repeatedly applies the rules of the calculus *except* $\mathbf{F}\to$ *and* $\mathbf{F_c}\neg$, thus obtaining a node set $\overline{S}$ whose final swff's have the form $\mathbf{T}a$, or $\mathbf{F}a$, or $\mathbf{F_c}a$, or $\mathbf{T_{cl}}a$, with a atomic, or $\mathbf{F}(A \to B)$, or $\mathbf{F_c}\neg A$. Moreover (due to the presence of the rules special-$\mathbf{F}\to$ and special-$\mathbf{F_c}\neg$, and of the $\mathbf{T_{cl}}$-rules and the $\mathbf{F_c}$-rules) for every swff H such that $H \equiv \mathbf{F}(A \to B)$ or $H \equiv \mathbf{F_c}\neg A$ and for every propositional variable p occurring in H, either $\mathbf{T_{cl}}p \in \overline{S}$ or $\mathbf{F_c}p \in \overline{S}$. Starting from the latter fact, one easily gets that any successor set Δ of $\overline{S}$ (corresponding to the application of $\mathbf{F}\to$ or $\mathbf{F_c}\neg$) gives rise to a node set $\overline{\Delta}$ *having the same certain part as* $\overline{S}$; and so on. Thus, one can construct a **Jn**-model $\underline{K}_{\mathbf{Jn}}(S)$ having a single final element Φ characterized by the certain part of its root $\overline{S}$.

Of course, also the calculus **Jn**-T^* can be translated into a cut-free and essentially contraction-free sequent calculus; the calculus is the following:

Axioms:

$$\frac{}{\Gamma, A \vdash A, \Delta} \qquad \frac{}{\Gamma, A, \neg A \vdash \Delta}$$

Rules for $\wedge$:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\text{L}\wedge \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\text{R}\wedge$$

Rules for $\vee$:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\text{L}\vee \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\text{R}\vee$$

Left Rules for →:

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\text{L→AN} \qquad \text{where } A \text{ is atomic or negated}$$

$$\frac{\Gamma, A \to (B \to C) \vdash \Delta}{\Gamma, (A \wedge B) \to C \vdash \Delta}\text{L→}\wedge \qquad \frac{\Gamma, A \to C, B \to C \vdash \Delta}{\Gamma, A \vee B \to C \vdash \Delta}\text{L→}\vee$$

$$\frac{\Gamma, B \to C \vdash A \to B, \Delta \qquad \Gamma, C \vdash \Delta}{\Gamma, (A \to B) \to C \vdash \Delta}\text{L→→}$$

Right Rule for →:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B, \Delta}\text{R→} \qquad \frac{\Gamma, \neg\neg A, \neg\neg B \vdash (A \to B), \Delta \qquad \Gamma, \neg\neg A, \neg B \vdash A \to B, \Delta}{\Gamma \vdash A \to B, \Delta}\text{special-R→}$$

Left Rules for ¬¬:

$$\frac{\Gamma, A \vdash}{\Gamma, \neg\neg A \vdash \Delta}\text{L}\neg\neg \qquad \frac{\Gamma, \neg A \vdash \Delta}{\Gamma, \neg\neg\neg A \vdash \Delta}\text{L}\neg\neg\neg \qquad \frac{\Gamma, \neg\neg A, \neg\neg B \vdash \Delta}{\Gamma, \neg\neg(A \wedge B) \vdash \Delta}\text{L}\neg\neg\wedge$$

$$\frac{\Gamma, \neg\neg A, \neg\neg B \vdash \Delta \qquad \Gamma, \neg\neg A, \neg B \vdash \Delta \qquad \Gamma, \neg A, \neg\neg B \vdash \Delta}{\Gamma, \neg\neg(A \vee B) \vdash \Delta}\text{L}\neg\neg\vee$$

$$\frac{\Gamma, \neg A, \neg\neg B \vdash \Delta \qquad \Gamma, \neg A, \neg B \vdash \Delta \qquad \Gamma, \neg\neg A, \neg\neg B \vdash \Delta}{\Gamma, \neg\neg(A \to B) \vdash \Delta}\text{L}\neg\neg\text{→}$$

Left Rules for ¬:

$$\frac{\Gamma, \neg A, \neg\neg B \vdash \Delta \qquad \Gamma, \neg A, \neg B \vdash \Delta \qquad \Gamma, \neg\neg A, \neg B \vdash \Delta}{\Gamma, \neg(A \wedge B) \vdash \Delta}\text{L}\neg\wedge$$

$$\frac{\Gamma, \neg A, \neg B \vdash \Delta}{\Gamma, \neg(A \vee B) \vdash \Delta}\text{L}\neg\vee \qquad \frac{\Gamma, \neg\neg A, \neg B \vdash \Delta}{\Gamma, \neg(A \to B) \vdash \Delta}\text{L}\neg\text{→}$$

Right Rule for ¬:

$$\frac{\Gamma, \neg\neg A \vdash}{\Gamma \vdash \neg A}\text{R}\neg$$

This completes our explanation of calculi for the interpolable intermediate propositional logics, which include also Intuitionistic Logic and Classical Logic. The former logic has been indirectly taken into account in the present section, in connection with the treatment of **Jn**; as for the latter, duplication-free tableau calculi or cut-free and contraction-free sequent calculi for it are universally known.

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