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**Many-valued Logics of Continuous t-norms  
and Their Functional Representation**

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# Introduction

The prevailing view in the development of logic up to XX century was that every proposition is either true or else false. This thesis was, however, already questioned in antiquity. For example in Aristotle's *De interpretatione* truth values of future contingents matters were discussed, while in medieval times truth indeterminacy was opposed by theological difficulties about divine foreknowledge (for an historical overview see [102, 80]).

Nevertheless, the beginning of formal many-valued logic is the work [78] of Jan Łukasiewicz in 1920 studying three valued logic, and the independent work of Post in 1921 [99]. Few years later, Heyting [70] introduced a three valued propositional calculus related with intuitionistic logic and Gödel [60] proposed an infinite hierarchy of finitely-valued systems: his aim is to show that there is no finitely-valued propositional calculus which is sound and complete for intuitionistic logic.

In the last few decades many-valued logics have been the object of a renewed interest: in 1965 Zadeh published the paper [115] where fuzzy sets are defined and the discipline of *Fuzzy Logic* began. Nowadays the various approaches to many-valued logics found in literature are competing as natural candidates to offer to the engineering discipline of Fuzzy Logic the theoretical foundations that have been lacking for several years.

Anyway, a strong condition distinguishes many-valued logic from fuzzy logic: fuzziness phenomena are not present at the meta-logical level, for example both the set of axioms and the related set of logical consequences in many-valued logics are crisp sets (see [95]). Logical and mathematical analysis on Fuzzy Logics with a fuzzy deduction apparatus can be found for example in Pavelka [98] and in [57], [62], [93].

The basic property of many-valued logics as studied in this thesis is the truth-functionality of their connectives: truth value of a formula only depends on the truth values of its subformulas. In this way we are excluding other non-classical logics as for example probabilistic [92], possibilistic [47]

and modal logics.

Triangular norms are the operations that are considered to fit as well as possible the notion of conjunction. When also continuity is required to connectives then the common fragment of all possible many-valued logics has been defined by Hájek and called *Basic Fuzzy Logic* in [67]: it is a propositional calculus that is sound and complete when formulas are interpreted in the interval  $[0, 1]$ , conjunction is interpreted as a continuous t-norm and implication is the corresponding residuum (see Chapter 1).

In this thesis we shall focus on the three main many-valued logics, namely Łukasiewicz, Gödel and Product logic. In particular we are interested in the truth tables of their formulas: since we suppose that the set of truth values is the interval  $[0, 1]$ , then truth tables will be functions from a suitable power of  $[0, 1]$  into  $[0, 1]$ .

In the first Chapter we shall give all the necessary definitions and notations. Many-valued logics are defined with a set of truth values in general different from  $[0, 1]$ , since in our analysis of Product logic we shall deal with an isomorphic version taking positive real numbers as truth values.

In the second Chapter we give a characterization of truth tables of the three main many-valued logics. Results for Łukasiewicz are well known in literature: McNaughton showed in [82] that truth tables of Łukasiewicz formulas are continuous piecewise linear functions, where each piece has integer coefficients. For Gödel and Product logic only existed algebraic descriptions (as in [72, 73] and [35, 36]) and the first explicit description of truth tables for Gödel logic appeared in [52].

One of the main advantages in having a functional representation for formulas of a given logic, is that we can try to check tautologies in a more direct way, using analytical and geometrical instruments.

For example, using such instruments, in [86] the satisfiability problem for Łukasiewicz logic is shown to be NP-complete. Indeed, truth tables of Łukasiewicz formulas have a very peculiar shape [82] and the information that they carry can be summarized in a finite number of points. This has been the starting point of the analysis in [8], where it has been shown that if we denote by  $\#\varphi$  the number of binary connectives and by  $n$  the number of variables in  $\varphi$ , a formula  $\varphi$  of Łukasiewicz logic is a tautology of infinite-valued logic if and only if it is a tautology in all finite-valued Łukasiewicz logics with a number of truth values less or equal to  $(\#\varphi/n)^n$ , if and only if it is a tautology of the  $2^{\#\varphi-1} + 1$ -valued logic. Very recent investigations aim to show that if  $\varphi$  is a tautology of infinite-valued Łukasiewicz logic then there exists  $m \leq \#\varphi$  such that  $\varphi$  is a tautology of  $m$ -valued Łukasiewicz

logic.

In the third Chapter we shall extend the above methods also to Gödel and Product logics and we shall use results in the fourth Section where sequents calculi are introduced for these logics.

Lukasiewicz logic has been the most investigated among t-norm based logics. This is due both to the continuity of its connectives (and then of truth tables) and to the deep results regarding its algebraic counterpart, MV-algebras.

Hence the rest of the thesis is mainly concerned with Lukasiewicz logic: in Chapter 5 we shall define Rational Lukasiewicz logic, introducing new unary connectives that allow rational slopes in truth tables [55]. In Chapter 6 we shall illustrate two main examples in which connectives of Łukasiewicz logic have a natural interpretation. These examples are used to discuss the probability of *fuzzy-events*, in particular we shall analyze subjective an conditional probability.

In the last Chapter a many-valued approach to collaborative filtering is presented, by using Rational Lukasiewicz logic. The main features are proposed and a basis for future experimentation is sketched. Result of this section have been published in [5].

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# Chapter 1

## Basic Definitions

### 1.1 Many-valued logics

We shall start giving a very general definition of propositional many valued logic, not specifying any interpretation for connectives.

**Definition 1.1.1** *The language for a propositional many-valued logic is given by a denumerable sequence  $X_1, X_2, \dots$  of propositional variables, a set  $C$  of connectives and a function  $\nu : C \rightarrow \mathbb{N}$ . A connective  $c \in C$  is  $n$ -ary if  $\nu(c) = n$ .*

*A many-valued propositional logic is a triple  $\mathcal{L} = (S, D, F)$ , where*

- $S$  is a non-empty set of truth-values,
- $D \subset S$  is the set of designated truth values,
- $F$  is a (finite) non-empty set of functions such that for any  $c \in C$  there exists  $f_c \in F$  with  $f_c : S^{\nu(c)} \rightarrow S$ .

Also if there are interesting examples of many-valued logics in which the set of truth values is not linearly ordered (for example, Belnap logic in [17]), we shall focus on logics in which the set  $S$  is linearly ordered.

Functions of  $F$  give the *truth tables* of the connectives of the logic.

**Definition 1.1.2** *A triple  $(S, D, F)$  is an infinite-valued logic if it is a many valued logic and  $S$  is an infinite set.  $(S, D, F)$  is a finite-valued logic if  $S$  is a finite set.*

If  $S$  is infinite and  $S_N = \{s_1, \dots, s_N\}$  is such that  $D \subseteq S_N \subseteq S$  and is closed with respect to all the functions in  $F$ , then the infinite-valued logic  $(S, D, F)$  naturally induces an  $N$ -valued logic  $(S_N, D, F')$  where each function in  $F'$  is the restriction to  $S_N$  of a function in  $F$ .

**Example 1.1.3** For each integer  $n > 0$ , let  $S_n$  be the set  $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ . With the above notation,

- *Boolean logic*  $\mathbf{B}$  can be written down as

$$\mathbf{B} = (\{0, 1\}, \{1\}, \{f_\wedge, f_\neg\}),$$

where

$$f_\wedge(x, y) = \min(x, y) \quad \text{and} \quad f_\neg(x) = 1 - x.$$

- *Lukasiewicz infinite-valued logic* is defined as

$$\mathbf{L}_\infty = ([0, 1], \{1\}, \{f_\oplus, f_\neg\}).$$

where

$$f_\oplus(x, y) = \min(1, x + y).$$

- *Kleene strong three valued logic* [76] is defined as

$$\mathbf{K} = (\{0, 1/2, 1\}, \{1\}, \{f_\neg, f_\vee, f_{\rightarrow_k}\})$$

where

$$f_\neg(x) = 1 - x, \quad f_\vee(x, y) = \max(x, y)$$

and

$\rightarrow_k$	0	1/2	1
0	1	1	1
1/2	1/2	1/2	1
1	0	1/2	1

- *Lukasiewicz  $(n+1)$ -valued logic* is defined as

$$\mathbf{L}_n = (S_n, \{1\}, \{f_\oplus, f_\neg\}),$$



- Gödel infinite-valued logic is given by

$$G_\infty = ([0, 1], \{1\}, \{f_\wedge, f_{\neg_G}, f_{\rightarrow_G}\}),$$

where

$$f_{\rightarrow_G}(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases} \quad \text{and} \quad f_{\neg_G}(x) = f_{\rightarrow_G}(x, 0).$$

- Product logic is

$$\Pi_\infty = ([0, 1], \{1\}, \{f_\cdot, f_{\neg_\Pi}, f_{\rightarrow_\Pi}\}),$$

where

$$f_\cdot(x, y) = xy \quad \text{and} \quad f_{\rightarrow_\Pi}(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise.} \end{cases}$$

**Definition 1.1.4** The set  $\text{Form}(S, D, F)$  of propositional formulas of a logic  $\mathcal{L} = (S, D, F)$ , is inductively defined as follows:

- Each  $X \in V$  is a formula.
- If  $*$   $\in C$ ,  $\nu(*) = k$  and  $\varphi_1, \dots, \varphi_k$  are formulas, then  $*\varphi_1 \cdots \varphi_k$  is a formula.

**Definition 1.1.5 (Satisfiability and Validity)** An assignment for  $L$  is any function  $v : V \rightarrow S$ . Assignments can be uniquely extended to the set of formulas as follows:

$$v(*\varphi_1 \cdots \varphi_k) = f_*(v(\varphi_1), \dots, v(\varphi_k)).$$

A formula  $\varphi$  is satisfied in  $\mathcal{L}$  by an assignment  $v$  if  $v(\varphi) \in D$ . A formula  $\varphi$  is valid in  $\mathcal{L}$  (or a tautology, in symbols  $\models_{\mathcal{L}} \varphi$ ) if  $\varphi$  is satisfied by all assignments, that is, if for every  $v$ ,  $v(\varphi) \in D$ .

Analogously, truth tables of connectives are extended to truth tables of formulas.

**Definition 1.1.6 (Truth tables)** The truth table of a formula  $\varphi$  whose variables are among  $X_1, \dots, X_n$ , is the function  $f_\varphi : [0, 1]^n \rightarrow [0, 1]$  such that  $f_\varphi(v(X_1), \dots, v(X_n)) = v(\varphi)$  for any assignment  $v$ .

**Proposition 1.1.7 (Function associated with a formula)** *Truth tables of formulas are inductively given by*

$$\begin{aligned} f_{X_i}(x_1, \dots, x_n) &= x_i \\ f_{*\varphi_1 \dots \varphi_k}(x_1, \dots, x_k) &= f_*(f_{\varphi_1}(x_1, \dots, x_n), \dots, f_{\varphi_k}(x_1, \dots, x_n)), \end{aligned}$$

for any  $(x_1, \dots, x_n) \in S^n$  and  $f_* \in F$ .

Assignments are canonically identified with points of  $S^n$ : if  $v(X_i) = x_i$  for all  $i \in \{1, \dots, n\}$ , then  $v(\varphi) = f_\varphi(x_1, \dots, x_n)$ .

If  $*$  is a binary connective then we write  $\varphi_1 * \varphi_2$  to denote  $*\varphi_1\varphi_2$ . Further, from now on, we shall use the same symbol to indicate a connective and its associated operation.

**Definition 1.1.8 (Logical Consequence)** *Let  $S$  be a linearly ordered lattice and  $D$  be an upward closed subset of  $S$  (i.e., if  $x \in D$  and  $y \geq x$  then  $y \in D$ ). We say that the finite set of formulas  $\Delta$  is a logical consequence of the finite set of formulas  $\Gamma$  (and we write  $\Gamma \models_{\mathcal{L}} \Delta$ ) if*

$$\inf_{\gamma \in \Gamma} v(\gamma) \in D \text{ implies } \sup_{\delta \in \Delta} v(\delta) \in D$$

for any  $v$  assignment for  $L$ . So,  $\Gamma \models_{\mathcal{L}} \Delta$  if, whenever  $v$  is an assignment satisfying all the formulas of  $\Gamma$ , then there exists at least one formula in  $\Delta$  that is satisfied by  $v$ .

**Definition 1.1.9 (Subformulas)** *For each formula  $\varphi$ , let  $Sub(\varphi)$  be the set of all subformulas of  $\varphi$ . If  $\psi \in Sub(\varphi)$  then we write  $\psi \preceq \varphi$ . Henceforth, different occurrences of the same subformula in  $\varphi$  shall be considered as different subformulas. Two occurrences  $\varphi$  and  $\psi$  of formulas are disjoint if neither  $\varphi \preceq \psi$  nor  $\psi \preceq \varphi$ . If  $\psi \preceq \varphi$  and  $\psi \neq \varphi$  we write  $\psi \prec \varphi$ .*

Let  $\mathcal{L} = (S, D, F)$  be a logic with only unary and binary connectives. If  $\varphi$  is a formula of  $\mathcal{L}$ , we shall denote by  $\text{var}(\varphi)$  the set of all variables occurring in  $\varphi$ . For each  $X \in \text{var}(\varphi)$ , let  $\#(X, \varphi)$ , giving the number of occurrences of  $X$  in  $\varphi$ , be inductively defined as follows:

- If  $\varphi = X$  then  $\#(X, \varphi) = 1$ . If  $\varphi = Y$  for some variable  $Y \neq X$ , then  $\#(X, \varphi) = 0$ .
- If  $\sim$  is a unary connective, then  $\#(X, \sim \psi) = \#(X, \psi)$ .
- If  $\star$  is a binary connective, then  $\#(X, \psi \star \vartheta) = \#(X, \psi) + \#(X, \vartheta)$ .

Then the total number of occurrences of variables in  $\varphi$  is given by:

$$\#(\varphi) = \sum_{X \in \text{var}(\varphi)} \#(X, \varphi).$$

Since now we have described propositional many valued logic from a semantical point of view, i.e., by means of truth tables.

The syntactic approach is the same as for propositional classical logic. A set of formulas called *axioms* is fixed and the inference rule is *modus ponens*: from  $\varphi$  and  $\varphi \rightarrow \psi$  we can infer  $\psi$ .

**Definition 1.1.10 (Proof and provability)** *If  $\Gamma$  is a set of formulas,  $\Gamma \vdash \varphi$  means that  $\Gamma$  proves  $\varphi$  (or  $\varphi$  is provable from  $\Gamma$ ), that is there exists a sequence of formulas  $\gamma_1, \dots, \gamma_u$  such that  $\gamma_u = \varphi$  and every  $\gamma_i$  either is an axiom, or belongs to  $\Gamma$  or is obtained from  $\gamma_{i_1}, \gamma_{i_2}$  ( $i_1, i_2 < i$ ) by modus ponens.  $\varphi$  is provable ( $\vdash \varphi$ ) if is provable from the emptyset.*

A logic satisfies the completeness theorem if the set of provable formulas coincides with the set of valid formulas.

## 1.2 Continuous t-norms and BL-algebras

In the above section we have introduced many-valued logics with *truth-functional* connectives: the truth value of the compound formula  $\varphi * \psi$  is determined only by the truth values of  $\varphi$  and  $\psi$ . In this section we shall focus our attention on many-valued logics having  $[0, 1]$  as set of truth values and where the only designated truth value is 1. Further, we shall fix an interpretation of connectives: a good candidate for the truth table of conjunction of two propositions should be a commutative and associative operation. It is also natural to assume that the truth degree of the conjunction of a proposition with a complete falsity should be completely false, and the conjunction of a proposition with a complete truth should not have smaller truth degree than the proposition has.

These properties are witnessed by the following operation (for an overview, see [24]).

**Definition 1.2.1 (t-norm)** *A t-norm is a binary operation  $*$  on  $[0, 1]$  such that*

- $*$  is commutative and associative, i.e., for all  $x, y, z \in [0, 1]$ ,

$$\begin{aligned}x * y &= y * x \\(x * y) * z &= x * (y * z),\end{aligned}$$

- $*$  is non-decreasing in both arguments

$$\begin{aligned}x_1 \leq x_2 &\text{ implies } x_1 * y \leq x_2 * y, \\y_1 \leq y_2 &\text{ implies } x * y_1 \leq x * y_2\end{aligned}$$

- $1 * x = x$  and  $0 * x = 0$  for all  $x \in [0, 1]$ .

**Example 1.2.2** The following are example of  $t$ -norms. All are continuous  $t$ -norms, with the exception of (iv).

(i) Lukasiewicz  $t$ -norm:  $x \odot y = \max(0, x + y - 1)$ .

(ii) Product  $t$ -norm:  $x \cdot y$  usual product between real numbers.

(iii) Gödel  $t$ -norm:  $x \wedge y = \min(x, y)$ .

(iv) Drastic  $t$ -norm:  $x *_D y = \begin{cases} 0 & \text{if } (x, y) \in [0, 1]^2 \\ \min(x, y) & \text{otherwise.} \end{cases}$

(v) The family of Frank  $t$ -norms [51] is given by:

$$x *_F^\lambda y = \begin{cases} x \odot y & \text{if } \lambda = 0 \\ x \cdot y & \text{if } \lambda = 1 \\ \min(x, y) & \text{if } \lambda = \infty \\ \log_\lambda \left( 1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1} \right) & \text{otherwise.} \end{cases}$$

An element  $x \in [0, 1]$  is *idempotent* with respect to a  $t$ -norm  $*$ , if  $x * x = x$ . For each continuous  $t$ -norm  $*$ , the set  $E$  of all idempotents is a closed subset of  $[0, 1]$  and hence its complement is a union of a set  $\mathcal{I}_{\text{Open}}(E)$  of countably many non-overlapping open intervals. Let  $[a, b] \in \mathcal{I}(E)$  if and only if  $(a, b) \in \mathcal{I}_{\text{Open}}(E)$ . For  $I \in \mathcal{I}(E)$  let  $*|I$  the restriction of  $*$  to  $I^2$ .

The following theorem [77, 51] characterizes all continuous  $t$ -norms as ordinal sums of Lukasiewicz, Gödel and product  $t$ -norms:

**Theorem 1.2.3** If  $*$ ,  $E, \mathcal{I}(E)$  are as above, then

- (i) for each  $I \in \mathcal{I}(E)$ ,  $*|I$  is isomorphic either to the Product  $t$ -norm or to Lukasiewicz  $t$ -norm.

(ii) If  $x, y \in [0, 1]$  are such that there is no  $I \in \mathcal{I}(E)$  with  $x, y \in I$ , then  $x * y = \min(x, y)$ .

When choosing a continuous t-norm as the truth table for conjunction, the following proposition enable us to obtain a truth table for implication:

**Proposition 1.2.4 (Residuum)** *Let  $*$  be a continuous t-norm. Then, for every  $x, y, z \in [0, 1]$ , the operation*

$$x \rightarrow_* y = \max\{z \mid x * z \leq y\}$$

is the the unique operation satisfying the condition

$$(x * z) \leq y \quad \text{if and only if} \quad x \leq (x \rightarrow_* y)$$

The operation  $\rightarrow_*$  is called *residuum* of the t-norm  $*$ .

**Example 1.2.5** *The following are residua of the three main continuous t-norms:*

	<b>T-norm</b>	<b>Residuum</b>
<b>L</b>	$x \odot y = \max(x + y - 1, 0)$	$x \rightarrow_{\odot} y = \min(1, 1 - x + y)$
<b>P</b>	$x \cdot y$ usual product of reals	$x \rightarrow_{\cdot} y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$
<b>G</b>	$x \wedge y = \min(x, y)$	$x \rightarrow_{\wedge} y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$

The problem of finding an appropriate axiomatization of many-valued logics based on continuous t-norm has been approached by introducing suitable classes of algebraic structures. In [71], monoidal logic is introduced and Hájek in [67] defines Basic (fuzzy) Logic. In the following we shall briefly describe some important features of Basic Logic.

In the notation of Section 1.1, Basic Fuzzy infinite-valued Logic (BL) is the triple  $([0, 1], \{1\}, \{f_*, f_{\rightarrow_*}, f_0\})$ , where  $f_*$  is a continuous t-norm,  $f_{\rightarrow_*}$  is its associates residuum and  $f_0$  is the function identically equal to 0.

Truth tables of other derived connectives are defined as follows:

$$x \wedge y = x * (x \rightarrow_* y) \tag{1.1}$$

$$x \vee y = ((x \rightarrow_* y) \rightarrow_* y) \wedge ((y \rightarrow_* x) \rightarrow_* x) \tag{1.2}$$

$$\neg x = x \rightarrow_* 0 \tag{1.3}$$

$$x \equiv y = (x \rightarrow_* y) * (y \rightarrow_* x) \tag{1.4}$$

From a syntactic point of view, formulas of Basic Propositional Fuzzy Logic are built in the usual way from the connectives of conjunctions ( $*$ ), implication ( $\rightarrow$ ), and from the constant 0. An axiom is a formula that can be written in any one of the following ways, where  $\varphi$ ,  $\psi$  and  $\chi$  denote arbitrary formulas:

- (A1)  $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
- (A2)  $(\varphi * \psi) \rightarrow \varphi$
- (A3)  $(\varphi * \psi) \rightarrow (\psi * \varphi)$
- (A4)  $(\varphi * (\varphi \rightarrow \psi) \rightarrow (\psi * (\psi \rightarrow \varphi)))$
- (A5a)  $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi * \psi) \rightarrow \chi)$
- (A5b)  $((\varphi * \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
- (A6)  $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
- (A7)  $0 \rightarrow \varphi$ .

In order to prove completeness, in [67] the author also introduced BL-algebras:

**Definition 1.2.6** *A BL-algebra is an algebra*

$$\mathbf{L} = (L, \cup, \cap, *, \rightarrow, 0, 1)$$

with four binary operations and two constants such that

- (i)  $(L, \cup, \cap, 0, 1)$  is a lattice with largest element 1 and least element 0 (with respect to the lattice ordering  $\leq$ ),
- (ii)  $(L, *, 1)$  is a commutative semigroup with the unit element 1, i.e.,  $*$  is commutative, associative,  $1 * x = x$  for all  $x$  (thus  $\mathbf{L}$  is a residuated lattice),
- (iii) the following conditions holds:

- (1)  $z \leq (x \rightarrow y)$  if and only if  $x * z \leq y$ , for all  $x, y, z \in L$
- (2)  $x \cap y = x * (x \rightarrow y)$
- (3)  $x \cap y = ((x \rightarrow y) \rightarrow y) \cap ((y \rightarrow x) \rightarrow x)$
- (4)  $(x \rightarrow y) \cap (y \rightarrow x) = 1$

**Example 1.2.7** *The unit interval  $[0, 1]$  equipped with a continuous  $t$ -norm and the corresponding residuum, is a BL-algebra.*

At first step, Hájek showed that a propositional formula is provable in BL if and only if it is a tautology in any linearly ordered BL-algebra. However, the *completeness theorem* of BL with respect to BL-algebras, i.e., that a

formula is provable in BL if and only if it is a tautology in  $[0, 1]$ , was left as an open problem in [67]. In [66] Hájek proved that such completeness theorem can be obtained provided two new axioms were added to the original axiomatic system of BL. In [33] the authors showed that these axioms are redundant, hence the original axiomatic of BL is sound and complete with respect to the algebraic structure induced by continuous t-norms on  $[0, 1]$ .

Instead of considering a general continuous t-norm, we shall focus now on Łukasiewicz, Gödel and Product t-norms. In the following sections we shall define the corresponding logics. In order to prove that axiomatizations of these three logics exactly describe the truth tables of the corresponding connectives in the interval  $[0, 1]$ , the same argument as for BL-algebras can be applied.

Let  $\mathcal{L}$  denote any of Łukasiewicz, Gödel and Product logic. Then a variety of algebraic structures  $\mathcal{A}^{\mathcal{L}}$  can be defined associated with  $\mathcal{L}$  and the following results can be proved (see for example [63], [67]):

- Examples of algebras in  $\mathcal{A}^{\mathcal{L}}$  are the unit interval  $[0, 1]$  with truth functions of connectives of  $\mathcal{L}$  as operations and the algebra of classes of provably equivalent formulas.
- If  $\varphi \in Form(\mathcal{L})$  is provable then  $\varphi = 1$  is valid in all algebras of  $\mathcal{A}^{\mathcal{L}}$ .
- Each algebra in  $\mathcal{A}^{\mathcal{L}}$  is a subalgebra of the direct product of some linearly ordered algebra.
- If  $\varphi = 1$  is valid in the algebra  $[0, 1]$  then it is valid in all linearly ordered algebras, in particular in the algebra of classes of formulas, which means that  $\varphi$  is a provable formula.

As a consequence of completeness theorem for logic  $\mathcal{L}$  with respect to algebras  $\mathcal{A}^{\mathcal{L}}$  we also have that the algebra of truth value of formulas of  $\mathcal{L}$  with  $n$  variables is the free algebra in the variety  $\mathcal{A}^{\mathcal{L}}$  over  $n$  generators.

### 1.2.1 Łukasiewicz logic and MV-algebras

Łukasiewicz infinite-valued propositional logic  $L_{\infty}$  is the triple  $([0, 1], \{1\}, \{f_{\odot}, f_{\rightarrow}, f_0\})$ , where

$$\begin{aligned} f_{\odot}(x, y) &= \max(0, x + y - 1) \\ f_{\rightarrow}(x, y) &= \min(1, 1 - x + y) \\ f_0 &= 0. \end{aligned}$$

Among infinite-valued systems, Łukasiewicz logic is the most extensively studied. A system of axioms for Łukasiewicz logic is furnished by Axioms for Basic Logic plus *double negation*:

$$\neg\neg\varphi \rightarrow \varphi,$$

where negation  $\neg$  is defined in (1.3) and is such that  $f_{\neg}(x) = 1 - x$ . Originally, in [79] Łukasiewicz infinite-valued logic was axiomatized (using implication and negation as the basic connectives) by the following schemata:

- L1)  $\varphi \rightarrow (\psi \rightarrow \varphi)$
- L2)  $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\varphi \rightarrow \theta))$
- L3)  $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$
- L4)  $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$
- L5)  $((\varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \varphi)) \rightarrow (\psi \rightarrow \varphi).$

Chang [28] and Meredith [83] proved independently that the last axiom is derivable from the others.

From  $\odot$  and  $\neg$  it is possible to define, in addition to connectives (1.1), (1.2), (1.4), the connective

$$x \oplus y = \neg(\neg x \odot \neg y)$$

such that  $f_{\oplus}(x, y) = \min(1, x + y)$ . Actually, any of the sets

$$\{\odot, \neg\}, \{\oplus, \neg\}, \{\rightarrow, \neg\}, \{\odot, \rightarrow\}$$

can be used to define all the other connectives.

In order to prove the completeness of this schemata of axioms with respect to semantics of the interval  $[0, 1]$ , Chang introduced *MV-algebras* in [27]. In the following we shall summarize some of the main results for the theory of MV-algebras. A standard reference is [32].

An MV-algebra is a structure  $A = (A, \oplus, \neg, 0, 1)$  satisfying the following equations:

$$\begin{aligned} x \oplus (y \oplus z) &= (x \oplus y) \oplus z \\ x \oplus y &= y \oplus x \\ x \oplus 0 &= x \\ x \oplus 1 &= 1 \\ \neg 0 &= 1 \\ \neg 1 &= 0 \end{aligned}$$



$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

As proved by Chang, boolean algebras coincide with MV-algebras satisfying the additional equation  $x \oplus x = x$  (idempotency). Each MV-algebra contains as a subalgebra the two-element boolean algebra  $\{0, 1\}$ . The set  $B(A)$  of all idempotent elements of an MV-algebra  $A$  is the largest boolean algebra contained in  $A$  and is called the *boolean skeleton* of  $A$ .

The monoids  $(A, \oplus, 0)$  and  $(A, \odot, 1)$  are isomorphic via the map

$$\neg : x \mapsto \neg x. \quad (1.5)$$

Further any MV-algebra  $A$  is equipped with the order relation

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1. \quad (1.6)$$

MV-algebras turn out to coincide with those BL-algebras satisfying the equation  $\neg\neg x = x$ .

**Example 1.2.8** (i) *The set  $[0, 1]$  equipped with operations*

$$x \oplus y = \min\{1, x + y\}, \quad x \odot y = \max\{0, x + y - 1\}, \quad \neg x = 1 - x. \quad (1.7)$$

*is an MV-algebra.*

(ii) *For each  $k = 1, 2, \dots$ , the set*

$$L_{k+1} = \left\{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\right\}, \quad (1.8)$$

*equipped with operations as in 1.7, is a linearly ordered MV-algebra (also called MV-chain).*

(iii) *If  $X$  is any set and  $A$  is an MV-algebra, the set of functions  $f : X \rightarrow A$  obtained by pointwise application of operations in  $A$  is an MV-algebra.*

(iv) *The set of all functions from  $[0, 1]^n$  into  $[0, 1]$  that are continuous and piecewise linear, and such that each linear piece has integer coefficients, and operations are obtained as pointwise application of operation in 1.7, is an MV-algebra (actually, the free MV-algebra over  $n$  free generators).*

Chang's Completeness Theorem states:

**Theorem 1.2.9** *An equation holds in every MV-algebra if and only if it holds in the MV-algebra  $[0, 1]$  equipped with operations  $x \oplus y = \min\{1, x + y\}$ ,  $x \odot y = \max\{0, x + y - 1\}$  and  $\neg x = 1 - x$ .*

This theorem was proved by Chang using quantifier elimination for totally ordered divisible abelian groups. There are several alternative proofs in literature: the syntactic proof by Rose and Rosser [105], the algebraic proof by Cignoli and Mundici [34] and the geometric proof by Panti [94].

Mundici in [85] constructed an equivalence functor  $\Gamma$  from the category of  $\ell$ -groups with strong unit to the category of MV-algebras:

A lattice-ordered group ( $\ell$ -group for short)  $G = (G, 0, -, +, \wedge, \vee)$  is an abelian group  $(G, 0, -, +)$  equipped with a lattice structure  $(G, \wedge, \vee)$  such that, for every  $a, b, c \in G$ ,  $c + (a \wedge b) = (c + a) \wedge (c + b)$ . An  $\ell$ -group is said to be totally ordered if the lattice-order is total. An element  $u \in G$  is a *strong unit* of  $G$  if for every  $x \in G$  there exists  $n \in \mathbb{N}$  such that  $nu \geq x$ . If  $G$  is an  $\ell$ -group and  $u$  is a strong unit for  $G$ , the MV-algebra  $\Gamma(G, u)$  has the form  $\{x \in G \mid 0 \leq x \leq u\}$  and operations are defined by  $x \oplus y = u \wedge x + y$  and  $\neg x = u - x$ . If  $A$  is an MV-algebra we shall denote by  $G_A$  the  $\ell$ -group corresponding to  $A$  via  $\Gamma$ .

In [41] the author shows that every MV-algebra is an algebra of functions taking values in an ultrapower of the interval  $[0, 1]$ .

## 1.2.2 Gödel Logic and Gödel algebras

Gödel infinite-valued propositional logic  $G_\infty$  is the triple  $([0, 1], \{1\}, \{f_\wedge, f_{\neg_G}\})$ , where

$$\begin{aligned} f_\wedge(x, y) &= \min(x, y) \\ f_{\neg_G}(x) &= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Finite-valued Gödel propositional logics  $G_n$  were introduced in [60] to prove that intuitionistic propositional logic cannot be viewed as a system of finite-valued logic. In [48], Dummett proved completeness of such system. Gödel propositional logic can be defined as the fragment of intuitionistic logic satisfying the axiom  $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$ . Theorems of Gödel logic are exactly those formulas which are valid in every linearly ordered Heyting algebra, where Heyting algebras are the structure naturally associated with intuitionistic logic. An analysis of Gödel logic can be found in [110, 14].

In Hájek framework, Gödel logic is obtained adding to axioms of Basic Logic the axiom

$$(G1) \quad \varphi \rightarrow (\varphi * \varphi)$$

stating the idempotency of  $*$ . Gödel algebras are BL-algebras satisfying the identity  $x * x = x$ .

### 1.2.3 Product Logic and PL algebras

Product Logic  $\Pi$  is the triple  $([0, 1], \{1\}, \{f, f_{\neg_G}\})$  where:

$$\begin{aligned} f(x, y) &= x \cdot y && \text{usual product of reals} \\ f_{\neg_G}(x) &= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In [37] it is shown that Axioms for Product Logic can be obtained by adding

$$(PI1) \quad \neg\neg\varphi \rightarrow ((\varphi \rightarrow \varphi \cdot \psi) \rightarrow \psi \cdot \neg\neg\psi)$$

to Axioms of Basic Logic.

Product logic algebras, or PL-algebras for short, were introduced in [68], where the completeness theorem for Product logic was proved. In [1] authors showed that the class of product algebras is the equivalent algebraic semantics (in the sense of [20]) of Product logic.

## 1.3 Notation and Geometrical definitions

In this Section we shall fix some notation and recall some basic notion of linear algebra, Euclidean geometry and topology that will be useful in the sequel.

By  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  we denote the set of natural, integer, rational and real numbers, respectively.  $\mathbb{Q}_+$  and  $\mathbb{R}_+$  are the set of greater or equal to zero rational and real numbers, respectively. Further,

$$S_n = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}.$$

Lukasiewicz, Gödel and Product infinite-valued logic will be denoted by  $L_\infty$ ,  $G_\infty$  and  $\Pi$ , respectively. Lukasiewicz and Gödel  $n + 1$ -valued logic will be denoted by  $L_n$  and  $G_n$ , respectively.

If  $\mathbf{x} = (h_1/k_1, \dots, h_n/k_n) \in \mathbb{R}_+^n$ , with  $0 \leq h_i \leq k_i$  ( $k_i \neq 0$ ) integers and  $\gcd(h_i, k_i) = 1$ , we denote by  $\text{den}(\mathbf{x})$  the least common multiple  $\text{lcm}(k_1, \dots, k_n)$ .

**Definition 1.3.1** If  $a_1/b_1 \dots a_n/b_n$  are distinct positive rational numbers in irreducible form, (i.e.,  $\gcd(a_i, b_i) = 1$  and  $b_i > 0$ ), then their Farey mediant is the rational number

$$\frac{a_1 + \dots + a_n}{b_1 + \dots + b_n}$$

The Farey mediant of distinct points  $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{Q}_+^m$  with  $m \geq n + 1$ , is obtained by coordinatewise application of the Farey mediant to components of  $\mathbf{x}_1, \dots, \mathbf{x}_n$ .

Note that the Farey mediant of  $\mathbf{x}_1, \dots, \mathbf{x}_n$  is a proper (i.e., different from each  $\mathbf{x}_1, \dots, \mathbf{x}_n$ ) convex combination of such points. Our present definition of Farey mediant differs from other definitions in the literature (for example the one given in [32]) since in general the point obtained by coordinatewise application of the Farey mediant is not in irreducible form. Anyway we are interested in the fact that for every choice of  $\mathbf{x}_1, \dots, \mathbf{x}_n$ , there always exists a convex combination of  $\mathbf{x}_1, \dots, \mathbf{x}_n$  having denominator less than or equal to the denominator of the Farey mediant of  $\mathbf{x}_1, \dots, \mathbf{x}_n$ .

A linear function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be *homogeneous* if  $f(\mathbf{0}) = 0$ . We say that a continuous piecewise linear function is homogeneous if and only if every linear piece is homogeneous. By  $E_n$  we mean the set  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  of  $n$ -dimensional unit vectors  $(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$ . We shall denote by  $\mathbf{0}$  the vector  $(0, \dots, 0)$  and by  $\mathbf{1}$  the vector  $(1, \dots, 1)$ . Given the Euclidean space  $\mathbb{R}^n$ , and a subset  $S \subseteq \mathbb{R}^n$ ,  $\dim(S)$  denotes the *dimension* of  $S$ . By definition  $\dim(\emptyset) = -1$ .

For any two vectors  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ , the *scalar product*  $\mathbf{x} \cdot \mathbf{y}$  is, as usual, the real number  $x_1y_1 + \dots + x_ny_n$ .

The set  $H^-$  of solutions  $\mathbf{x} = (x_1, \dots, x_n)$  of an inequality of the form  $\mathbf{a} \cdot \mathbf{x} \leq b$ , for  $\mathbf{a} \in \mathbb{R}^n$ ,  $b \in \mathbb{R}$ , is called an (*affine*) *halfspace* of  $\mathbb{R}^n$ . The solutions of the corresponding equation  $\mathbf{a} \cdot \mathbf{x} = b$  form the *supporting hyperplane*  $H$  of  $H^-$ .

A *polyhedron*  $P$  is the set-theoretic intersection  $P = \bigcap_{i \in I} H_i^-$  of a finite number of halfspaces. Each supporting hyperplane  $H_j$  ( $j \in I$ ) such that  $H_j \cap P \neq \emptyset$  is called a *bounding hyperplane* of  $P$ . Note that  $P = \bigcap \{H_i^- : \dim(H_i \cap P) = \dim(P) - 1\}$ .

By a *facet* of  $P$  we mean any  $(\dim(P) - 1)$ -dimensional polyhedron  $F$ , arising as the intersection of  $P$  with one of its bounding hyperplanes. The set of *faces* of a polyhedron  $P$  is defined as follows:

- $\emptyset$  and  $P$  are faces of  $P$ .
- Each facet of a face of  $P$  is a face of  $P$ .

Set-theoretic inclusion makes the set  $\mathbf{F}(P)$  of faces of  $P$  into a lattice  $(\mathbf{F}(P), \subseteq)$ .

A *polyhedral complex*  $C$  is a set of polyhedra such that

- If  $A \in C$  then  $\mathbf{F}(A) \subseteq C$ .
- If  $A, B \in C$  then  $A \cap B$  is a face of both  $A$  and  $B$ , i.e.,  $A \cap B \in \mathbf{F}(A) \cap \mathbf{F}(B)$ .

Each polyhedron in  $C$  is called a *cell* of  $C$ . Let  $0 \leq k \leq n$  be integers such that  $n$  is the maximum dimension of cells in  $C$ . The set  $C^{(k)}$  of all  $k$ -dimensional cells of  $C$  is called the  *$k$ -skeleton* of  $C$ . Let us denote by  $\mathbf{F}(C)$  the set of faces of cells belonging to  $C$ .

Finally, we shall deal with topological notions. Every  $l$ -dimensional space will be considered equipped with the topology induced by the Euclidean metric (which is the same as the natural topology on  $\mathbb{R}^l$ ). Let  $F$  be an  $l$ -dimensional face of a cell in a polyhedral complex.  $F$  is a closed set in the  $l$ -dimensional space containing  $F$ . By the *open  $l$ -dimensional face* of a cell we mean the biggest set contained in a face of the cell that is open with respect to the topology of the  $l$ -dimensional space containing such face. In other words, if  $F \in \mathbf{F}(C)$  then  $F \setminus \bigcup\{G \in \mathbf{F}(C) \mid G \subseteq F \text{ and } G \neq F\}$  (the *relative interior* of  $F$ ) is an open face.

See [50],[74] for further references.

## Chapter 2

# Functional Representation

The aim of this section is to describe the truth tables of Łukasiewicz, Gödel and Product formulas. In the next chapters these results will be used to analyze different aspects of the above logics.

In Section 2.1 we shall recall well known results for Łukasiewicz logic and we shall give a description of the free MV-algebra over one variable by means of discrete functions (following [44]). In Section 2.2 we shall give a characterization of truth tables of formulas of Gödel logic, while in Section 2.3 we shall focus attention on truth tables of formulas of Product logic.

### 2.1 Łukasiewicz logic

The work on functional representation of many-valued logics began with McNaughton [82] in 1951, who described the class of functions associated with Łukasiewicz logic.

**Definition 2.1.1** *A McNaughton function  $f : [0, 1]^n \rightarrow [0, 1]$  is a continuous, piecewise linear function such that each piece has integer coefficients: that is, there exist finitely many polynomials  $p_1, \dots, p_{m_f}$  each  $p_i$  being of the form  $p_i(x_1, \dots, x_n) = a_{i1}x_1 + \dots + a_{in}x_n + b_i$ , with  $a_{i1}, \dots, a_{in}, b_i$  integers, such that, for any  $\mathbf{x} \in [0, 1]^n$ , there exists  $j \in \{1, \dots, m_f\}$  for which  $f(\mathbf{x}) = p_j(\mathbf{x})$ .*

**Theorem 2.1.2 (McNaughton theorem)** *A function  $f : [0, 1]^n \rightarrow [0, 1]$  is a truth table of a Łukasiewicz formula if and only if it is a McNaughton function.*

Mundici [89] was the first to give a constructive proof of McNaughton Theorem 2.1.2, describing an algorithm that for each continuous piecewise

linear function  $f$  finds a formula  $\varphi$  such that  $f$  is the truth table of  $\varphi$ . Since then, different proofs have been given (see for example the ones in [3] and in [93]).

The study of McNaughton functions gives many insights in Łukasiewicz logic:

- it allows a representation of Łukasiewicz formulas in normal form, by means of Schauder hats and unimodular triangulations corresponding tightly to the branch of algebraic geometry known as *desingularization of toric varieties* [12, 4];
- it was a source of inspiration for the geometrical proof of completeness theorem in [94];
- it was used to show that the satisfiability problem for Łukasiewicz logic is NP-complete (see [86]);
- it allowed a reduction of infinite-valued Łukasiewicz logic to finite-valued logic [8].

Further, McNaughton Theorem gives us a concrete geometric representation of free MV-algebras:

**Theorem 2.1.3** *The free MV-algebra over  $n$  generators is isomorphic to the MV-algebra  $\mathcal{M}_n$  obtained by the pointwise application of Łukasiewicz connectives to McNaughton functions.*

### 2.1.1 A discrete free MV-algebra over one generator

Formulas of Łukasiewicz infinite-valued propositional calculus with only one variable, have been often investigated (see, for example, [2],[89]) for their immediate geometrical interpretation.

In this section we construct a set of discrete functions and we equip it with operations yielding an isomorphism with the set of McNaughton functions of one variable. Results of this section appeared in [44].

Let  $\mathcal{M}_1$  be the class of McNaughton functions of one variable, i.e., the class of functions from  $[0, 1]$  in  $[0, 1]$  that are continuous, piecewise linear where each piece has integer coefficients. Let  $f \in \mathcal{M}_1$ . By Definition 2.1.1 of McNaughton function, for every integer  $n > 0$  and for  $i = 0, 1, \dots, n$ ,  $f(i/n) \in S_n$ .

**Definition 2.1.4** *A node for a McNaughton function  $f$  is a rational number  $r$  such that  $f(r)$  is a point in which  $f$  is not differentiable. Further, 0 and 1 are nodes for every function  $f \in \mathcal{M}_1$ .*

Let  $K(f)$  be the set of nodes for  $f$ . By Definition 2.1.1,  $K(f)$  is a finite set. By  $den(r)$  we denote the minimum denominator of a rational number  $r$ . Let  $K(f) = \{r_1, \dots, r_s\}$ . If  $n = \text{lcm}(den(r_1), \dots, den(r_s))$ , then we set

$$Discr : f \in \mathcal{M}_1 \mapsto Discr(f) \in S_n^{S_n}$$

such that  $Discr(f)$  is the restriction of  $f$  to  $S_n$ . Let us denote by

$$\mathcal{D} = Discr(\mathcal{M}_1) = \{g \mid g = Discr(f) \text{ with } f \in \mathcal{M}_1\}.$$

The set  $\mathcal{D}$  is strictly included in the set of all discrete functions obtained as restrictions of McNaughton functions to a finite domain. Further,  $\mathcal{D}$  is disjoint from  $\mathcal{M}_1$ .

**Example 2.1.5** Consider the identity function  $\iota \in \mathcal{M}_1$ . The restrictions  $\iota_1 = \iota|_{S_1} : x \in S_1 \rightarrow x \in S_1$  and  $\iota_2 = \iota|_{S_2} : x \in S_2 \rightarrow x \in S_2$  are such that  $\iota_1 \in \mathcal{D}$  and  $\iota_2 \notin \mathcal{D}$ .

For every  $g : S_n \rightarrow S_n$  such that  $g \in \mathcal{D}$ , the integer  $n$  will be called the *dimension* of  $g$  and denoted by  $dim(g)$ .

To every function  $g : S_n \rightarrow S_n$  we associate a continuous function  $Cont(g) : [0, 1] \rightarrow [0, 1]$  such that:

- $Cont(g)(\frac{i}{n}) = g(\frac{i}{n})$ ;
- for every  $x \in [\frac{i}{n}, \frac{i+1}{n}]$ ,

$$Cont(g)(x) = g(\frac{i}{n}) + (nx - i)(g(\frac{i+1}{n}) - g(\frac{i}{n})).$$

**Proposition 2.1.6** Let  $f \in \mathcal{M}_1$  and let  $n > 0$ . Then:

- i)  $Cont(Discr(f)) = f$ ;
- ii) There exists  $g : S_n \rightarrow S_n$  such that  $Discr(Cont(g)) \neq g$ .
- iii) If  $g \in \mathcal{D}$  then  $Discr(Cont(g)) = g$ .

**Proof.**

- i) Let  $n$  be the least common multiple of the denominators of all elements in  $K(f)$ . By definition,  $Discr(f)(x) = f(x)$  for every  $x \in S_n$  and in particular for every node of  $f$ . The function  $Cont(Discr(f))$  is formed by linear interpolation of points  $(x, Discr(f)(x)) = (x, f(x))$  with  $x \in S_n$ , and so it coincides with  $f$ .



- ii) Consider, for example,  $\iota_2 : x \in S_2 \mapsto x \in S_2$ . Then  $Cont(\iota_2) : x \in [0, 1] \mapsto x \in [0, 1]$  and  $K(Cont(\iota_2)) = \{0, 1\}$  so that  $Discr(Cont(\iota_2)) : x \in S_1 \mapsto x \in S_1$  and  $Discr(Cont(\iota_2)) = \iota_1 \neq \iota_2$ .
- iii) If  $g = Discr(f)$ , with  $f \in \mathcal{M}_1$ , then  $Cont(g) = f$  and  $Discr(Cont(g)) = g$ .

■

### Description of the class $\mathcal{D}$

For every function  $g : S_n \rightarrow S_n$ , let

$$\pi(g) = \left( n \cdot \left( g\left(\frac{1}{n}\right) - g(0) \right), n \cdot \left( g\left(\frac{2}{n}\right) - g\left(\frac{1}{n}\right) \right), \dots, n \cdot \left( g(1) - g\left(\frac{n-1}{n}\right) \right) \right)$$

and

$$\sigma(g) = \left( g(0), \frac{2}{n} \cdot g\left(\frac{1}{n}\right) - \frac{1}{n} \cdot g\left(\frac{2}{n}\right), \dots, g\left(\frac{n-1}{n}\right) - \frac{n-1}{n} \cdot g(1) \right).$$

Then, for every  $i \in \{0, \dots, n-1\}$ ,  $y = \pi(g)(i) \cdot x + \sigma(g)(i)$  is the equation of the line between points  $\left(\frac{i}{n}, g\left(\frac{i}{n}\right)\right)$  and  $\left(\frac{i+1}{n}, g\left(\frac{i+1}{n}\right)\right)$ . Note that  $\pi$  is always a vector of integer numbers.

**Definition 2.1.7** *If  $g : S_n \rightarrow S_n$ , an element  $\frac{r}{n} \in S_n$  with  $r \neq 0, n$ , will be called a node for  $g$ , if*

$$g\left(\frac{r-1}{n}\right), g\left(\frac{r}{n}\right), g\left(\frac{r+1}{n}\right)$$

*are not collinear. Elements 0 and 1 are supposed to be nodes of every function.*

As for McNaughton functions, for every function  $g : S_n \rightarrow S_n$ , we will denote by  $K(g)$  the set of nodes of  $g$ .

**Definition 2.1.8** *A function  $g : S_n \rightarrow S_n$  is a discrete McNaughton function if:*

- $\sigma(g)$  is a vector of integer numbers.
- The least common multiple of nodes denominators is equal to  $n$ .

**Theorem 2.1.9** *The class  $\mathcal{D} = Discr(\mathcal{M}_1)$  coincides with the class of discrete McNaughton functions.*

**Proof.** Let  $g \in \mathcal{D}$ . Then  $g = Discr(f)$  where  $f$  is a McNaughton function. Note that  $K(g) = K(f)$  and so each linear piece of  $f$  is expressed by equation  $y = \pi(g)(i)x + \sigma(g)(i)$ , so that both  $\pi(g)$  and  $\sigma(g)$  are integer vectors. Further, since  $Cont(g) = Cont(Discr(f)) = f$ , by definition  $n$  is equal to the least common multiple of denominators of its nodes.

Vice-versa let  $g : S_n \rightarrow S_n$  be a discrete McNaughton function. Then  $Cont(g)$  is a McNaughton function such that  $n$  is the least common multiple of nodes denominators and so  $g \in \mathcal{D}$ . ■

Let us denote by  $\mathcal{QD}$  the set of discrete functions  $g : S_n \rightarrow S_n$  such that:

- $\sigma(g)$  is a vector of integer numbers.
- The least common multiple of nodes denominators is less or equal to  $n$ .

Clearly  $\mathcal{D} \subset \mathcal{QD}$ . We will denote by  $fitt$  the “fitter” function

$$fitt : g \in \mathcal{QD} \rightarrow fitt(g) \in \mathcal{D}$$

such that  $fitt(g)$  is the restriction of  $g : S_n \rightarrow S_n$  to  $S_h$ , where  $h$  is the least common multiple of nodes denominators and hence  $h$  divides  $n$  and  $S_h \subseteq S_n$ .

In other words, if  $g \in \mathcal{D}$  then  $fitt(g) = g$  and if  $g : S_n \rightarrow S_n$  and the least common multiple of nodes denominators is  $h < n$ , then  $fitt(g) : x \in S_h \rightarrow g(x) \in S_h$ .

If  $k$  is a multiple of  $m$ , say  $k = lm$ , we can transform a vector  $u \in \mathbb{Z}^m$  into a vector  $enl(u, k) \in \mathbb{Z}^k$  in the following way. For every  $i = 1, \dots, k$  we set  $enl(u, k)(i) = u(h + 1)$  if  $hl < i \leq (h + 1)l$ , where  $h = 0, \dots, m - 1$ . For example, if  $u = (1, 2, 3) \in \mathbb{Z}^3$  and  $k = 12$ , then

$$enl(u, 12) = (1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3).$$

We will now introduce an operation between discrete McNaughton functions. Given  $f_n, g_m \in \mathcal{D}$ , let  $k = \text{lcm}(n, m)$ . Let

$$\pi = enl(\pi(f_n), k) + enl(\pi(g_m), k) \in \mathbb{Z}^k$$

and denote by  $h$  the least common multiple of absolute value of elements of  $\pi$ . Further, set  $l = \text{lcm}(h, k)$ .

For each element  $\frac{i}{l} \in S_l$ , we denote by  $rn(\frac{i}{l})$  the greatest element of  $S_n$  less or equal to  $\frac{i}{l}$ , and by  $ln(\frac{i}{l})$  the least element of  $S_n$  greater or equal to  $\frac{i}{l}$ . Let  $f_n^l$  be the extension of  $f_n$  to elements in  $S_l$ , that is:

$$f_n^l(\frac{i}{l}) = \begin{cases} f_n(\frac{i}{n}) & \text{if } \frac{i}{l} = \frac{j}{n} \\ n \cdot [f_n(ln(\frac{i}{l})) - f_n(rn(\frac{i}{l}))] \cdot (\frac{i}{l} - rn(\frac{i}{l})) + f_n(rn(\frac{i}{l})) & \text{otherwise} \end{cases}$$

Analogously we can define  $g_m^l$ . Then we set

$$f_n \uplus g_m = fitt(f_n^l \oplus g_m^l) \quad (2.1)$$

where the operation  $\oplus$  is the operation in the finite MV-algebra  $S_l$ . Note that the dimension of  $f_n \uplus g_m$  is less or equal to  $l$ . Further let  $0_1 : x \in \{0, 1\} \mapsto 0$ .

**Example 2.1.10** *Let*

$$f_2 : \begin{cases} 0 & \mapsto 1 \\ 1/2 & \mapsto 1 \\ 1 & \mapsto 0 \end{cases} \quad \text{and} \quad g_3 : \begin{cases} 0 & \mapsto 1 \\ 1/3 & \mapsto 0 \\ 2/3 & \mapsto 1 \\ 1 & \mapsto 1 \end{cases}$$

We have  $\pi(f_2) = (0, -2)$ ,  $\sigma(f_2) = (1, 2)$ ,  $\pi(g_3) = (-3, 3, 0)$ ,  $\sigma(g_3) = (1, -1, 1)$ ,  $\text{lcm}(K(f_2)) = 2$  and  $\text{lcm}(K(g_3)) = 3$  so  $f_2$  and  $g_3$  are in  $\mathcal{D}$ . Then  $k = 6$  and since

$$\begin{aligned} enl(\pi(f_2), 6) &= (0, 0, 0, -2, -2, -2) \\ enl(\pi(g_3), 6) &= (-3, -3, 3, 3, 0, 0), \end{aligned}$$

$\pi = (-3, -3, 3, 1, -2, -2)$  and so  $l = 6$ . It is easy to verify that,

$$f_2^6 = \begin{cases} 0 & \mapsto 1 \\ 1/6 & \mapsto 1 \\ 1/3 & \mapsto 1 \\ 1/2 & \mapsto 1 \\ 2/3 & \mapsto 2/3 \\ 5/6 & \mapsto 1/3 \\ 1 & \mapsto 0 \end{cases} \quad \text{and} \quad g_3^6 = \begin{cases} 0 & \mapsto 1 \\ 1/6 & \mapsto 1/2 \\ 1/3 & \mapsto 0 \\ 1/2 & \mapsto 1/2 \\ 2/3 & \mapsto 1 \\ 5/6 & \mapsto 1 \\ 1 & \mapsto 1 \end{cases}$$

Then  $f_2^6 \oplus g_3^6(x) = 1$  for every  $x \in S_6$  and so  $f_2 \uplus g_3 : x \in \{0, 1\} \mapsto 1$ .

**Corollary 2.1.11** *If  $f, g \in \mathcal{M}_1$  then*

$$(i) \text{ Discr}(f) \uplus \text{ Discr}(g) = \text{ Discr}(f \oplus g);$$

$$(ii) \neg(Discr(f)) = Discr(\neg f);$$

$$(iii) Discr(\mathbf{0}) = 0_1.$$

**Proof.**

- (i) In the definition of  $\uplus$ , we enlarge the domain of  $Discr(f)$  and  $Discr(g)$  to  $S_l$ . What we have to show is that such  $l$  is a multiple of denominators of new nodes arising in the sum. Let  $n = \dim(f)$  and  $m = \dim(g)$ , and let  $Discr(f) \uplus Discr(g) = \text{fitt}(Discr(f)^l \oplus Discr(g)^l)$  where, as defined in the construction above,  $l = \text{lcm}(n, m, \pi(1), \dots, \pi(k))$ . If  $k = \text{lcm}(n, m)$ , we have that  $f(x) + g(x) = 1$  if and only if

$$(enl(\pi(f_n), k)x + enl(\sigma(f_n), k)) + (enl(\pi(g_m), k)x + enl(\sigma(g_m), k)) = 1$$

if and only if

$$x = \frac{1 - (enl(\sigma(f_n), k) + enl(\sigma(g_m), k))}{enl(\pi(f_n), k) + enl(\pi(g_m), k)}.$$

Since  $K(f \oplus g) = K(f) \cup K(g) \cup \{x \in [0, 1] \mid f(x) + g(x) = 1\}$ , the denominator of every node of  $f \oplus g$  divides  $l$ . The function  $\text{fitt}$  restricts  $Discr(f)^l \oplus Discr(g)^l$  to exactly the least common multiple of its nodes denominators and so the dimension of  $Discr(f) \uplus Discr(g)$  is exactly the least common multiple of nodes denominators of  $f \oplus g$ .

- (ii),(iii) Note that the negation does not change any denominator of nodes of a function. The claims follow by definition of  $Discr$ . ■

**Theorem 2.1.12** *The MV-algebra  $(\mathcal{M}_1, \oplus, \neg, \mathbf{0})$  is isomorphic with  $(\mathcal{D}, \uplus, \neg, 0_1)$ .*

**Proof.** By Corollary 2.1.11 we know that  $Discr$  is an homomorphism of MV-algebras. By Proposition 2.1.6, the function  $Cont' : f \in \mathcal{D} \rightarrow Cont(f) \in \mathcal{M}_1$  is the inverse of  $Discr$  so  $Discr$  is an isomorphism. ■

## 2.2 Gödel logic

Linearly ordered Heyting algebras have been called  $\mathcal{L}$ -algebras by Horn in [72], where free  $\mathcal{L}$ -algebras have been examined and described. In this section we shall describe truth tables of Gödel formulas, thus giving a functional description of free  $\mathcal{L}$ -algebras. Results of this section have been published in [52].

Let us start with an example: let  $\varphi = \neg Y \vee (X \wedge Y)$ . The truth table  $f_\varphi(x, y)$  of  $\varphi$  is given by Figure 2.1, and is equal to 1 when  $y = 0$ , is equal to  $x$  when  $x \leq y$  and  $y > 0$  and it is equal to  $y$  when  $0 < y \leq x$ .

We hence introduce a subdivision of  $[0, 1]^n$  taking into account the possible orders between components of each point  $\mathbf{x} \in [0, 1]^n$ .

Let  $Perm_1^n$  denote the set of permutations of  $\{1, \dots, n\}$ . For every  $j \in \{1, \dots, n\}$  and  $\sigma \in Perm_1^n$  we consider the set

$$C_j^\sigma = \left\{ \mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n \mid \begin{array}{l} x_{\sigma(1)} = 0, \quad \dots, \quad x_{\sigma(j)} = 0 \\ 0 < x_{\sigma(j+1)} \leq 1, \quad \dots, \quad 0 < x_{\sigma(n)} \leq 1 \end{array} \right\}$$

Further let, for every  $\sigma \in Perm_1^n$ ,

$$C_0^\sigma = C_0 = \{ \mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n \mid 0 < x_1 \leq 1, \dots, 0 < x_n \leq 1 \}$$

The (non disjoint) union of all  $C_j^\sigma$  when  $j = 0, \dots, n$  and  $\sigma$  is a permutation of  $\{1, \dots, n\}$ , is the hypercube  $[0, 1]^n$ .

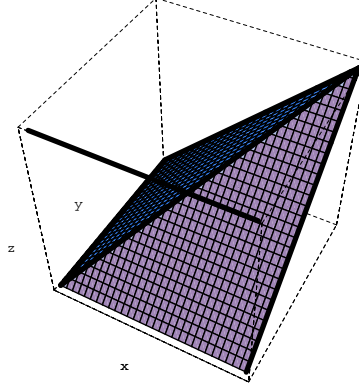
**Example 2.2.1** *If  $n = 2$  there are only two permutations, so that the square  $[0, 1]^2$  is partitioned as follows:*

$$\begin{aligned} C_0 &= \{(x, y) \mid x > 0 \text{ and } y > 0\} \\ C_1^{(12)} &= \{(x, y) \mid x = 0 \text{ and } y > 0\} \\ C_1^{(21)} &= \{(x, y) \mid y = 0 \text{ and } x > 0\} \\ C_2^{(12)} = C_2^{(21)} &= \{(0, 0)\} \end{aligned}$$

The different possible strict orders between  $x_{\sigma(j+1)}, \dots, x_{\sigma(n)}$  and the number of identical variables determine a further partition of each  $C_j^\sigma$ .

Let  $\sigma'$  be a permutation of  $\{\sigma(j+1), \dots, \sigma(n)\}$  and let  $k \in \{j+1, \dots, n-1\}$  and  $i \in \{1, \dots, n-k\}$ . Consider subsets of  $C_j^\sigma$  defined by

$$C_j^\sigma(\sigma') = \bigcup_{h=j+1}^n \left\{ \mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n \mid \begin{array}{l} x_{\sigma(1)} = \dots = x_{\sigma(j)} = 0 \\ 0 < x_{\sigma'\sigma(j+1)} < \dots < \\ x_{\sigma'\sigma(h)} = \dots = x_{\sigma'\sigma(n)} = 1 \end{array} \right\}$$


 Figure 2.1: Truth table of the Gödel formula  $\varphi = \neg Y \vee (X \wedge Y)$ 

and

$$D_j^\sigma(\sigma', k, i) = \bigcup_{h=j+1}^n \left\{ \mathbf{x} \in [0, 1] \left| \begin{array}{l} x_{\sigma(1)} = \dots = x_{\sigma(j)} = 0 \\ 0 < x_{\sigma'\sigma(j+1)} < \dots < x_{\sigma'\sigma(k)} = \\ \dots = x_{\sigma'\sigma(k+i)} < \dots < \\ < x_{\sigma'\sigma(h)} = \dots = x_{\sigma'\sigma(n)} = 1 \end{array} \right. \right\}.$$

We have

$$C_j^\sigma = \bigcup_{\sigma' \in \text{Perm}_{\sigma(j+1)}^{\sigma(n)}} \left( C_j^\sigma(\sigma') \cup \bigcup_{k=j+1}^{n-1} \bigcup_{i=1}^{n-k} D_j^\sigma(\sigma', k, i) \right)$$

and

$$\bigcap_{\sigma' \in \text{Perm}_{\sigma(j+1)}^{\sigma(n)}} C_j^\sigma(\sigma') \cap \bigcup_{\substack{\sigma' \in \text{Perm}_{\sigma(j+1)}^{\sigma(n)} \\ k \in \{j+1, \dots, n-1\}, \\ i \in \{1, \dots, n-k\}}} D_j^\sigma(\sigma', i, k) = \mathbf{y}_j^\sigma \in \{0, 1\}^n$$

where  $\mathbf{y}_j^\sigma = (y_1, \dots, y_n)$  is such that

$$\begin{aligned} y_{\sigma(1)} &= \dots = y_{\sigma(j)} = 0, \\ y_{\sigma(j+1)} &= \dots = y_{\sigma(n)} = 1. \end{aligned} \tag{2.2}$$

**Example 2.2.2** In case  $n = 3$  consider the set

$$C_1^{(213)} = \{(x, y, z) \mid y = 0, 0 < x \leq 1, 0 < z \leq 1\}.$$

$C_1^{(213)}$  is covered by the following subsets:

$$\begin{aligned} C_1^{(213)}(13) &= \{(x, 0, z) \mid 0 < x < z < 1\} \cup \{(x, 0, 1) \mid 0 < x < 1\} \cup \{(1, 0, 1)\} \\ C_1^{(213)}(31) &= \{(x, 0, z) \mid 0 < z < x < 1\} \cup \{(1, 0, z) \mid 0 < z < 1\} \cup \{(1, 0, 1)\} \\ D_1^{(213)}((13), 2, 1) &= \{(x, 0, z) \mid 0 < x = z < 1\} \cup \{(1, 0, 1)\} \\ D_1^{(213)}((31), 2, 1) &= D_1^{(213)}((13), 2, 1). \end{aligned}$$

In case  $n = 7$  we have for example,

$$D_2^\sigma(\sigma', 5, 1) = \left\{ (x_1, \dots, x_7) \mid \begin{array}{l} x_{\sigma(1)} = x_{\sigma(2)} = 0 \\ 0 < x_{\sigma'\sigma(3)} < x_{\sigma'\sigma(4)} < \\ x_{\sigma'\sigma(5)} = x_{\sigma'\sigma(6)} < x_{\sigma'\sigma(7)} = 1 \end{array} \right\}.$$

Let  $\mathcal{C}_j^\sigma = \{C_j^\sigma(\sigma'), D_j^\sigma(\sigma', k, i) \mid \sigma' \in \text{Perm}_{\sigma(j+1)}^{\sigma(n)}, k = j + 1, \dots, n, i = 0, \dots, n - k\}$  and

$$\mathcal{C} = \bigcup_{\substack{\sigma \in \text{Perm}_1^n \\ j=1, \dots, n}} \mathcal{C}_j^\sigma. \quad (2.3)$$

Recall that the *projection over the  $i$ -th variable* is a function  $\pi_i : [0, 1]^n \rightarrow [0, 1]$  such that for every  $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n$ ,  $\pi_i(\mathbf{x}) = x_i$ .

**Theorem 2.2.3** *The restriction of  $f_\varphi$  on every  $C \in \mathcal{C}$  is either equal to 0 or to 1 or is a projection.*

**Proof.** First note that if  $f_\varphi$  is linear over a region  $C$  then in that region  $f_\varphi$  must be identically equal either to 0 or to 1 or to a projection, for there are not arithmetical operations. The proof follows by induction.

- The case  $\varphi = X_i$  is trivial.
- If  $\varphi = \neg\psi$  then by induction hypothesis  $f_\psi$  over every  $C \in \mathcal{C}$  is either equal to 0 or to 1 or to a projection and for every  $\mathbf{x} \in [0, 1]^n$

$$f_\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } f_\psi(\mathbf{x}) = 0 \\ 0 & \text{if } f_\psi(\mathbf{x}) > 0 \end{cases}.$$

If  $C \in \mathcal{C}$  then there exists  $C_j^\sigma$  such that  $C \subseteq C_j^\sigma$ . Then for every  $\mathbf{x} = (x_1, \dots, x_n) \in C$

$$\begin{aligned} x_{\sigma(1)} = 0, & \quad \dots, \quad x_{\sigma(j)} = 0 \\ 0 < x_{\sigma(j+1)} \leq 1, & \quad \dots, \quad 0 < x_{\sigma(n)} \leq 1 \end{aligned}$$

and so the restriction of  $f_\psi$  to  $C$  is either identically equal to 0 or is always different from 0, so that the restriction of  $f_\varphi$  to  $C$  is either identically equal to 0 or is identically equal to 1.

- If  $\varphi = \psi_1 \wedge \psi_2$ , trivial.
- If  $\varphi = \psi_1 \rightarrow \psi_2$  then

$$f_\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } f_{\psi_1}(\mathbf{x}) \leq f_{\psi_2}(\mathbf{x}) \\ f_{\psi_2}(\mathbf{x}) & \text{otherwise.} \end{cases}$$

and  $f_{\psi_1}, f_{\psi_2}$  are 0, 1 or projections on every  $C \in \mathcal{C}$ . Different cases are possible, depending on whether restrictions of  $f_{\psi_1}, f_{\psi_2}$  to  $C \in \mathcal{C}$  are identically equal to 0 or to 1 or are projections. In any case, for every  $\mathbf{x} \in C$ , for every  $C \in \mathcal{C}$ , only one condition between  $(f_{\psi_1}(\mathbf{x}) \leq f_{\psi_2}(\mathbf{x}))$  or  $(f_{\psi_1}(\mathbf{x}) > f_{\psi_2}(\mathbf{x}))$  or  $f_{\psi_1}(\mathbf{x}) = f_{\psi_2}(\mathbf{x}) = 1$  is verified and the claim holds. ■

We shall now prove the inverse of the previous theorem, namely that if  $f$  is a function such that, restricted to an element of  $\mathcal{C}$  is either equal to 0 or to 1 or to a projection, then there exists a Gödel formula  $\varphi$  such that  $f$  is the truth table of  $\varphi$ .

For every  $C_j^\sigma$  with  $j > 0$ , the Gödel formula

$$\vartheta_j^\sigma = \neg X_{\sigma(1)} \wedge \dots \wedge \neg X_{\sigma(j)} \wedge \neg \neg (X_{\sigma(j+1)} \wedge \dots \wedge X_{\sigma(n)})$$

is such that  $f_{\vartheta_j^\sigma}$  is the characteristic function of  $C_j^\sigma$  and

$$\vartheta_0 = \neg \neg (X_1 \wedge \dots \wedge X_n)$$

is such that  $f_{\vartheta_0}$  is the characteristic function of  $C_0$ .

Hence we can independently find formulas  $\psi_j^\sigma$  associated with the restriction of  $f$  to the different  $C_j^\sigma$  and then merge them by

$$f(\mathbf{x}) = \bigvee_{j,\sigma} (f_{\psi_j^\sigma} \wedge f_{\vartheta_j^\sigma}). \quad (2.4)$$

In general it is not possible to find a formula whose truth table is the characteristic function of  $C_j^\sigma(\sigma')$  or  $D_j^\sigma(\sigma', k, i)$ . In order to cope with this fact we will introduce a kind of *weakly characteristic* function of such regions.



Let

$$\varphi_i^J = \bigwedge_{j \in J} (X_j \rightarrow X_i) \rightarrow X_j \quad (2.5)$$

and

$$\delta_J = \bigwedge_{\substack{i, j \in J \\ i \neq j}} (X_i \rightarrow X_j).$$

Let  $f_{\varphi_i^J}$  be the function associated to  $\varphi_i^J$ . Then  $f_{\varphi_i^J}(\mathbf{x})$  is equal to 1 if and only if  $\mathbf{x} = (x_1, \dots, x_n)$  is such that  $x_i$  is strictly less than  $x_j$  for all  $j \in J$  or  $x_i = x_j = 1$  for every  $j \in J$ . Indeed

$$(x_j \rightarrow x_i) \rightarrow x_j = \begin{cases} 1 & \text{if } x_i < x_j \\ x_j & \text{otherwise.} \end{cases}$$

Let  $f_{\delta^J}$  be the function associated to  $\delta^J$ . Then  $f_{\delta^J}(\mathbf{x})$  is equal to 1 if and only if  $\mathbf{x} = (x_1, \dots, x_n)$  is such that  $x_i = x_j$  for all  $i, j \in J$ . Indeed, for every  $i$  and  $j$

$$(x_i \rightarrow x_j) \wedge (x_j \rightarrow x_i) = \begin{cases} 1 & \text{if } x_i = x_j \\ x_i & \text{if } x_i < x_j \\ x_j & \text{if } x_j < x_i. \end{cases}$$

Functions associated to  $\varphi_i^J$  and  $\delta_J$  are very similar to characteristic functions of the following sets:

$$A_i^J = \{\mathbf{x} \in [0, 1] \mid x_i < x_j \text{ for every } j \in J\} \cup \\ \{\mathbf{x} \in [0, 1] \mid x_i = x_j = 1 \text{ for every } j \in J\}$$

and

$$B^J = \{\mathbf{x} \in [0, 1] \mid x_i = x_j \text{ for every } i, j \in J\}$$

respectively, since, for every  $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n$ ,

$$f_{\varphi_i^J}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in A_i^J \\ \min\{x_t\}_{t \in J} & \text{otherwise} \end{cases} \\ f_{\delta^J}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in B^J \\ \min\{x_t\}_{t \in J} & \text{otherwise.} \end{cases}$$

We associate with the region  $C_j^\sigma(\sigma')$  the formula

$$\chi_{C_j^\sigma(\sigma')} = \varphi_{\sigma'\sigma(j+1)}^{I_{j+1}} \wedge \dots \wedge \varphi_{\sigma'\sigma(n-1)}^{I_{n-1}} \quad (2.6)$$

where for every  $h = j + 1, \dots, n - 1$ ,  $I_h = \{\sigma'\sigma(h + 1), \dots, \sigma'\sigma(n)\}$ . By (2.5),

- if  $\mathbf{x} \in C_j^\sigma(\sigma')$  then  $f_{\chi_{C_j^\sigma(\sigma')}}(\mathbf{x}) = 1$ ;
- if  $\mathbf{x} \in C_j^\sigma \setminus C_j^\sigma(\sigma')$  then there exists  $i > j + 1$  such that
  - $x_{\sigma(1)} = \dots = x_{\sigma(j)} = 0$
  - $0 < x_{\sigma'\sigma(j+1)} < \dots < x_{\sigma'\sigma(i-1)}$
  - the other components have an order different from the order of components of elements of  $C_j^\sigma(\sigma')$ , that is, there exists  $\sigma''$  permutation of  $\sigma'\sigma(i), \dots, \sigma'\sigma(n)$  (different from the identity) such that  $x_{\sigma''\sigma'\sigma(i)} \leq \dots \leq x_{\sigma''\sigma'\sigma(n)}$ .

Then  $f_{\chi_{C_j^\sigma(\sigma')}}(\mathbf{x}) = x_{\sigma''\sigma'\sigma(i)}$ .

With the region  $D_j^\sigma(\sigma', k, i)$  is associated the formula

$$\begin{aligned} \chi_{D_j^\sigma(\sigma', k, i)} = & \varphi_{\sigma'\sigma(j+1)}^{I_{j+1}} \wedge \dots \wedge \varphi_{\sigma'\sigma(k)}^{I_{k+i}} \wedge \\ & \wedge \delta_{\{\sigma'\sigma(k), \dots, \sigma'\sigma(k+i)\}} \wedge \\ & \varphi_{\sigma'\sigma(k+i)}^{I_{k+i}} \wedge \dots \wedge \varphi_{\sigma'\sigma(n-1)}^{I_{n-1}} \end{aligned}$$

where  $I_i = \{\sigma'\sigma(i + 1), \dots, \sigma'\sigma(n)\}$ . Analogously,  $f_\chi$  takes value 1 over  $D_j^\sigma(\sigma'\sigma, k, i)$  and otherwise is equal to  $x_{\sigma''\sigma'\sigma(i)}$  where  $\sigma''$  is a permutation of  $\sigma'\sigma(i), \dots, \sigma'\sigma(n)$  (different from the identity) such that  $x_{\sigma''\sigma'\sigma(i)} \leq \dots \leq x_{\sigma''\sigma'\sigma(n)}$ .

**Example 2.2.4** Consider, for  $n = 3$ , the set  $C_0(123)$  of points  $(x_1, x_2, x_3)$  such that  $0 < x_1 < x_2 < x_3 < 1$  or  $0 < x_1 < x_2 < x_3 = 1$  or  $0 < x_1 < x_2 = x_3 = 1$  or  $0 < x_1 = x_2 = x_3 = 1$ . Then

$$\begin{aligned} \chi_{C_0(123)} = & (X_2 \rightarrow X_1) \rightarrow X_2 \quad \wedge \quad (X_3 \rightarrow X_1) \rightarrow X_3 \\ & \wedge \quad (X_3 \rightarrow X_2) \rightarrow X_3. \end{aligned}$$

If  $\mathbf{x} = (x_1, x_2, x_3) \in C_0(132)$  is such that  $0 < x_1 < x_3 < x_2$  (and so  $\mathbf{x} \notin C_0(123)$ ), then  $f_{\chi_{C_0(123)}}(\mathbf{x}) = x_3$ . Note that  $C_0(123) \cap C_0(132) = \{(x_1, 1, 1) \mid 0 < x_1 \leq 1\}$ .

**Lemma 2.2.5** *Let  $f : [0, 1]^n \rightarrow [0, 1]$  be a function such that for every  $C \in \mathcal{C}$*

$$\begin{aligned} f|_C &= 0 \text{ or} \\ f|_C &= 1 \text{ or} \\ f|_C &= x_h \end{aligned}$$

*with  $h \in \{1, \dots, n\}$ . Let  $\mathbf{y}_j^\sigma$  as in Equation (2.2). Then, for every  $\sigma \in \text{Perm}_1^n$  and  $j = 1, \dots, n$ , the function  $f_{\psi_j^\sigma}$  where  $\psi_j^\sigma$  is defined by*

$$\psi_j^\sigma = \bigvee_{C \in \mathcal{C}_j^\sigma} (\chi_C \wedge f|_C) \quad (2.7)$$

*if  $f(\mathbf{y}_j^\sigma) = 1$ , and by 0 if  $f(\mathbf{y}_j^\sigma) = 0$ , is such that  $f_{\psi_j^\sigma}(\mathbf{x}) = f(\mathbf{x})$  for every  $\mathbf{x} \in C_j^\sigma$ .*

**Proof.** If  $f(\mathbf{y}_j^\sigma) = 0$  then  $f|_{C_j^\sigma} = 0$  and so the claim is true.

Otherwise, let  $f(\mathbf{y}_j^\sigma) = 1$ . Then for every  $C \in \mathcal{C}_j^\sigma$ ,  $f|_C > 0$ . In this case, for every  $\sigma' \in \text{Perm}_{\sigma(j+1)}^{\sigma(n)}$ ,  $k = j+1, \dots, n$  and  $i = 0, \dots, n-k$  the restriction of  $f$  to  $C_j^\sigma(\sigma')$  and  $D_j^\sigma(\sigma', k, i)$  cannot be equal to 0, otherwise the function  $f$  would not be well defined.

Let  $\mathbf{x} = (x_1, \dots, x_n)$  such that  $x_{\sigma(1)} = \dots = x_{\sigma(j)} = 0$ . As an intermediate step we shall prove that, if  $\mathbf{x} \in C_j^\sigma(\sigma')$ ,

$$f|_{C_j^\sigma(\sigma')}(\mathbf{x}) \geq \bigvee_{\substack{C \in \mathcal{C}_j^\sigma \\ C \neq C_j^\sigma(\sigma')}} f_{\chi_C}(\mathbf{x}) \wedge f|_C(\mathbf{x}).$$

Indeed, suppose by contradiction that there exists  $C \neq C_j^\sigma(\sigma')$  such that, for  $\mathbf{x} \in C_j^\sigma(\sigma')$ ,

$$f|_{C_j^\sigma(\sigma')}(\mathbf{x}) < f_{\chi_C}(\mathbf{x}) \wedge f|_C(\mathbf{x}) \leq 1.$$

- If  $f_{\chi_C}(\mathbf{x}) > f|_C(\mathbf{x})$ , then let  $f_{\chi_C} = x_{\sigma''\sigma(i)}$ . By definition of  $\chi_C$ ,  $x_{\sigma''\sigma(i)}$  is the smallest component of  $\mathbf{x}$  that does not respect the order of components of elements of  $C$ . Since  $f_{\chi_C}(\mathbf{x}) > f|_C(\mathbf{x}) > 0$ , then  $x_{\sigma''\sigma(i)} > x_{\sigma'(j)}$  and the intersection  $C \cap C_j^\sigma(\sigma')$  contains all points of the form  $(0, \dots, 0, x_{\sigma'(j+1)}, \dots, x_{\sigma'(i-1)}, 1, \dots, 1)$ . Restrictions  $f|_C$  and  $f|_{C_j^\sigma(\sigma')}$  must coincide over  $C \cap C_j^\sigma(\sigma')$ , and are different from 0 and 1. So both are equal to one variable among  $x_{\sigma'(j+1)}, \dots, x_{\sigma'(i-1)}$ . This is a contradiction.
- If  $f_{\chi_C}(\mathbf{x}) \leq f|_C(\mathbf{x})$ , then  $f_{\chi_C}(\mathbf{x}) \wedge f|_C(\mathbf{x}) = f_{\chi_C}(\mathbf{x})$  and so  $f_{\chi_C}(\mathbf{x}) > f|_{C_j^\sigma(\sigma')} > 0$ . Reasoning as above, we again get a contradiction.

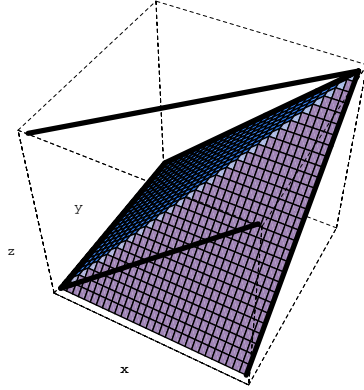


Figure 2.2: Truth table of the Gödel formula  $\varphi$  in Example 2.2.7

The same happens for  $D_j^\sigma(\sigma', k, i)$ , namely

$$f|_{D_j^\sigma(\sigma', k, i)}(\mathbf{x}) \geq \bigvee_{\substack{C \in \mathcal{C}_j^\sigma \\ C \neq D_j^\sigma(\sigma', k, i)}} f|_C(\mathbf{x}) \wedge f|_C(\mathbf{x}).$$

So  $f(\mathbf{x}) = \bigvee_{C \in \mathcal{C}_j^\sigma} f|_C(\mathbf{x}) \wedge f|_C(\mathbf{x})$  for every  $\mathbf{x} \in C_j^\sigma$  and hence  $f = f_{\psi_j^\sigma}$ . ■

**Theorem 2.2.6** *The algebra  $\mathcal{F}$  of functions  $f : [0, 1]^n \rightarrow [0, 1]$  such that for every  $C \in \mathcal{C}$*

$$\begin{aligned} f|_C &= 0 \text{ or} \\ f|_C &= 1 \text{ or} \\ f|_C &= x_h \text{ with } h \in \{1, \dots, n\}. \end{aligned}$$

*equipped with the pointwise operations of  $\wedge$ ,  $\vee$  and  $\rightarrow$  is the free  $\mathcal{L}$ -algebra over  $n$  generators.*

**Proof.** By Lemma 2.2.5 and equations (2.4),(2.6),(2.2), for every function  $f \in \mathcal{F}$ , there exists a formula  $\varphi$  such that  $f \equiv f_\varphi$ . That is equivalent to saying that  $\mathcal{F}$  is the  $\mathcal{L}$ -algebra generated by the projection functions  $\pi_i : \mathbf{x} \in [0, 1]^n \rightarrow x_i \in [0, 1]$ . In [73] it is proved that the variety of  $\mathcal{L}$ -algebras is generated by  $[0, 1]$ . Then, from a result of universal algebra (see [38]) it follows that  $\mathcal{F}$  is the free algebra generated by the  $n$  projections. ■

**Example 2.2.7** For  $n = 2$  consider the function in Figure 2.2. In order to find the formula  $\varphi$  having such truth table, we can consider the following regions

$$\begin{aligned} C_0(12) &= \{(x, y) \mid 0 < x < y \leq 1\} \cup \{(1, 1)\} \\ C_0(21) &= \{(x, y) \mid 0 < y < x \leq 1\} \cup \{(1, 1)\} \\ D_0 &= \{(x, x) \mid x > 0\} \\ C_1 &= \{(x, y) \mid y = 0\} \end{aligned}$$

whose weakly characteristic functions are respectively given by:

$$\begin{aligned} \varphi_1 &= (Y \rightarrow X) \rightarrow Y & \varphi_2 &= (X \rightarrow Y) \rightarrow X \\ \delta &= (X \rightarrow Y) \wedge (Y \rightarrow X) & \theta &= \neg Y. \end{aligned}$$

Then

$$\varphi = (\varphi_1 \wedge X) \vee (\varphi_2 \wedge Y) \vee \delta \vee (\theta \wedge X)$$

and this is equivalent to  $((X \rightarrow Y) \wedge (Y \rightarrow X)) \vee (\neg Y \wedge X)$ .

### 2.2.1 Extending Gödel logic

Consider now the logic  $G + \Delta$ , obtained adding to Gödel logic the unary operator  $\Delta$  interpreted by

$$v(\Delta(\varphi)) = \begin{cases} 1 & \text{if } v(\varphi) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The connective  $\Delta$  was axiomatized in [14]. Functions associated with formulas of  $G + \Delta$  are much easier to describe than functions associated with formulas of Gödel logic. For the sake of notation, we shall consider here only the case of  $n = 2$ .

Consider the following regions, where  $i, j = 1, 2$  and  $i \neq j$ :

$$\begin{aligned} A_{ij} &= \{(x_1, x_2) \mid x_i = 0 \ 0 < x_j < 1\} \\ B_{ij} &= \{(x_1, x_2) \mid x_i = 0 \ x_j = 1\} \\ C_{ij} &= \{(x_1, x_2) \mid 0 < x_i < x_j < 1\} \\ D &= \{(x_1, x_2) \mid 0 < x_1 = x_2 < 1\}, \\ 0 &= \{(0, 0)\}, & I &= \{(1, 1)\}. \end{aligned}$$

Truth tables of the following formulas are characteristic functions of the above regions:

$$\begin{aligned}
\alpha_{ij} &= \neg x_i \wedge \neg \Delta x_j \wedge \neg \neg x_j \\
\beta_{ij} &= \neg x_i \wedge \Delta x_j \\
\gamma_{ij} &= \Delta((x_j \rightarrow x_i) \rightarrow x_j) \wedge \neg \neg x_i \wedge \neg \Delta x_j \\
\delta &= \Delta((x_1 \rightarrow x_2) \wedge (x_2 \rightarrow x_1)) \wedge \neg \Delta x_1 \wedge \neg \neg x_1 \wedge \neg \Delta x_2 \wedge \neg \neg x_2 \\
\omega &= \neg x_1 \wedge \neg x_2 \quad \text{and} \quad \iota = \Delta x_1 \wedge \Delta x_2
\end{aligned}$$

Then truth tables of formulas of  $G + \Delta$  precisely coincide with all functions such that on every  $A_{ij}, B_{ij}, C_{ij}, D, O$  and  $I$  are either equal to 0 or to 1 or to a projection.

This result can be easily extended to higher dimensions.

## 2.3 Product Logic

In order to deal with product logic by means of piecewise linear functions, we shall introduce an infinite-valued logic  $\Sigma$  whose domain of interpretation is the set of real positive numbers plus a distinct symbol for infinity, and connectives are interpreted as sum and difference.

In this section we shall define the following objects, that are needed both to describe truth tables of product formulas and to prove reducibility results in the next Chapter:

- Indexes for regions in which the truth tables are linear,
- Regions of linearity,
- Restriction of truth tables functions to regions of linearity (*cases* of the functions),
- Equations for boundaries of regions of linearity.

### 2.3.1 Some Preliminaries

Let  $\Sigma = (\mathbb{R}_+^*, \{0\}, \{+, \rightarrow_\Sigma, \neg_\Sigma\})$  denote a logic in which

- $\mathbb{R}_+^* = [0, \infty]$  is the set of positive real numbers plus a distinct symbol  $\infty$  (for infinity) such that for every  $x \in \mathbb{R}_+, x < \infty$ ;
- $x + y$  is the usual sum of real numbers;

- $x \rightarrow_{\Sigma} y = \begin{cases} 0 & \text{if } x \geq y \\ y - x & \text{otherwise,} \end{cases}$  where  $\infty - x = \infty$  if  $x \in \mathbb{R}_+$ ;
- $\neg_{\Sigma} x = x \rightarrow_{\Sigma} \infty = \begin{cases} 0, & \text{if } x = \infty \\ \infty, & \text{otherwise.} \end{cases}$

Each formula of the Product logic  $\Pi_{\infty}$  can be *translated* in a formula of  $\Sigma$  and the map

$$\iota : x \in [0, 1] \rightarrow \begin{cases} \log(x^{-1}), & \text{if } x > 0 \\ \infty & \text{otherwise.} \end{cases}$$

(where logarithms are taken to the base e) is such that  $\varphi$  is a tautology of  $\Pi_{\infty}$  if and only if  $\iota(\varphi)$  is a tautology of  $\Sigma$ . Hence in the rest of this Section and in the following Chapter we shall investigate  $\Sigma$  instead if  $\Pi_{\infty}$ .

If  $\varphi$  is a formula we shall denote by  $n\varphi$  the formula  $\underbrace{\varphi + \dots + \varphi}_{n \text{ times}}$ .

In the Figure 2.3 truth tables of formulas  $(X \rightarrow Y^2) \cdot (Y \rightarrow X^3)$  and  $(X \rightarrow_{\Sigma} 2Y) + (Y \rightarrow_{\Sigma} 3X)$  are represented. Note that in the first graph there is a discontinuity in the point  $(0, 0)$  and that in the second graph the values at points having components equal to  $\infty$  are omitted.

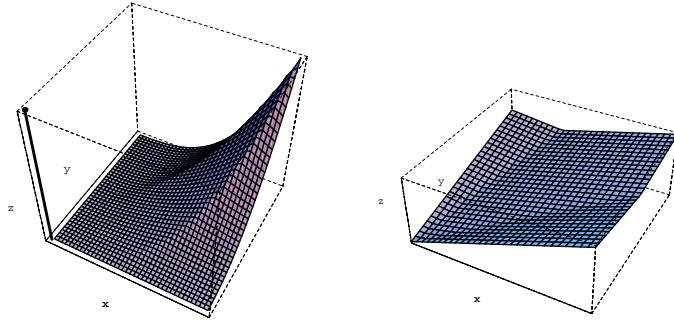


Figure 2.3: Truth table of  $(X \rightarrow Y^2) \cdot (Y \rightarrow X^3)$  and  $(X \rightarrow_{\Sigma} 2Y) + (Y \rightarrow_{\Sigma} 3X)$

In the rest of this section, we shall omit the indexes  $\Sigma$  and we shall simply write  $\neg$  and  $\rightarrow$ .

**Definition 2.3.1 (Indexes)** For every formula  $\varphi$ , the set of indexes  $J(\varphi)$  associated with  $\varphi$  is the set of strings over the two-element alphabet  $\{A, B\}$  given by the following inductive stipulation:

- $J(X_i) = \{\epsilon\}$  for every variable  $X_i$  of  $\varphi$  (where  $\epsilon$  denotes the empty string).
- $J(\neg\varphi) = J(\varphi)$ .
- $J(\varphi + \psi) = \{\sigma_1\sigma_2 \mid \sigma_1 \in J(\varphi), \sigma_2 \in J(\psi)\}$ .
- $J(\varphi \rightarrow \psi) = \{\sigma_1\sigma_2A, \sigma_1\sigma_2B \mid \sigma_1 \in J(\varphi), \sigma_2 \in J(\psi)\}$ .

**Definition 2.3.2 (Regions of linearity)** For every  $\sigma \in J(\varphi)$  we inductively define the subset  $\text{cell}_\varphi(\sigma)$  of  $(\mathbb{R}_+^*)^n$  as follows:

- $\text{cell}_{X_i}(\epsilon) = (\mathbb{R}_+^*)^n$ .
- $\text{cell}_{\neg\varphi}(\sigma) = \text{cell}_\varphi(\sigma)$ .
- $\text{cell}_{\varphi+\psi}(\sigma_1\sigma_2) = \text{cell}_\varphi(\sigma_1) \cap \text{cell}_\psi(\sigma_2)$ .
- $\text{cell}_{\varphi\rightarrow\psi}(\sigma_1\sigma_2A) = \text{cell}_\varphi(\sigma_1) \cap \text{cell}_\psi(\sigma_2) \cap \{\mathbf{x} \mid f_\varphi(\mathbf{x}) - f_\psi(\mathbf{x}) \geq 0\}$ ,
- $\text{cell}_{\varphi\rightarrow\psi}(\sigma_1\sigma_2B) = \text{cell}_\varphi(\sigma_1) \cap \text{cell}_\psi(\sigma_2) \cap \{\mathbf{x} \mid f_\varphi(\mathbf{x}) - f_\psi(\mathbf{x}) \leq 0\}$ .

The set  $C^{(n)}(\varphi)$  is defined by

$$C^{(n)}(\varphi) = \{\text{cell}_\varphi(\sigma) \mid \sigma \in J(\varphi), \dim(\text{cell}_\varphi(\sigma)) = n\}. \quad (2.8)$$

We shall denote by  $I(\varphi)$  the subset of  $J(\varphi)$  whose elements are indexes of cells in  $C^{(n)}(\varphi)$ .

We further let  $C(\varphi)$  be the set of polyhedra obtained by adding to  $C^{(n)}(\varphi)$  all the faces of polyhedra in  $C^{(n)}(\varphi)$ , in symbols,

$$P \in C(\varphi) \quad \text{if and only if} \quad \text{there exists } A \in C^{(n)}(\varphi) \text{ such that } P \in \mathbf{F}(A).$$

Since  $(\mathbb{R}_+^*)^n$  is a closed hypercube with faces given by

$$\{\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n \mid x_{i_1} = 0, x_{i_2} = \infty \text{ with } i_1 \in J_1, i_2 \in J_2, \\ J_1, J_2 \subseteq \{1, \dots, n\} \text{ and } J_1 \cup J_2 \neq \emptyset\},$$

then every open face  $F$  of a cell of  $C^{(n)}(\varphi)$  is either contained in  $\mathbb{R}_+^n$ , or is contained in a hyperplane of equation  $x_i = \infty$ .



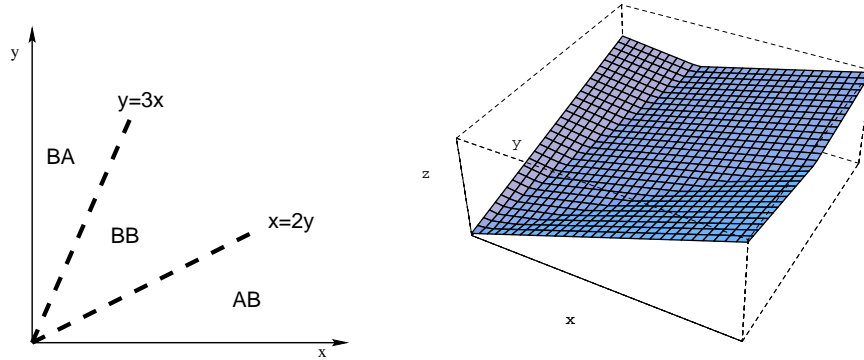


Figure 2.4: Regions of linearity and truth table of  $\varphi$  in Example 2.3.4

**Definition 2.3.3 (Cases of functions)** For each  $\sigma \in I(\varphi)$ , the restriction of  $f_\varphi$  to  $\text{cell}_\varphi(\sigma)$  is called the case of  $f_\varphi$  on  $\text{cell}_\varphi(\sigma)$  and is denoted by  $c_\varphi(\sigma)$ .

**Example 2.3.4** Consider the formula  $\varphi = (X \rightarrow 2Y) + (Y \rightarrow 3X)$ . Then,

$$\begin{aligned} J(X) = J(Y) &= \{\epsilon\} \\ J(2X) = J(3Y) &= \{\epsilon\} \\ J(X \rightarrow 2Y) &= \{A, B\} \\ J(Y \rightarrow 3X) &= \{A, B\} \\ J((X \rightarrow 2Y) + (Y \rightarrow 3X)) &= \{AA, AB, BA, BB\} \end{aligned}$$

The regions of linearity of Definition 2.3.2 are (see Figure 2.4)

$$\begin{aligned} \text{cell}_{X \rightarrow 2Y}(A) &= \{(x, y) \mid x \geq 2y\} \\ \text{cell}_{X \rightarrow 2Y}(B) &= \{(x, y) \mid x \leq 2y\} \\ \text{cell}_{Y \rightarrow 3X}(A) &= \{(x, y) \mid y \geq 3x\} \\ \text{cell}_{Y \rightarrow 3X}(B) &= \{(x, y) \mid y \leq 3x\} \\ \text{cell}_\varphi(AA) &= \emptyset \\ \text{cell}_\varphi(AB) &= \{(x, y) \mid x \geq 2y\} \\ \text{cell}_\varphi(BA) &= \{(x, y) \mid y \geq 3x\} \\ \text{cell}_\varphi(BB) &= \{(x, y) \mid x/2 \leq y \leq 3x\} \end{aligned}$$

Cases of function  $f_\varphi$  are

$$\begin{aligned} c_\varphi(AB)(x, y) &= 3x - y \\ c_\varphi(BB)(x, y) &= 2x + y \\ c_\varphi(BA)(x, y) &= 2y - x \end{aligned}$$

See also Example 3.3.2.

**Lemma 2.3.5** For each  $k \in \{1, \dots, n\}$  and for each open  $k$ -dimensional face  $F$  of elements of  $C(\varphi)$ ,  $f_\varphi$  is linear over  $F$ .

**Proof.** The proof proceeds by induction on the complexity of (= number of occurrences of connectives in)  $\varphi$ , as follows:

If  $\varphi = X_i$  then we are done.

If  $\varphi = \neg\psi$ , let  $F$  be any open face of  $\text{cell}_\varphi(\sigma) = \text{cell}_\psi(\sigma) \in C^{(n)}(\varphi)$ . By definition, for every  $\mathbf{x} \in F$ , we can write

$$c_\varphi(\sigma)(\mathbf{x}) = \begin{cases} 0 & \text{if } c_\psi(\sigma)(\mathbf{x}) = \infty, \\ \infty & \text{if } c_\psi(\sigma)(\mathbf{x}) \leq \infty. \end{cases}$$

By induction hypothesis,  $c_\psi(\sigma)$  is linear over  $F$ . If  $c_\psi(\sigma)$  is constant over  $F$ , say  $c_\psi(\sigma)(\mathbf{x}) = c$  for all  $\mathbf{x} \in F$ , then either  $c = \infty$  or  $c \neq \infty$  and, in any case,  $c_\varphi(\sigma)$  is linear. Otherwise, the function  $c_\psi(\sigma)$  is a non-constant linear function over  $F$  and the set  $\{c_\psi(\sigma)(\mathbf{x}) \mid \mathbf{x} \in F\}$  is an open set of  $(\mathbb{R}_+^*)^n$ , whence it does not contain the extremal point  $\infty$ . In this case we can write  $c_\varphi(\sigma)(\mathbf{x}) = \infty$  for all  $\mathbf{x} \in F$ . By induction hypothesis,  $c_\varphi(\sigma)$  is a linear function over  $F$ .

If  $\varphi = \psi_1 + \psi_2$ , then let  $\sigma = \sigma_1\sigma_2$  and  $F$  be an open face of  $\text{cell}_\varphi(\sigma) = \text{cell}_{\psi_1}(\sigma_1) \cap \text{cell}_{\psi_2}(\sigma_2)$ . Then  $F$  is contained in the open faces of  $\text{cell}_{\psi_1}(\sigma_1)$  and  $\text{cell}_{\psi_2}(\sigma_2)$ . Precisely one of the identities  $c_\varphi(\sigma)(\mathbf{x}) = g_1^*(c_{\psi_1}(\sigma_1), c_{\psi_2}(\sigma_2))(\mathbf{x})$  and  $c_\varphi(\sigma)(\mathbf{x}) = g_2^*(c_{\psi_1}(\sigma_1), c_{\psi_2}(\sigma_2))(\mathbf{x})$  holds true for all  $\mathbf{x} \in F$ . By induction hypothesis  $c_\varphi(\sigma)$  is linear.

If  $\varphi = \psi_1 \rightarrow \psi_2$ , then let  $\sigma = \sigma_1\sigma_2A$ . Then

$$c_\varphi(\sigma_1\sigma_2A)(\mathbf{x}) = 0$$

and the claim is trivial.

If  $\sigma = \sigma_1\sigma_2B$  then

$$c_\varphi(\sigma_1\sigma_2B)(\mathbf{x}) = c_{\psi_2}(\sigma_2)(\mathbf{x}) - c_{\psi_1}(\sigma_1)(\mathbf{x})$$

By induction hypothesis,  $c_\varphi(\sigma_1\sigma_2B)$  is linear over every open face of  $\text{cell}_\varphi(\sigma)$ . ■

### 2.3.2 Description of functions

For each function  $f_\varphi : (\mathbb{R}_+^*)^n \rightarrow \mathbb{R}_+^*$  associated with a formula  $\varphi$  of  $\Sigma$  logic, we shall study separately its restriction to facets of cells wholly contained in  $\mathbb{R}_+^n$  and its restriction to facets of cells with a non-empty intersection with  $(\mathbb{R}_+^*)^n \setminus (\mathbb{R}_+^n)$ .

Note that if  $\varphi = \psi_1 \star \psi_2$  (with  $\star$  any binary connective) and if  $\sigma = \sigma' \sigma'' C$  with  $C \in \{\epsilon, A, B\}$ ,  $\sigma' \in I(\psi_1)$  and  $\sigma'' \in I(\psi_2)$ , then  $\text{cell}_{\psi_1}(\sigma') \supseteq \text{cell}_\varphi(\sigma)$  and  $\text{cell}_{\psi_2}(\sigma'') \supseteq \text{cell}_\varphi(\sigma)$  and so any open face  $F$  of  $\text{cell}_\varphi(\sigma)$  is jointly contained in an open face of  $\text{cell}_{\psi_1}(\sigma')$  and in an open face of  $\text{cell}_{\psi_2}(\sigma'')$ .

**Proposition 2.3.6** *Let  $\sigma \in I(\varphi)$  and let  $F$  be an open face of  $\text{cell}_\varphi(\sigma)$  such that  $F \subseteq \mathbb{R}_+^n$ . Then we have:*

- (i) *either for every  $\mathbf{x} \in F$ ,  $c_\varphi(\sigma)(\mathbf{x})$  is equal to  $\infty$ ,*
- (ii) *or the restriction of  $c_\varphi(\sigma)$  to  $F$  is a linear and homogeneous function, with integer coefficients, taking values in  $\mathbb{R}_+$ .*

**Proof.** The proof easily proceeds by induction on the complexity of  $\varphi$ . ■

**Lemma 2.3.7** *The restriction of the function  $f_\varphi$  to  $\mathbb{R}_+^n$  is either identically equal to  $\infty$ , or is continuous with respect to the natural topology of  $\mathbb{R}_+$ .*

**Proof.** Again by induction on the complexity of  $\varphi$ .

- If  $\varphi = X_i$  the claim is trivial, since  $f_\varphi(\mathbf{x}) = x_i$ .
- If  $\varphi = \neg_\Sigma \psi$  then, for all  $\mathbf{x} \in \mathbb{R}_+^n$ ,

$$f_\varphi(\mathbf{x}) = \begin{cases} 0 & \text{if } f_\psi(\mathbf{x}) = \infty \\ \infty & \text{if } f_\psi(\mathbf{x}) < \infty. \end{cases}$$

By induction hypothesis, either for all  $\mathbf{x} \in \mathbb{R}_+^n$ ,  $f_\psi(\mathbf{x}) = \infty$  and then  $f_\varphi(\mathbf{x}) = 0$ , or  $f_\psi$  is continuous in  $\mathbb{R}_+^n$  and then  $f_\psi(\mathbf{x}) < \infty$  and  $f_\varphi(\mathbf{x}) = \infty$ .

- If  $\varphi = \psi_1 + \psi_2$ . The claim easily follows by induction hypothesis, since  $f_\varphi = f_{\psi_1} + f_{\psi_2}$ .
- If  $\varphi = \psi_1 \rightarrow_\Sigma \psi_2$  then, for every  $\mathbf{x} \in \mathbb{R}_+^n$ ,

$$f_\varphi(\mathbf{x}) = \begin{cases} 0 & \text{if } f_{\psi_2}(\mathbf{x}) - f_{\psi_1}(\mathbf{x}) \leq 0 \\ f_{\psi_2}(\mathbf{x}) - f_{\psi_1}(\mathbf{x}) & \text{otherwise.} \end{cases}$$

The claim follows by induction hypothesis.

■

In the following Lemma we shall examine the behavior of functions at points having some component equal to  $\infty$ .

**Lemma 2.3.8** *Let  $F$  be an open face of  $(\mathbb{R}_+^*)^n$  such that  $F \subseteq \{\mathbf{x} \mid x_i = \infty\} \subseteq (\mathbb{R}_+^*)^n \setminus (\mathbb{R}_+)^n$ . Then*

(i) *either for every  $\mathbf{x} \in F$ ,  $f_\varphi(\mathbf{x}) = \infty$  or  $f_\varphi(\mathbf{x}) = 0$ ,*

(ii) *or there exists a formula  $\vartheta$  such that for every  $\mathbf{x} = (x_1, \dots, x_n) \in F$ ,*

$$f_\varphi(\mathbf{x}) = f_\vartheta(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

*where  $\text{var}(\vartheta) \subseteq \text{var}(\varphi) \setminus \{X_i\}$  and  $\#(\vartheta) \leq \#(\varphi)$ .*

**Proof.** Let  $F \subseteq \{\mathbf{x} \mid x_i = \infty\}$  and if  $\mathbf{x} = (x_1, \dots, x_n) \in F$ , let  $\mathbf{x}^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in (\mathbb{R}_+^*)^{n-1}$ . The proof proceeds by induction on the complexity of  $\varphi$ . Let us consider the case  $\varphi = \psi_1 \rightarrow_\Sigma \psi_2$ . Since for every  $\mathbf{x} \in F$ ,

$$f_\varphi(\mathbf{x}) = \begin{cases} f_{\psi_2}(\mathbf{x}) - f_{\psi_1}(\mathbf{x}) & \text{if } f_{\psi_1}(\mathbf{x}) \leq f_{\psi_2}(\mathbf{x}) \\ 0 & \text{otherwise,} \end{cases}$$

it is sufficient to consider only the case in which  $f_{\psi_1}(\mathbf{x}) \leq f_{\psi_2}(\mathbf{x})$ . Then

- If  $f_{\psi_1}(\mathbf{x}) = 0$ , then  $f_\varphi(\mathbf{x}) = f_{\psi_2}(\mathbf{x})$  and by induction hypothesis the claim is settled.
- If  $f_{\psi_2}(\mathbf{x}) = \infty$  then if  $f_{\psi_1}(\mathbf{x}) = \infty$ ,  $f_\varphi(\mathbf{x}) = 0$  otherwise  $f_\varphi(\mathbf{x}) = \infty$ .
- If  $f_{\psi_1}(\mathbf{x}) = f_{\vartheta_1}(\mathbf{x}^i)$  and  $f_{\psi_2}(\mathbf{x}) = f_{\vartheta_2}(\mathbf{x}^i)$ , with  $\text{var}(\vartheta_1) \subseteq \text{var}(\psi_1) \setminus \{X_i\}$ ,  $\text{var}(\vartheta_2) \subseteq \text{var}(\psi_2) \setminus \{X_i\}$ ,  $\#(\vartheta_1) \leq \#(\psi_1)$  and  $\#(\vartheta_2) \leq \#(\psi_2)$ , then  $f_\varphi(\mathbf{x}) = f_{\vartheta_1 \rightarrow_\Sigma \vartheta_2}(\mathbf{x}^i)$  with

$$\text{var}(\vartheta_1 \rightarrow_\Sigma \vartheta_2) \subseteq (\text{var}(\psi_1) \cup \text{var}(\psi_2)) \setminus \{X_i\} = \text{var}(\varphi) \setminus \{X_i\}$$

and  $\#(\vartheta_1 \rightarrow_\Sigma \vartheta_2) \leq \#(\varphi)$ .

The other cases are similar. ■

Iterated applications of the above theorem yields

**Proposition 2.3.9** *Let  $F$  be an open face of  $(\mathbb{R}_+^*)^n$  such that every  $\mathbf{x} \in F$  has exactly  $1 \leq k \leq n$  infinite components, say,  $x_{i_1}, \dots, x_{i_k} = \infty$ . Then either  $f_\varphi(\mathbf{x}) = \infty$ , or else there exists a formula  $\vartheta$  satisfying the following two conditions:*

- $\text{var}(\vartheta) = \text{var}(\varphi) \setminus \{X_{i_1}, \dots, X_{i_k}\}$  and  $\#(\vartheta) \leq \#(\varphi)$ ,
- for every  $\mathbf{x} \in F$ , letting  $\mathbf{x}^{i_1, \dots, i_k}$  denote the element of  $\mathbb{R}_+^{n-k}$  obtained by deleting from  $\mathbf{x}$  all the  $i_1$ th,  $\dots$ ,  $i_k$ th components, then  $f_\varphi(\mathbf{x}) = f_\vartheta(\mathbf{x}^{i_1, \dots, i_k}) < \infty$ .

■

### 2.3.3 Characterization of functions

Description of truth tables of Product formulas is given by the following

**Theorem 2.3.10** *Let  $f : (\mathbb{R}_+^*)^n \rightarrow \mathbb{R}_+^*$  be a function such that*

- (i) *The restriction of  $f$  to  $\mathbb{R}_+^n$  is either identically equal to  $\infty$ , or is continuous with respect to the natural topology of  $\mathbb{R}_+$ , homogeneous piecewise linear and such that each linear piece has integer coefficients.*
- (ii) *The restriction of  $f$  to each open subset  $F$  of  $(\mathbb{R}_+^*)^n \setminus (\mathbb{R}_+)^n$  is either equal to  $\infty$ , or, for every  $\mathbf{x} \in F$ , we have the identity  $f(\mathbf{x}) = g(x_{i_1}, \dots, x_{i_k})$ , where  $x_{i_1}, \dots, x_{i_k}$  are exactly the components of  $\mathbf{x}$  different from  $\infty$  and  $g : (\mathbb{R}_+)^k \rightarrow \mathbb{R}_+$  is a continuous, homogeneous, piecewise linear function such that each linear piece has integer coefficients.*

*Then there exists a formula  $\varphi$  of  $\Sigma$  logic such that  $f$  is the truth table of  $\varphi$ .*

In order to prove Theorem 2.3.10 we shall adapt to our context the same machinery used in [93] to prove McNaughton Theorem. For more details see [93] and [52].

We need some preliminary results. For the sake of simplicity, let us call *product function* any continuous, homogeneous, piecewise linear function with integer coefficients. Regions  $D$  in which a product function is linear are union of *polyhedral cones*, i.e., without loss of generality they can be expressed as

$$D = \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{A} \cdot \mathbf{x} \leq 0\}$$

where  $\mathbf{A}$  is an integer  $(m \times n)$  matrix and  $\leq$  is defined componentwise.

Consider the restriction to  $\mathbb{R}_+^n$ : firstly we shall describe formulas having homogeneous linear functions as truth tables, then we examine how to glue different linear functions together.

**Fact 1.** For every  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ , consider the function

$$g(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n \vee 0$$

with  $a_i \in \mathbb{Z}$ . Let us suppose, without loss of generality, that  $a_{i_1}, \dots, a_{i_r}$  are positive numbers and  $a_{i_{r+1}}, \dots, a_{i_n}$  are negative. Then the formula

$$\lambda = ((-a_{i_{r+1}})X_{i_{r+1}} + \dots + (-a_{i_n})X_{i_n}) \rightarrow (a_{i_1}X_{i_1} + \dots + a_{i_r}X_{i_r})$$

is such that  $f_\lambda(\mathbf{x}) = g(\mathbf{x})$  for every  $\mathbf{x} \in \mathbb{R}_+^n$ .

**Fact 2.** Suppose that polyhedral cones  $D_1, D_2 \subseteq \mathbb{R}_+^n$  and product functions  $g_1$  and  $g_2$  are such that  $D_1 = \{\mathbf{x} \mid g_1(x) = 0\}$  and  $D_2 = \{\mathbf{x} \mid g_2(x) = 0\}$ . Let  $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  be such that both restrictions  $f|_{D_1}$  and  $f|_{D_2}$  coincide respectively on  $D_1$  and  $D_2$  with product functions  $h_1$  and  $h_2$ . For every  $p$  and  $q$ , let

$$h^{p,q}(\mathbf{x}) = (pg_1(\mathbf{x}) \rightarrow h_1(\mathbf{x})) \vee (qg_2(\mathbf{x}) \rightarrow h_2(\mathbf{x})). \quad (2.9)$$

Clearly  $h^{p,q}$  is a product function. We want to prove that there exist  $\bar{p}$  and  $\bar{q}$  such that the restriction  $f|_{D_1 \cup D_2}$  is equal to  $h^{\bar{p},\bar{q}}$  on  $D_1 \cup D_2$ .

Since  $g_1, g_2, h_1, h_2$  are product functions, it is possible to find regions  $P_1, \dots, P_m$  such that for every  $i = 1, \dots, m$  each of  $g_1, g_2, h_1, h_2$  is linear on  $P_i$ : it is enough to intersect regions of linearity of each product function.

Let us consider the inequality

$$((h_1(\mathbf{x}) - p_i g_1(\mathbf{x})) \vee 0) - h_2(\mathbf{x}) \leq 0. \quad (2.10)$$

If  $\mathbf{x} \in P_i \cap D_2$  then  $p_i$  can be chosen (depending on  $\mathbf{x}$ ) so large that the inequality (2.10) holds true. The dependence on  $\mathbf{x}$  can be eliminated by considering that the function  $((h_1(\mathbf{x}) - p_i g_1(\mathbf{x})) \vee 0) - h_2(\mathbf{x})$  is linear on  $P_i \cap D_2$ .

Let  $\bar{p} = \max_{1 \leq i \leq m} p_i$  where  $p_i$  satisfy (2.10). Then, for every  $\mathbf{x} \in D_2$ ,

$$\bar{p} g_1(\mathbf{x}) \rightarrow h_1(\mathbf{x}) \leq h_2(\mathbf{x})$$

and hence, for every natural number  $q$  and  $\mathbf{x} \in D_2$ ,

$$\begin{aligned} h^{\bar{p},q}(\mathbf{x}) &= ((\bar{p}g_1(\mathbf{x}) \rightarrow h_1(\mathbf{x})) \vee (qg_2(\mathbf{x}) \rightarrow h_2(\mathbf{x}))) = \\ &= ((\bar{p}g_1(\mathbf{x}) \rightarrow h_1(\mathbf{x})) \vee h_2(\mathbf{x})) = h_2(\mathbf{x}) = f(\mathbf{x}). \end{aligned}$$

Analogously, there exists  $\bar{q}$  such that for any  $\mathbf{x} \in D_1$ ,  $h^{\bar{p}, \bar{q}}(\mathbf{x}) = f(\mathbf{x})$ . Then the restriction  $f|_{D_1 \cup D_2}$  of  $f$  to  $D_1 \cup D_2$  is a product function.

**Fact 3.** Let  $D = \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{A} \cdot \mathbf{x} \leq 0\}$  and  $\mathbf{a}_1, \dots, \mathbf{a}_m$  be rows of  $\mathbf{A}$ . By Fact 1, for every expression  $\mathbf{a}_i \cdot \mathbf{x}$  there is a formula  $\pi(i, \mathbf{x})$  such that the truth table of  $\pi(i, \mathbf{x})$  is  $0 \vee (\mathbf{a}_i \cdot \mathbf{x})$ . Then the formula

$$\varphi_D = \pi(1, \mathbf{x}) \wedge \dots \wedge \pi(m, \mathbf{x}) \quad (2.11)$$

is such that  $D = \{\mathbf{x} \mid f_{\varphi_D}(\mathbf{x}) = 0\}$ .

We have now all the necessary tools for proving Theorem 2.3.10:

**Proof.** Let  $f : (\mathbb{R}_+^*)^n \rightarrow \mathbb{R}_+^*$  be a function satisfying (i) and (ii) of Theorem 2.3.10. Just as for Gödel formulas in Section 2.2, for every region  $I \subseteq (\mathbb{R}_+^*)^n \setminus \mathbb{R}_+^n$  such that

$$I = \left\{ (x_1, \dots, x_n) \mid \begin{array}{l} x_{i_1} = \infty, \dots, x_{i_j} = \infty \\ x_{i_{j+1}} < \infty, \dots, x_n < \infty \end{array} \right\},$$

the formula

$$\vartheta^I = \neg X_{i_1} \vee \dots \vee \neg X_{i_j} \vee \neg \neg (X_{i_{j+1}} \vee \dots \vee X_{i_n})$$

is such that  $f_{\vartheta^I}(\mathbf{x}) = 0$  for every  $\mathbf{x} \in I$  and  $f_{\vartheta^I}(\mathbf{x}) = \infty$  for  $\mathbf{x} \notin I$ . Further, the formula

$$\delta = \neg \neg (X_1 \vee \dots \vee X_n)$$

is such that  $f_\delta(\mathbf{x}) = 0$  for  $\mathbf{x} \in \mathbb{R}_+^n$  and  $f_\delta(\mathbf{x}) = \infty$  otherwise.

This fact allows us to find independently formulas  $\alpha^I$  associated with the restriction of  $f$  to the different  $I$  and to find the formula  $\beta$  associated to the restriction of  $f$  to  $\mathbb{R}_+^n$ , and then to merge them in the following way:

$$f(\mathbf{x}) = \bigwedge_I (f_{\vartheta^I} \vee f_{\alpha^I}) \wedge (f_\beta \vee f_\delta). \quad (2.12)$$

In this way the problem is reduced to finding the formula  $\beta$  corresponding to the restriction of  $f$  to  $\mathbb{R}_+^n$ .

If  $f|_{\mathbb{R}_+^n}$  is equal to  $\infty$  then we can set for example  $\beta = X_1 \vee \neg X_1$  and the claim is settled.

If  $f|_{\mathbb{R}_+^n}$  is a product function then there exist polyhedral cones  $D_i = \{\mathbf{x} \mid \mathbf{A}_i \cdot \mathbf{x} \leq 0\}$  for  $i = 1, \dots, u$ , such that  $f$  is linear and homogeneous on each  $D_i$ . By Fact 3 there exist formulas  $\varphi_{D_i}$  as in Equation (2.11). Applying several times Fact 2 we get the conclusion.  $\blacksquare$

**Example 2.3.11** Consider regions

$$\begin{aligned} D_1 &= \{(x, y) \mid 2y - x \leq 0\} & D_2 &= \{(x, y) \mid x - 2y \leq 0\} \\ I_1 &= \{(x, y) \mid x < \infty, y = \infty\} & I_2 &= \{(x, y) \mid x = \infty, y < \infty\} \\ I_3 &= \{(x, y) \mid x = \infty, y = \infty\} \end{aligned}$$

and the function  $f : (\mathbb{R}_+^*)^2 \rightarrow \mathbb{R}_+^*$  defined by

$$\begin{aligned} f|_{D_1} &= 2x - 2y & f|_{D_2} &= x \\ f|_{I_1} &= \infty & f|_{I_2} &= y \\ f|_{I_3} &= \infty. \end{aligned}$$

Then

$$\begin{aligned} \vartheta^{I_1} &= \neg X \wedge \neg\neg Y & \vartheta^{I_2} &= \neg Y \wedge \neg\neg X \\ \vartheta^{I_3} &= \neg X \wedge \neg Y & \delta &= \neg\neg(X \vee Y). \end{aligned}$$

Let  $\varphi_{D_1} = X \rightarrow 2Y$  and  $\varphi_{D_2} = 2Y \rightarrow X$ : the restrictions of  $f$  to  $D_1$  and  $D_2$  are expressed by formulas  $\chi_1 = 2Y \rightarrow 2X$  and  $\chi_2 = X$ . Then Equation (2.9) becomes

$$\chi^{p,q} = (p\varphi_{D_1} \rightarrow \chi_1) \vee (q\varphi_{D_2} \rightarrow \chi_2)$$

and by direct calculation formula  $\chi^{1,2}$  turns out to satisfy  $f_{\chi^{1,2}}(x, y) = f(x, y)$ , for every  $(x, y) \in D_1 \cup D_2 = \mathbb{R}_+^2$ . Then the formula

$$\varphi = (\vartheta^{I_1} \vee X \vee \neg X) \wedge (\vartheta^{I_2} \vee Y) \wedge (\vartheta^{I_3} \vee X \vee \neg X) \wedge (\delta \vee \chi^{1,2})$$

has  $f$  as truth table.

Recall that  $\Sigma$  is isomorphic to Product Logic via the transformation

$$\iota : x \in [0, 1] \rightarrow \begin{cases} \log(x^{-1}), & \text{if } x > 0 \\ \infty & \text{otherwise.} \end{cases}$$

Then description of truth tables of Product logic can be obtained by applying the inverse of  $\iota$  to functions described in Theorem 2.3.10.



## Chapter 3

# Finite-valued reductions

The satisfiability problem of infinite-valued Lukasiewicz logic was proved to be NP-complete by Mundici in [86]. In that paper, the author showed that the decision problem of infinite-valued propositional Lukasiewicz logic  $L_\infty$  can be reduced to the decision problem of a suitable set of finite-valued Lukasiewicz logics. More precisely, a formula  $\varphi$  is valid in  $L_\infty$  if and only if, for each  $i \in \{1, \dots, 2^{2^{\#(\varphi)}}\}$ ,  $\varphi$  is valid in  $(i + 1)$ -valued logic  $L_i$ . Here,  $\#(\varphi)$  denotes the total number of occurrences of variables in  $\varphi$ .

Strengthening this result, in [8] the authors showed that the tautologousness of  $\varphi$  in  $L_\infty$  can be checked in *exactly one*  $(m + 1)$ -valued logic  $L_m$ , for  $m = 2^{\#(\varphi)-1}$ .

In this Chapter we extend the methods and results of [8] to Gödel and Product logic, as well as to logics obtained by a combination of Gödel and Lukasiewicz connectives (resp., Gödel and Product connectives). While results for Gödel logic are already known (see [67]), our method is fresh and the reduction of formulas of Product Logic to finite-valued logics is new here. As an application, we shall define in Chapter 4 a calculus for all these infinite-valued logics. The results of this chapter have appeared in [10, 11].

### 3.1 Introduction

Recall that in Definition 1.1.1 a many-valued propositional logic is defined as a triple  $\mathcal{L} = (S, D, F)$ , where

- $S$  is a non-empty set of *truth-values*,
- $D \subset S$  is the set of *designated* truth values,

- $F$  is a (finite) non-empty set of functions such that for any  $f \in F$  and for any integer  $\nu(f) > 0$ ,  $f : S^{\nu(f)} \rightarrow S$  and for every  $c \in C$  there exists  $f_c \in F$ .

**Definition 3.1.1 (Critical points)** *If  $\varphi$  is a formula of the logic  $(S, D, F)$ , a set of critical points for the function  $f_\varphi$  is a subset  $C \subseteq S^n$  (where  $n = |\text{var}(\varphi)|$ ) such that*

$$f_\varphi(S^n) \subseteq D \quad \text{if and only if} \quad f_\varphi(C) \subseteq D.$$

**Definition 3.1.2 (Denominator upper bound)** *A function  $b_{\mathcal{L}} : \text{Form}(\mathcal{L}) \rightarrow \mathbb{N}$  is a denominator upper bound for the logic  $\mathcal{L}$ , if for every formula  $\varphi$ , there exists a set  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  of critical points for  $f_\varphi$  such that*

$$\max \text{den}(\mathbf{x}_i) \leq b_{\mathcal{L}}(\varphi).$$

Our general construction leading to a sequent calculus for a given logic  $\mathcal{L}$  has several steps as follows:

- Find a formulation of  $\mathcal{L}$  where the function  $f_\varphi$  determined by any formula  $\varphi$  is piecewise linear with integer coefficients.
- By induction on the complexity of  $\varphi$ , define indexes to keep track of the *building process* of  $\varphi$  from its subformulas; and correspondingly keep track of all functions, polyhedral complexes, and bounding hyperplanes given by these subformulas.
- Given a formula  $\varphi$ , find a finite set of critical points for  $\varphi$ .
- From the relationship between bounding hyperplanes and subformulas, obtain an upper bound on the denominator of critical points, only depending on the length and on the number of variables of  $\varphi$ .
- Use continuity arguments, and their generalizations, to reduce the problem whether  $\varphi$  is a tautology in  $\mathcal{L}$  to the corresponding problem for  $\varphi$  in a unique, effectively determined, finite-valued logic.
- Use this finite-valued reduction to construct a sequent calculus for the infinite-valued logic  $\mathcal{L}$ .

In this section we recall the main ideas of the method given in [8] to reduce the tautology problem in infinite-valued Łukasiewicz logic to its finite-valued counterpart.

For each McNaughton function (Definition 2.1.1)  $f : [0, 1]^n \rightarrow [0, 1]$  there is a polyhedral complex  $C(f)$  such that  $f$  is linear over each cell of  $C(f)$ ,  $\bigcup C(f) = [0, 1]^n$ , and each vertex (0-cell) of  $C(f)$  is a point of  $([0, 1] \cap \mathbb{Q})^n$ .

If  $\varphi$  is not a tautology of  $\mathbb{L}_\infty$  and  $|\text{var}(\varphi)| = n$ , then there exists a point  $\mathbf{x} \in [0, 1]^n$  such that  $f_\varphi(\mathbf{x}) < 1$ . By linearity it follows that there is a vertex  $\mathbf{v}$  of  $C(f_\varphi)$  such that  $f_\varphi(\mathbf{v}) < 1$ , and we have that  $\varphi$  is not a tautology of  $\mathbb{L}_{\text{den}(\mathbf{v})}$ .

Note that we can obtain  $\mathbf{v}$  as the solution of a system  $M\mathbf{v} = \mathbf{b}$  of linear equations. The rows of the system are the equations of the hyperplanes whose intersection is  $\mathbf{v}$ .

We can build the polyhedral complex  $C(\varphi) = C(f_\varphi)$  by inductively combining the complexes associated with subformulas of  $\varphi$ . With any variable  $X_i$  of  $\varphi$  we associate the complex  $C(f_{X_i})$  containing  $[0, 1]^n$  together with all its faces. Since negation  $\neg$  does not introduce new subdivisions, we must only consider the disjunctive connective  $\oplus$ . It turns out that the only new bounding hyperplanes of full-dimensional cells that can be introduced in the complex associated to a subformula  $\psi \oplus \vartheta$  of  $\varphi$ , have equations of the form  $g_1(\mathbf{x}) + g_2(\mathbf{x}) = 1$ , where  $g_1$  and  $g_2$  are linear polynomials coinciding respectively with  $f_\psi$  and  $f_\vartheta$  on suitably determined cells  $c_1 \in C(f_\psi)$  and  $c_2 \in C(f_\vartheta)$ .

Thus, all the entries in the system matrix  $M$  are integers, whence  $\text{den}(\mathbf{v}) \leq |\det(M)|$ . We are now in a position to give an upper bound to  $|\det(M)|$ .

The analysis in [8] of subformulas of  $\varphi$  yields a matrix  $M'$  such that  $|\det(M')| = |\det(M)|$  and the sum of the absolute values of the entries in  $M'$  does not exceed the total number of occurrences of variables in  $\varphi$ . A straightforward computation now yields the desired upper bound on  $\text{den}(\mathbf{v})$ , namely  $b_L = (\#\varphi/n)^n$ , where  $n$  is the number of variables of  $\varphi$ . Therefore,

$$\models_{\mathbb{L}_\infty} \varphi \quad \text{if and only if} \quad \models_{\mathbb{L}_m} \varphi \quad \text{for each } m \in \{1, \dots, b_L\}.$$

One further application of the continuity and differentiability properties of the function associated to  $\varphi$  yields the equivalence

$$\models_{\mathbb{L}_\infty} \varphi \quad \text{if and only if} \quad \models_{\mathbb{L}_{2^{\#\varphi}-1}} \varphi.$$

As a final step, it can be shown that the smallest basis  $s$  such that  $b_L < s^{\#\varphi}$  is  $e^{1/e}$  where  $e = 2.71828\dots$  is the basis of natural logarithms.

Since McNaughton functions are continuous, the set of critical points for  $\varphi$  coincides with the set of vertices of  $C(f_\varphi)$ . As we shall see, the continuity

assumption can be weakened without essential prejudice to the applicability of our method.

Our method depends on the existence of *nice* relations between the case functions  $c_\varphi(\sigma)$  and the elements of  $D(\varphi, \sigma)$ . In the following sections, we shall show what these nice relations are for Gödel logic and for Product logic.

## 3.2 Gödel Logic

The results proved in this section are already known and can be proved straightforwardly as is done, for instance, in [67]. Since functions associated with Gödel formulas are geometrically simple, we use Gödel logic to exemplify our method.

Let  $\varphi$  be a formula of Gödel logic, such that  $|\text{var}(\varphi)| = n$ . Then the truth table  $f_\varphi$  of  $\varphi$  is a (piecewise linear) function from  $[0, 1]^n$  into  $[0, 1]$  described by Lemma 2.2.5. In Section 2.2, the set of regions of linearity is denoted by  $\mathcal{C}$  (Equation (2.3)) and elements of  $\mathcal{C}$  are subspaces of  $[0, 1]^n$  limited by hyperplanes of equations  $x_i = x_j$  or  $x_i = 0$  or  $x_i = 1$ .

Hence, vertexes of elements of  $\mathcal{C}$  are vertexes of the hypercube  $[0, 1]^n$ .

**Corollary 3.2.1** *The function  $f_\varphi$  is identically equal to 1 if and only if*

- (i)  $f_\varphi(\mathbf{x}) = 1$  for every vertex  $\mathbf{x}$  of  $[0, 1]^n$ , and
- (ii)  $f_\varphi(\mathbf{x}) = 1$  for at least one point  $\mathbf{x}$  in every element of  $\mathcal{C}$ .

**Proof.** By Lemma 2.2.5 the function  $f_\varphi$  is linear over every element of  $\mathcal{C}$ . Conditions (i) and (ii) imply that  $f_\varphi$  is equal to 1 on every  $C \in \mathcal{C}$  and hence identically over  $[0, 1]^n$ . ■

The following theorem is a well known consequence of a result in [60]. We give here an alternative proof.

**Theorem 3.2.2** *For every formula  $\varphi$  of Gödel logic,*

$$\models_{G_\infty} \varphi \quad \text{if and only if} \quad \models_{G_{n+1}} \varphi$$

where  $n = |\text{var}(\varphi)|$ .

**Proof.** By Corollary 3.2.1 the minimal set of critical points for a formula  $\varphi$  of Gödel logic, consists of vertices of  $\mathcal{C}$  (i.e., vertices of the hypercube  $[0, 1]^n$ )

and of one point in each element of  $\mathcal{C}$ . A point in an  $n$ -dimensional element of  $\mathcal{C}$  can be obtained as the Farey mediant (Definition 1.3.1) of vertices of such cell, and hence in this case it has denominator equal to  $n + 1$ . Suppose that  $C$  is an element of  $\mathcal{C}$  with dimension less than  $n$ . Then points in  $C$  either have some of their coordinates equal to 0 or to 1, or have two or more of their coordinates one equal to another. Let us denote by  $J_0, J_1$  the subsets of  $\{1, \dots, n\}$  such that, for every  $\mathbf{x} = (x_1, \dots, x_n) \in C$ ,

- $x_{j_0} = 0$ , for every  $j_0 \in J_0$ ;
- $x_{j_1} = 1$ , for every  $j_1 \in J_1$ .

Further, denote by  $J_2^1, \dots, J_2^t$  subsets of  $\{1, \dots, n\}$  such that, if  $r \in \{1, \dots, t\}$ ,

- $x_{j_2} = x_{j'_2}$ , for every  $j_2, j'_2 \in J_2^r$  and  $j_2 \neq j'_2$ .

Since  $\dim(C) < n$  then  $J_0 \cup J_1 \cup \bigcup_{r=1}^t J_2^r \neq \emptyset$ .

For every point  $\mathbf{p} \in F$ , let  $\mathbf{p}' = (p'_1, \dots, p'_n) \in F'$  be defined in the following way:

- for every  $j_0 \in J_0, j_1 \in J_1$ , we set  $p'_{j_0} = 0, p'_{j_1} = 1$ .
- for every  $r = 1, \dots, t$  choose a component  $p^r$  in each  $\{p_i \mid i \in J_2^r\}$ . Then set, for every  $j_2, j'_2 \in J_2^r$ ,  $p'_{j_2} = p'_{j'_2} = p^r$ .
- in all the other cases set  $p'_i = p_i$ .

The denominator of point  $\mathbf{p}'$  divides  $\text{den}(\mathbf{p})$ .

So, in every open face there is a point of denominator dividing  $n + 1$ , and hence  $f_\varphi \equiv 1$  if and only if the restriction of  $f_\varphi$  to  $S_{n+1}$  is identically equal to 1. ■

### 3.3 Product Logic

Things for Product Logic are a slight more complicate. Firstly, truth tables involve more objects in their definition than Gödel formulas do. Further, no finite subset of  $[0, 1]$ , other than  $\{0, 1\}$  and its subsets, is closed under the product operation, hence there are not finite-valued Product Logics. We will show that for every formula  $\varphi$  of product logic  $\Pi$ , one can always define a finite-valued logic  $\Sigma_n = \Sigma_n(\#\varphi, |\text{var}(\varphi)|)$  such that  $\varphi$  is a tautology in  $\Pi$  if and only if a suitably transformed formula  $\tilde{\varphi}$  is a tautology in  $\Sigma_n$ .

We shall then use the description of truth tables of Theorem 2.3.10, and the relationship between boundary elements and cases of functions as defined in Definition 2.3.3. We shall then provide an upper bound for the denominator of critical points. Using a new unary connective that transforms sums into truncated sums we shall finally give finite-valued approximations for the tautology problem of formulas in product logic.

The denominator  $\text{den}(\mathbf{p})$  of a rational point  $\mathbf{p} \in (\mathbb{R}_+^*)^n$  having  $i$  many components equal to  $\infty$ , is defined as the denominator of the projection of  $\mathbf{p}$  into  $(\mathbb{R}_+)^{n-i}$  along those components. In other words,  $\infty$  is considered as having denominator equal to 1.

The work in the following subsections is organized as follows:

- In Section 3.3.1 we investigate the relations between boundaries and subformulas of  $\varphi$ . Further we show that, in each cell of  $C(\varphi)$  the polynomial associated with  $\varphi$  can be described in terms of polynomials associated with subformulas of  $\varphi$ .
- In Section 3.3.2 critical points are examined. A critical point of  $\varphi$  arises as the solution of a system where the rows are given by suitable subformulas of  $\varphi$ . Using Lemmas of Section 3.3.1 we can appropriately manipulate such rows in order to obtain an upper bound on the denominator of critical points.
- In Section 3.3.3 we reduce the problem of tautologousness to a class of finite-valued logics.

### 3.3.1 Subformulas and cell boundaries

Proposition 2.3.9 allows us to reduce the study of the behavior of functions in faces of the form  $\{\mathbf{x} \in (\mathbb{R}_+^*)^n \mid x_{i_1} = \dots = x_{i_t} = \infty\}$  to the study of functions defined over  $\mathbb{R}_+^{n-t}$ . Then we can safely limit ourselves to the study of piecewise linear functions defined over  $\mathbb{R}_+^n$  and we can accordingly consider cells as polyhedral subsets of  $\mathbb{R}_+^n$ .

**Definition 3.3.1** For every linear polynomial  $q(\mathbf{x}) = \mathbf{a}\mathbf{x} + c$ , we shall denote by  $\pi(q)$  the vector  $\mathbf{a}$ . Further, if  $f_\varphi(\mathbf{x}) = \infty$  for every  $\mathbf{x} \in \mathbb{R}_+^n$ , we set  $\pi(c_\varphi(\sigma)) = \mathbf{0}$  for each  $\sigma \in I(\varphi)$ .

**Example 3.3.2** The Product formula  $(X \rightarrow Y) \cdot (X \rightarrow Z)$  becomes the  $\Sigma$  formula  $\varphi = (X \rightarrow_\Sigma Y) + (X \rightarrow_\Sigma Z)$ . The index set of  $\varphi$  is

$$I(\varphi) = \{AA, AB, BA, BB\},$$

and the corresponding cells are

$$\begin{aligned} \text{cell}_{AA}(\varphi) &= \{(x, y, z) \in \mathbb{R}_+^3 \mid x \geq y, x \geq z\} \\ \text{cell}_{AB}(\varphi) &= \{(x, y, z) \in \mathbb{R}_+^3 \mid x \geq y, x \leq z\} \\ \text{cell}_{BA}(\varphi) &= \{(x, y, z) \in \mathbb{R}_+^3 \mid x \leq y, x \geq z\} \\ \text{cell}_{BB}(\varphi) &= \{(x, y, z) \in \mathbb{R}_+^3 \mid x \leq y, x \leq z\} \end{aligned}$$

Further cases of functions are given by

$$\begin{aligned} c_\varphi(AA)(x, y, z) &= 0 \quad \text{and} \quad \pi(c_\varphi(AA)) = (0, 0, 0) \\ c_\varphi(AB)(x, y, z) &= y - x \quad \text{and} \quad \pi(c_\varphi(AB)) = (-1, 1, 0) \\ c_\varphi(BA)(x, y, z) &= z - x \quad \text{and} \quad \pi(c_\varphi(BA)) = (-1, 0, 1) \\ c_\varphi(BB)(x, y, z) &= y - 2x + z \quad \text{and} \quad \pi(c_\varphi(BB)) = (-2, 1, 1). \end{aligned}$$

Note also that

$$\begin{aligned} \text{if } x = \infty \quad \text{then} \quad f_\varphi(x, y, z) &= 0 \\ \text{if } y = \infty \text{ and } x \neq \infty \quad \text{then} \quad f_\varphi(x, y, z) &= \infty \\ \text{if } z = \infty \text{ and } z \neq \infty \quad \text{then} \quad f_\varphi(x, y, z) &= \infty. \end{aligned}$$

**Notation.** Recall that we have denoted by  $I(\varphi)$  the subset of  $J(\varphi)$  whose elements are indexes of full dimensional cells in  $C(\varphi)$  (see Definition 2.3.1). Let further:

$$I_C(\varphi) = \begin{cases} \{\sigma_1\sigma_2A \in I(\varphi) \mid \sigma_1 \in I(\psi_1), \sigma_2 \in I(\psi_2)\} & \text{if } \varphi = \psi_1 \rightarrow_\Sigma \psi_2 \\ \{\sigma_1\sigma_2 \mid \sigma_1 \in I_C(\psi_1), \sigma_2 \in I_C(\psi_2)\} & \text{if } \varphi = \psi_1 + \psi_2 \\ I_C(\psi) & \text{if } \varphi = \neg\psi \end{cases}$$

$$I_{\overline{C}}(\varphi) = I(\varphi) \setminus I_C(\varphi).$$

The set  $I_C(\varphi)$  is the set of indexes of regions in which the function  $f_\varphi$  is a constant function. For instance, in the above Example we have  $I_C(\varphi) = \{AA\}$ .

**Definition 3.3.3 (Boundaries)** For each  $\sigma \in I(\varphi)$  we inductively define the set  $D(\varphi, \sigma) \subseteq \mathbb{Z}^{n+1}$  as follows:

- $D(X_i, \epsilon) = \pm(E_n \times \{0, 1\})$ .
- If  $\varphi = \neg\psi$  and  $\sigma \in I(\psi)$ , then  $D(\neg\varphi, \sigma) = D(\varphi, \sigma)$ .
- If  $\varphi = \psi_1 + \psi_2$  and  $\sigma = \sigma_1\sigma_2$ , then  $D(\psi_1 + \psi_2, \sigma_1\sigma_2) = D(\psi_1, \sigma_1) \cup D(\psi_2, \sigma_2)$ .

- If  $\varphi = \psi_1 \rightarrow \psi_2$  and  $\sigma = \sigma_1\sigma_2C$  ( $C \in \{A, B\}$ ) let  $D_{\sigma_1\sigma_2}$  defined as follows:

If  $\dim(\text{cell}_{\psi_1}(\sigma_1) \cap \text{cell}_{\psi_2}(\sigma_2)) < n$  then  $D_{\sigma_1\sigma_2} = \emptyset$ ; otherwise  $D_{\sigma_1\sigma_2}$  is the set of all the elements  $(\mathbf{a}, c)$  such that

- $\gcd(\mathbf{a}, c) = \gcd(a_1, \dots, a_n, c) = 1$ , where  $\mathbf{a} = (a_1, \dots, a_n)$ ;
- for every  $\mathbf{x} \in \text{cell}_{\psi_1}(\sigma_1) \cap \text{cell}_{\psi_2}(\sigma_2)$ ,

$$c_{\psi_1}(\sigma_1)(\mathbf{x}) - c_{\psi_2}(\sigma_2)(\mathbf{x}) = 0 \quad \text{if and only if} \quad \mathbf{a} \cdot \mathbf{x} = c.$$

Then  $D(\psi_1 \rightarrow \psi_2, \sigma_1\sigma_2A) = D(\psi_1 \rightarrow \psi_2, \sigma_1\sigma_2B) = D(\psi_1, \sigma_1) \cup D(\psi_2, \sigma_2) \cup D_{\sigma_1\sigma_2}$ .

Given  $\sigma \in I(\varphi)$ , suppose that the vector  $(\mathbf{a}, c)$  satisfies  $\gcd(\mathbf{a}, c) = 1$ ; in addition suppose that the set  $\{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = c\}$  contains a facet of  $\text{cell}_\varphi(\sigma)$ . Then  $(\mathbf{a}, c)$  belongs to the set  $D(\varphi, \sigma)$ . Accordingly, by a slight abuse of language, each element of  $D(\varphi, \sigma)$  will be called a *boundary element* of  $\text{cell}_\varphi(\sigma)$ .

Each element of  $D(\varphi, \sigma)$  represents the equation of a bounding hyperplane of  $\text{cell}_\varphi(\sigma)$ . If we consider the formula  $\varphi$  as inductively built up from its subformulas then every element of  $D(\varphi, \sigma)$  must be “introduced” by a subformula of  $\varphi$ . The next Lemma establishes some relations between boundaries and subformulas of  $\varphi$ .

**Lemma 3.3.4** *Let  $\sigma \in I(\varphi)$ . If  $(\mathbf{a}, c) \in D(\varphi, \sigma) \setminus \pm(E_n \times \{0, 1\})$ , then*

- $c = 0$ ;
- there exists a subformula  $\vartheta = \vartheta_1 \rightarrow_\Sigma \vartheta_2$  of  $\varphi$  and  $\tau = \tau_1\tau_2C \in I(\vartheta)$  (where  $C \in \{A, B\}$ ) such that  $\text{cell}_\varphi(\sigma) \subseteq \text{cell}_\vartheta(\tau)$  and there exists  $d \in \mathbb{Z}$  such that  $d\mathbf{a} = \pi(c_\vartheta(\tau_1\tau_2B))$ .

**Proof.**

Let  $(\mathbf{a}, c) \in D(\varphi, \sigma)$ . We shall proceed by induction on the complexity of  $\varphi$ .

- In case  $\varphi = X_i$ , the proof is immediate.
- If  $\varphi = \neg_\Sigma \psi$  or  $\varphi = \psi_1 + \psi_2$  the result follows from Definition 3.3.3 and from induction.



- If  $\varphi = \psi_1 \rightarrow_{\Sigma} \psi_2$  and  $\sigma = \sigma_1\sigma_2A$  then either  $(\mathbf{a}, c) \in D(\psi_1, \sigma_1) \cup D(\psi_2, \sigma_2)$  and then the desired result immediately follows by induction, or else, for every  $\mathbf{x} \in \text{cell}_{\psi_1}(\sigma_1) \cap \text{cell}_{\psi_2}(\sigma_2)$ , we have

$$\mathbf{a} \cdot \mathbf{x} = c \quad \text{if and only if} \quad c_{\psi_2}(\sigma_2)(\mathbf{x}) - c_{\psi_1}(\sigma_1)(\mathbf{x}) = 0.$$

Since  $c_{\psi_1 \rightarrow_{\Sigma} \psi_2}(\sigma_1\sigma_2B) = c_{\psi_2}(\sigma_2) - c_{\psi_1}(\sigma_1)$ , then

$$\mathbf{a} \cdot \mathbf{x} = c \quad \text{if and only if} \quad c_{\psi_1 \rightarrow_{\Sigma} \psi_2}(\sigma_1\sigma_2B)(\mathbf{x}) = 0.$$

By Propositions 2.3.6 and 2.3.9,  $c = 0$ . Since  $\mathbf{a} = (a_1, \dots, a_n)$  satisfies the identity  $\gcd(a_1, \dots, a_n) = 1$ , then there exists an integer  $d$  such that  $d\mathbf{a} = \pi(c_{\psi_1 \rightarrow_{\Sigma} \psi_2}(\sigma_1\sigma_2B))$ .

- If  $\varphi = \psi_1 \rightarrow_{\Sigma} \psi_2$  and  $\sigma = \sigma_1\sigma_2B$  then one similarly gets  $d\mathbf{a} = \pi(c_{\psi_1 \rightarrow_{\Sigma} \psi_2}(\sigma_1\sigma_2B))$ .

■

**Example 3.3.5** Consider the formula  $\varphi$  of Example 3.3.2 and let  $\sigma = BA \in I(\varphi)$ . Then

$$\begin{aligned} D(\varphi, BA) &= D(X \rightarrow_{\Sigma} Y, B) \cup D(X \rightarrow_{\Sigma} Z, A) = \\ &= \pm(E_n \times \{0, 1\}) \cup \{(-1, 1, 0, 0)\} \cup \{(-1, 0, 1, 0)\}. \end{aligned}$$

Choose  $(\mathbf{a}, c) = (-1, 0, 1, 0)$ . We have  $A \in I(X \rightarrow_{\Sigma} Z)$ ,  $\text{cell}_{\varphi}(BA) \subseteq \text{cell}_{X \rightarrow_{\Sigma} Z}(A) = \{(x, y, z) \mid x \geq z\}$  and  $c_{X \rightarrow_{\Sigma} Z}(B) = z - x$ . Further,  $\pi(c_{X \rightarrow_{\Sigma} Z}(B)) = (-1, 0, 1)$ .

In the following Lemma we shall show how, upon arbitrarily fixing an index  $\sigma \in I(\varphi)$  and a subformula  $\psi$  of  $\varphi$ , it is always possible to decompose  $\pi(c_{\varphi}(\sigma))$  in terms of  $\psi$  and of others pairwise disjoint occurrences  $\vartheta_1, \dots, \vartheta_h$  of subformulas of  $\varphi$ .

**Lemma 3.3.6** *Let  $\psi \preceq \varphi$  and  $\sigma \in I(\varphi)$ . Then there exists  $\tau \in I(\psi)$  with  $\text{cell}_{\varphi}(\sigma) \subseteq \text{cell}_{\psi}(\tau)$ , together with a (possibly empty) set  $\{\vartheta_1, \dots, \vartheta_t\}$  of pairwise disjoint occurrences of subformulas of  $\varphi$  also disjoint from  $\psi$ , such that*

$$(i) \quad \pi(c_{\varphi}(\sigma)) = \delta\pi(c_{\psi}(\tau)) + \sum_{i=1}^t \pm\pi(c_{\vartheta_k}(\rho_k)), \quad \text{where } \delta \text{ is in } \{-1, 0, 1\}, \\ \rho_k \in I_{\overline{C}}(\vartheta_k) \text{ is such that } \text{cell}_{\varphi}(\sigma) \subseteq \text{cell}_{\vartheta_k}(\rho_k);$$

(ii) whenever  $\alpha$  is a subformula of  $\varphi$  such that there exists  $k$  with  $\vartheta_k \prec \alpha$ , then also  $\psi \prec \alpha$ .

**Proof.** By structural induction on the complexity of  $\varphi$ .

- If  $\varphi = X_i$  and  $\sigma = \epsilon$  the claims are trivial.
- If  $\varphi = \neg_{\Sigma}\psi$ , then for every  $\mathbf{x} \in \text{cell}_{\varphi}(\sigma)$ , we either have  $c_{\varphi}(\sigma)(\mathbf{x}) = 0$  or  $c_{\varphi}(\sigma)(\mathbf{x}) = \infty$  and in both cases  $\pi(c_{\varphi}(\sigma)) = \mathbf{0}$ . Condition (ii) trivially holds.
- If  $\varphi = \psi + \vartheta$  and  $\sigma = \sigma_1\sigma_2$ , with  $\sigma_1 \in I(\psi)$  and  $\sigma_2 \in I(\vartheta)$ , then  $\pi(c_{\varphi}(\sigma)) = \pi(c_{\psi}(\sigma_1)) + \pi(c_{\vartheta}(\sigma_2))$ . The case  $\vartheta + \psi$  is similar.
- If  $\varphi = \alpha_1 + \alpha_2$  with  $\sigma = \sigma_1\sigma_2$ ,  $\sigma_1 \in I(\alpha_1)$  and  $\sigma_2 \in I(\alpha_2)$ , then  $\pi(c_{\varphi}(\sigma)) = \pi(c_{\alpha_1}(\sigma_1)) + \pi(c_{\alpha_2}(\sigma_2))$ . Since  $\psi$  is a subformula of either  $\alpha_1$  or  $\alpha_2$ , then the claim holds by applying the induction hypothesis to the one among  $\alpha_1$  and  $\alpha_2$  where  $\psi$  occurs.
- If  $\varphi = \alpha_1 \rightarrow_{\Sigma} \alpha_2$  and  $\sigma = \sigma_1\sigma_2A$  then  $c_{\varphi}(\sigma) = 0$  and the claim trivially follows.
- If  $\varphi = \psi \rightarrow_{\Sigma} \vartheta$  and  $\sigma = \sigma_1\sigma_2B$  with  $\sigma_1 \in I(\psi)$  and  $\sigma_2 \in I(\vartheta)$ , then  $\pi(c_{\varphi}(\sigma)) = \pi(c_{\vartheta}(\sigma_2)) - \pi(c_{\psi}(\sigma_1))$ . The case  $\vartheta \rightarrow_{\Sigma} \psi$  is similar.
- If  $\varphi = \alpha_1 \rightarrow_{\Sigma} \alpha_2$  and  $\sigma = \sigma_1\sigma_2B$  several different cases are possible.
  - If  $\psi \preceq \alpha_1$  and  $c_{\alpha_1}(\sigma_1) > 0$  then arguing by induction we have

$$\pi(c_{\alpha_1}(\sigma_1)) = \delta\pi(c_{\psi}(\tau)) + \sum \pm\pi(c_{\vartheta_j}(\rho_j))$$

for suitable  $\tau, \{\vartheta_j\}_j, \{\rho_j\}_j$ . If there exists  $\alpha$  subformula of  $\varphi$  and  $k$  with  $\vartheta_k \prec \alpha$  then we also have  $\psi \prec \alpha$ . If  $c_{\alpha_2}(\sigma_2) > 0$

$$\pi(c_{\varphi}(\sigma)) = \pi(c_{\alpha_2}(\sigma_2)) - \left( \delta\pi(c_{\psi}(\tau)) + \sum \pm\pi(c_{\vartheta_j}(\rho_j)) \right).$$

Otherwise,

$$\pi(c_{\varphi}(\sigma)) = -\delta\pi(c_{\psi}(\tau)) - \sum \pm\pi(c_{\vartheta_j}(\rho_j)).$$

One argues similarly for  $\psi \preceq \alpha_2$  and  $c_{\alpha_2}(\sigma_2) > 0$ .

- If  $\psi \preceq \alpha_1$  and  $c_{\alpha_1}(\sigma_1) = 0$  then  $c_{\varphi}(\sigma) = c_{\alpha_2}(\sigma_2)$ .  $\psi$  does not occur in the decomposition and then  $\delta = 0$ .

The case  $\psi \preceq \alpha_2$  and  $c_{\alpha_2}(\sigma_2) = 0$  is similar.

■

The following Lemma is a generalization of the previous one to  $n$  disjoint occurrences  $\psi_1, \dots, \psi_n$  of subformulas of  $\varphi$ .

**Lemma 3.3.7** *Let  $\psi_1, \dots, \psi_m$  be pairwise disjoint occurrences of subformulas of  $\varphi$  and let  $\sigma \in I(\varphi)$ . Then there exists  $J \subseteq \{1, \dots, m\}$  and a unique  $\tau_i \in I_{\overline{C}}(\psi_i)$  for every  $i \in J$  such that  $\text{cell}_\varphi(\sigma) \subseteq \text{cell}_{\psi_i}(\tau_i)$  and*

$$\pi(c_\varphi(\sigma)) = \sum_{i \in J} \pm \pi(c_{\psi_i}(\tau_i)) + \sum_{k \in K} \pm \pi(c_{\vartheta_k}(\rho_k)),$$

where

- $K = \{1, \dots, t\}$  is a (possibly empty) finite set,
- for every  $k, k' \in K$ ,  $\vartheta_k, \vartheta_{k'}$  are pairwise disjoint occurrences of subformulas of  $\varphi$  also disjoint from each  $\psi_1, \dots, \psi_m$ ,
- $\rho_k \in I_{\overline{C}}(\vartheta_k)$  are such that  $\text{cell}_\varphi(\sigma) \subseteq \text{cell}_{\vartheta_k}(\rho_k)$ .

**Proof.** Lemma 3.3.6 yields  $\tau_1 \in I(\psi_1)$ ,  $\{\vartheta_{1,i}\}_i$  pairwise disjoint occurrences of subformulas of  $\varphi$  and  $\{\rho_{1,i} \in I(\vartheta_{1,i})\}_i$  such that

$$\pi(c_\varphi(\sigma)) = \delta_1 \pi(c_{\psi_1}(\tau_1)) + \sum_i \pm \pi(c_{\vartheta_{1,i}}(\rho_{1,i})),$$

where  $\delta_1 \in \{-1, 0, 1\}$  and whenever  $\alpha$  is a subformula of  $\varphi$  and  $k$  satisfies  $\vartheta_{1,k} \prec \alpha$ , then also  $\psi_1 \prec \alpha$ .

Since  $\psi_2$  is disjoint from  $\psi_1$ , then  $\psi_1 \not\preceq \psi_2$  and so there exists  $j$  such that  $\psi_2 \preceq \vartheta_{1,j}$ . Arguing in the same way for  $\pi(c_{\vartheta_{1,j}}(\rho_{1,j}))$ , we get

$$\begin{aligned} \pi(c_\varphi(\sigma)) &= \\ &\delta_1 \pi(c_{\psi_1}(\tau_1)) + \delta_2 \pi(c_{\psi_2}(\tau_2)) + \sum \pm \pi(c_{\vartheta_{2,i}}(\rho_{2,i})) + \sum_{i \neq j} \pm \pi(c_{\vartheta_{1,i}}(\rho_{1,i})). \end{aligned}$$

By similarly handling  $\psi_3, \dots, \psi_m$ , one obtains the desired conclusion. ■

**Example 3.3.8** Consider  $\varphi = (X \rightarrow_\Sigma Y) + (X \rightarrow_\Sigma Z)$  as in Example 3.3.2 and 3.3.5. Let  $\psi$  be the occurrence of  $X$  in  $X \rightarrow_\Sigma Y$  and let  $\sigma = BB \in I(\varphi)$ . Then

$$\begin{aligned} \pi(c_\varphi)(BB) &= (-2, 1, 1) = -(1, 0, 0) + (0, 1, 0) + (-1, 0, 1) \\ &= -\pi(c_X(\epsilon)) + \pi(c_Y(\epsilon)) + \pi(c_{X \rightarrow_\Sigma Z}(B)). \end{aligned}$$

### 3.3.2 Upper bounds for denominators of critical points

By Lemma 2.3.8, the restriction of  $f_\varphi$  to any open face  $F \subseteq (\mathbb{R}_+^*)^n \setminus (\mathbb{R}_+)^n$  contained in a hyperplane of equation  $x_i = \infty$  is either equal to 0, or is equal to  $\infty$ , or is equal to a function, defined on  $(\mathbb{R}_+^*)^{n-1}$ , associated with a formula  $\vartheta$  of length less or equal to  $\#(\varphi)$ , having the additional property that the variable  $X_i$  does not occur in  $\vartheta$ . If the restriction of  $f_\varphi$  to  $F$  is 0 or  $\infty$ , then a set  $C$  of critical points can be constructed in such a way that  $C$  contains just one point of  $F$ . On the other hand, if for every  $\mathbf{x} \in F$ ,  $f_\varphi(\mathbf{x}) = f_\vartheta(\mathbf{x}^i)$  and  $C'$  is a set of critical points for  $f_\vartheta$ , then a set  $C$  of critical points can be constructed in such a way that  $C \cap F = \{\mathbf{x} \in (\mathbb{R}_+^*)^n \mid \mathbf{x}^i \in C' \text{ and } x_i = \infty\}$ .

Suppose now that the function  $f_\varphi$  associated with the formula  $\varphi$  is not identically equal to  $\infty$  over  $\mathbb{R}_+^n$ . Then by Lemma 2.3.7, the restriction of  $f_\varphi$  to  $\mathbb{R}_+^n$  is continuous with respect to the natural topology on  $\mathbb{R}_+$ ; by Proposition 2.3.6 and Lemma 3.3.4, the restriction of  $f_\varphi$  to  $\mathbb{R}_+^n$  consists of homogeneous pieces with homogeneous boundaries. In order to test whether  $f_\varphi$  is identically equal to 0, it is enough to check that each linear piece  $l$  of  $f_\varphi$  (defined over an open set) is equal to 0 at just one point  $\mathbf{x}_l$  in the interior of the cell other than the origin  $\mathbf{0}$ . Let  $C_l$  be the cell in which  $f_\varphi$  is equal to  $l$  and let  $k$  be such that the intersection between  $C_l$  and  $\{\mathbf{x} \mid x_k = 1\}$  is non-empty. Since each linear piece  $l$  is homogeneous, we can choose  $\mathbf{x}_l$  to be the intersection of  $(n-1)$  many boundaries of  $C_l$  and the hyperplane of equation  $x_k = 1$ . Hence,  $\mathbf{x}_l$  is the solution of a system  $M\mathbf{x} = \mathbf{b}$  where  $M$  is an  $(n \times n)$ -matrix of integer numbers, where we can safely suppose that the first row is equal to  $\mathbf{e}_k$  and the other rows  $\mathbf{l}_i$  are such that  $(\mathbf{l}_i, 0) \in D(\varphi, \sigma)$ , and where  $\mathbf{b} = \mathbf{e}_1$ .

We want to determine an upper bound for the denominator of the solution of such a system.

By Proposition 2.3.6, if the function  $f_\varphi$  is not identically equal to  $\infty$  over  $\mathbb{R}_+^n$ , then for every  $\sigma \in I(\varphi)$  and every open face  $F$  of  $\text{cell}_\varphi(\sigma)$  contained in  $\mathbb{R}_+^n$ , the restriction  $c_\varphi(\sigma)$  of  $f_\varphi$  to  $F$  is a linear function. For every  $\mathbf{x} \in F$  and  $\mathbf{v} \in \pm(E_n)$  one can now consider the directional derivative  $\frac{df_\varphi}{d\mathbf{v}}(\mathbf{x})$ . Indeed,  $\frac{df_\varphi}{d\mathbf{v}}(\mathbf{x})$  exists for all  $\mathbf{x} \in \mathbb{R}_+^n$ , with trivial modifications when  $\mathbf{x}$  lies on the border of  $\mathbb{R}_+^n$ .

**Lemma 3.3.9** *For each formula  $\varphi$  in the variables  $X_1, \dots, X_n$  and each*

point  $\mathbf{x} \in \mathbb{R}_+^n$ , if  $\mathbf{v} \in \{\mathbf{e}_i, -\mathbf{e}_i\}$  is such that  $\frac{df_\varphi}{d\mathbf{v}}(\mathbf{x})$  exists then

$$\left| \frac{df_\varphi}{d\mathbf{v}}(\mathbf{x}) \right| \leq \#(X_i, \varphi).$$

**Proof.** We can safely assume that the function  $f_\varphi$  is not equal to  $\infty$  on  $\mathbb{R}_+^n$  (for otherwise, since the function is constant, we have  $|df_\varphi/d\mathbf{v}(\mathbf{x})| = 0$  and we are done).

For  $\mathbf{x} \in \mathbb{R}_+^n$ , there exists  $\sigma \in I(\varphi)$  such that  $f_\varphi(\mathbf{x}) = c_\varphi(\sigma)(\mathbf{x}) = a_{\sigma,1}x_1 + \dots + a_{\sigma,n}x_n$  and  $\varepsilon > 0$  such that  $\{\mathbf{x} + \lambda\mathbf{v} \mid 0 \leq \lambda < \varepsilon\} \subseteq \text{cell}_\varphi(\sigma)$ . By induction on the complexity of  $\varphi$  we have  $|a_{\sigma,i}| \leq \#(X_i, \varphi)$ . It follows that

$$\left| \frac{df_\varphi}{d\mathbf{v}}(\mathbf{x}) \right| = |a_{\sigma,i}| \leq \#(X_i, \varphi).$$

■

**Theorem 3.3.10** *Let  $\mathbf{p}$  be a vertex of an  $n$ -dimensional cell of  $C^{(n)}(\varphi) \cap [0, 1]^n$ . Then there exists an  $(n-1) \times (n-1)$  matrix  $M_{\mathbf{p}}$  whose entries  $a_{i,j}$  are integers, satisfying the inequalities  $\sum_{i,j=1}^{n-1} |a_{i,j}| \leq \#(\varphi) - 1$  and  $\text{den}(\mathbf{p}) \leq |\det(M_{\mathbf{p}})|$ .*

**Proof.** Observe that  $\mathbf{p}$  is the solution of a system  $M\mathbf{x} = \mathbf{b}$  with  $M$  an  $n \times n$  integer matrix. We can safely suppose that the first row is equal to  $\mathbf{e}_k$  and  $\mathbf{b} = (1, 0, \dots, 0) = \mathbf{e}_1$ , and the remaining rows  $\mathbf{l}_2, \dots, \mathbf{l}_n$  of  $M$  satisfy the condition  $(\mathbf{l}_2, 0), \dots, (\mathbf{l}_n, 0) \in D(\varphi, \sigma)$  for suitable  $\sigma \in I(\varphi)$ . Hence

$$\mathbf{p} = M^{-1}\mathbf{b} = \frac{1}{\det(M)}\tilde{M}\mathbf{b},$$

where each entry of  $\tilde{M}$  is an integer. Then  $\text{den}(\mathbf{p}) \leq |\det(M)|$ .

Since  $(\mathbf{l}_i, 0) \in D(\varphi, \sigma)$  for each  $i \geq 2$ , then, by Lemma 3.3.4, we either have  $\mathbf{l}_i \in \pm E_n$  or there exists  $\psi_i = \psi_{i_1} \rightarrow_\Sigma \psi_{i_2} \preceq \varphi$ ,  $\tau'_{i_1}\tau'_{i_2}C \in I(\psi_i)$ , ( $C \in \{A, B\}$ ) with  $\text{cell}_\varphi(\sigma) \subseteq \text{cell}_{\psi_i}(\tau'_{i_1}\tau'_{i_2}C)$  and  $d_i \in \mathbb{Z}$  such that  $d_i\mathbf{l}_i = \pi(c_{\psi_i}(\tau'_{i_1}\tau'_{i_2}B))$ . We shall now define a matrix  $M_p$  with the same determinant as  $M$ , in such a way to relate this determinant with the length of the formulas corresponding to the rows of  $M_p$ . For our current purpose of finding an upper bound of  $\text{den}(\mathbf{p})$ , we may safely assume that each  $d_i$  is equal to  $\pm 1$ .

In case there do not exist in  $M$  two distinct rows  $\mathbf{l}_r, \mathbf{l}_s$  such that  $\psi_r \preceq \psi_s$  and there is no row  $\mathbf{l}_i = \pm \mathbf{e}_t$  (for any  $\mathbf{e}_t \in E_n$ ), then we let  $M_{\mathbf{p}} = (a'_{i,j})$  be the  $(n-1) \times (n-1)$ -matrix obtained from  $M = (a_{i,j})_{i,j=1,\dots,n}$  by deleting the first

row and the  $k$ -th column. Since  $\psi_i$  for  $i = 2, \dots, n$  are disjoint occurrences of subformulas of  $\varphi$ , then  $\sum_{i=2}^n \#(\psi_i) \leq \#(\varphi)$ . Further,  $|\det(M_{\mathbf{p}})| = |\det(M)|$  and, since  $|a_{ij}| \leq \#(X_j, \psi_i)$  by Lemma 3.3.9, then

$$\sum_{i,j=1}^{n-1} |a'_{ij}| = \sum_{i=2, j=1, j \neq k}^n |a_{ij}| \leq \sum_{i=2}^{n-1} (\#(\psi_i) - \#(X_k, \psi_i)) \leq \#(\varphi) - 1.$$

Otherwise, if the subformulas  $\psi_i$  are not pairwise disjoint or there is a row  $\mathbf{l}_i = \pm \mathbf{e}_t$ , we let  $P \subseteq \{2, \dots, n\}$  be such that

$$p \in P \quad \text{if and only if} \quad \mathbf{l}_p \notin \pm E_n.$$

Let further  $h \in P$  be such that there is no  $h'$  with  $\psi_h \preceq \psi_{h'}$ . Let  $\psi_{k_1}, \dots, \psi_{k_u}$  be such that, for every  $i \in \{1, \dots, u\}$ ,

- $k_i \in P$ ;
- $\psi_{k_i} \preceq \psi_h$ ;
- there is no other formula  $\psi_{k'}$  ( $k' \in P$ ) such that  $\psi_{k_i} \preceq \psi_{k'} \preceq \psi_h$ .

Applying Lemma 3.3.7 to  $\psi_h, \psi_{k_1}, \dots, \psi_{k_u}$  and  $\tau_h \in I(\psi_h)$ , there exists  $J \subseteq \{k_1, \dots, k_u\}$  such that for each  $k_i \in J$  there exists a unique  $\omega_i \in I_{\overline{C}}(\psi_{k_i})$  with  $\text{cell}_{\psi_h}(\tau_h) \subseteq \text{cell}_{\psi_{k_i}}(\omega_i)$  and

$$\pi(c_{\psi_h}(\tau_h)) = \sum_{k_j \in J} \pm \pi(c_{\psi_{k_j}}(\omega_j)) + \sum_{i=1}^s \pm \pi(c_{\vartheta_i}(\rho_i))$$

where  $\vartheta_1, \dots, \vartheta_s$  are pairwise disjoint occurrences of subformulas of  $\psi_h$  also disjoint from each of the occurrences  $\psi_{k_1}, \dots, \psi_{k_u}$ , and  $\rho_i \in I_{\overline{C}}(\vartheta_i)$  with  $\text{cell}_{\psi_h}(\tau_h) \subseteq \text{cell}_{\vartheta_i}(\rho_i)$  for every  $i = 1, \dots, s$ .

By the elementary properties of determinants, replacing in  $M$  the row  $\mathbf{l}_h$  by  $\sum_{j=1}^s \pm \pi(c_{\vartheta_j}(\rho_j))$  we obtain a matrix  $M'$  such that  $|\det(M')| = |\det(M)|$ .

Repeating the above procedure finitely many times, we obtain an  $n \times n$  matrix  $\overline{M}_p$  such that each row  $\mathbf{l}_i$  ( $i \in P$ ) of  $\overline{M}_p$  has the form

$$\sum_{h \in H^i} \pm \pi(c_{\vartheta_h^i}(\rho_h^i))$$

and, for any two distinct rows  $\mathbf{l}_i, \mathbf{l}_j \in \overline{M}_p$  ( $i, j \in P$ ) we have  $\vartheta_{h_i}^i \not\leq \vartheta_{k_j}^j$  for every  $h_i \in H^i$  and  $k_j \in H^j$ .

Consider now any row  $\mathbf{l}_i \in \overline{M}_p$  (with  $\mathbf{l}_i \neq \mathbf{l}_1$ ) such that  $\mathbf{l}_i = \pm \mathbf{e}_t \in \pm E_n$ . Then, by subtracting suitable multiples of  $\mathbf{l}_i$  from the remaining rows, we get a matrix  $\overline{M}'_p$  such that all entries in its  $t$ -th column are 0, except for the  $(i, t)$ th entry which is  $\pm 1$ , while all the other columns and the determinant are left unchanged. Repeating this procedure for all rows  $\mathbf{l}_i \in \pm E_n$  except for the first one, we finally obtain a matrix  $M'_p = \{m_{ij} \in \mathbb{Z} : 1 \leq i, j \leq n\}$  with the following property: there exist indexes  $j_1, \dots, j_t$  such that in every row  $(m_{ij})_{j=1}^n$ , elements of columns indexed by  $j_1, \dots, j_t$  coincide respectively with the  $j_1$ th,  $\dots$ ,  $j_t$ th elements of the vector  $\sum_{h \in H^i} \pm \pi(c_{\vartheta_h^i}(\rho_h^i))$  where for every  $h \in H^i$  and  $i = 1, \dots, n$ , the occurrences  $\vartheta_h^i$  are pairwise disjoint subformulas of  $\varphi$ . Further,  $m_{ij} = 0$  for  $j \neq j_1, \dots, j_t$ .

Let  $M_{\mathbf{p}} = (a''_{ij})$  be the  $(n-1) \times (n-1)$  matrix obtained from  $M'_p$  by deleting the first row and the  $k$ th column.

Then  $\det(M_{\mathbf{p}}) = \det(M)$  and since  $|m_{i,j}| \leq \sum_{h \in H^i} \#(X_j, \vartheta_h^i)$ ,

$$\sum_{i,j=1}^{n-1} |a''_{ij}| \leq \sum_{i=2}^{n-1} \sum_{h \in H^i} (\#\vartheta_h^i - \#(X_k, \vartheta_h^i)) \leq \#(\varphi) - 1.$$

■

**Theorem 3.3.11** *Let  $\varphi$  be a formula in the variables  $X_1, \dots, X_n$  (with  $n \geq 2$ ) and let  $\mathbf{p}$  be a vertex of an  $n$ -dimensional cell of  $C^{(n)}(\varphi) \cap [0, 1]^n$ . Then*

$$\text{den}(\mathbf{p}) \leq \left( \frac{\#(\varphi) - 1}{n - 1} \right)^{n-1}.$$

If  $n = 1$  then  $\text{den}(\mathbf{p}) = 1$ .

**Proof.**

Let  $M_{\mathbf{p}} = (\mathbf{l}_i)$  be the  $(n-1) \times (n-1)$  matrix defined in Theorem 3.3.10, with  $\mathbf{l}_i = (a_{i1}, \dots, a_{i(n-1)})$  such that  $\sum_{i,j=1}^{n-1} |a_{ij}| \leq \#(\varphi) - 1$  and  $\text{den}(\mathbf{p}) \leq |\det(M_{\mathbf{p}})|$ . By Hadamard's inequality,  $|\det(M_{\mathbf{p}})| \leq \prod_{i=1}^{n-1} \|\mathbf{l}_i\|$ , where  $\|\mathbf{l}_i\| = \sqrt{a_{i1}^2 + \dots + a_{i(n-1)}^2} \leq |a_{i1}| + \dots + |a_{i(n-1)}|$ . Hence,

$$\text{den}(\mathbf{p}) \leq \prod_{i=1}^{n-1} \sum_{j=1}^{n-1} |a_{ij}| \leq \prod_{i=1}^{n-1} \frac{\#(\varphi) - 1}{n - 1} = \left( \frac{\#(\varphi) - 1}{n - 1} \right)^{n-1}.$$

■

Note that

$$\left(\frac{\#(\varphi) - 1}{n - 1}\right)^{n-1} = 2^{(n-1)\log((\#(\varphi)-1)/(n-1))} < 2^{\#(\varphi)-1}.$$

As in the case of Łukasiewicz logic, the smallest basis  $s$  such that  $((\#(\varphi) - 1)/(n - 1))^{n-1} < s^{\#(\varphi)-1}$  is  $e^{1/e}$ .

**Remark.** The function  $f_\varphi$  is homogeneous. So, in our investigation of denominator upper bounds of  $f_\varphi$ , we can equivalently restrict our attention to  $([0, 1] \cup \{\infty\})^n$  or to any  $([0, a] \cup \{\infty\})^n$  with  $a > 1$ . In the last case, the bound could be smaller, but the cardinality of the set of critical points (and then the number of values of the logic which we want to reduce to) remains the same.

**Theorem 3.3.12** *If  $\mathbf{q} \in (\mathbb{R}_+^*)^n$  is such that  $f_\varphi(\mathbf{q}) > 0$ , then there exists  $\mathbf{p} \in (([0, 1] \cap \mathbb{Q}) \cup \{\infty\})^n$  such that  $f_\varphi(\mathbf{p}) > 0$  and  $\text{den}(\mathbf{p})$  divides  $2^{\#(\varphi)-1}$ .*

**Proof.** Skipping all trivialities, we can safely suppose that the function  $f_\varphi$  is not identically equal to  $\infty$  over  $\mathbb{R}_+^n$ . Thus  $f_\varphi$  is continuous over  $\mathbb{R}_+^n$ . Suppose that there exists a point  $\mathbf{q} \in \mathbb{R}_+^n$  at which  $f_\varphi$  assumes a value strictly greater than 0. Since  $f_\varphi$  consists of linear homogeneous pieces, we can safely suppose that  $\mathbf{q}$  is a vertex of  $C^{(n)}(\varphi) \cap [0, 1]^n \subseteq \mathbb{Q}^n$ . Let  $q_k$  be the component of  $\mathbf{q}$  equal to 1. Among all points of the form  $\mathbf{x} = (x_1/2^{\#(\varphi)-1}, \dots, x_n/2^{\#(\varphi)-1})$ , let  $\mathbf{p} = (p_1, \dots, p_n)$  be the one closest to  $\mathbf{q}$  (with respect to Euclidean distance) such that  $p_k = 1$ . By way of contradiction, suppose that  $f_\varphi(\mathbf{p}) = 0$ . Since  $f_\varphi(\mathbf{q}) \geq 1/\text{den}(\mathbf{q})$ , by Lemma 3.3.11, we get

$$f_\varphi(\mathbf{q}) - f_\varphi(\mathbf{p}) \geq \left(\frac{n-1}{\#(\varphi)-1}\right)^{n-1}.$$

On the other hand, since  $\mathbf{p}$  is the closest point to  $\mathbf{q}$  among all points with denominator equal to  $2^{\#(\varphi)-1}$ , then

$$|p_i - q_i| \leq \frac{1}{2 \cdot 2^{\#(\varphi)-1}},$$

where  $p_i$  and  $q_i$  are the  $i$ th coordinates of  $\mathbf{p}$  and  $\mathbf{q}$ , respectively. Since  $p_k = q_k = 1$ , an application of Lemma 3.3.9 yields

$$f_\varphi(\mathbf{q}) - f_\varphi(\mathbf{p}) \leq \sum_{i=1}^n (|p_i - q_i| \#(X_i, \varphi)) = \sum_{i=1, i \neq k}^n (|p_i - q_i| \#(X_i, \varphi)),$$



and then

$$\left(\frac{n-1}{\#(\varphi)-1}\right)^{n-1} \leq \sum_{i=1, i \neq k}^n \left(\frac{1}{2} \frac{1}{2^{\#(\varphi)-1}} \#(X_i, \varphi)\right) \leq \frac{1}{2^{\#(\varphi)-1}} \frac{\#(\varphi)-1}{2},$$

whence

$$\frac{\#(\varphi)-1}{2} \left(\frac{\#(\varphi)-1}{n-1}\right)^{n-1} \geq 2^{\#(\varphi)-1}. \quad (3.1)$$

The desired conclusion now follows by noting that such inequality is never satisfied for  $\#(\varphi), n = 2, 3, \dots$ . Indeed, letting  $g = (\#(\varphi) - 1)/(n - 1)$ , we have

$$\frac{\#(\varphi)-1}{2} g^{(\#(\varphi)-1)/g} \geq 2^{\#(\varphi)-1}$$

if and only if

$$\frac{\log_2(\#(\varphi)-1)-1}{\#(\varphi)-1} \geq \frac{g - \log_2 g}{g}.$$

Let  $\log x$  denote the logarithm of  $x$  to the base  $e$ . The real-valued function  $f_1(x) = (\log_2 x - 1)/x$  reaches its maximum value  $1/(2e \log 2)$  for  $x = 2e$ . The function  $f_2(x) = (x - \log_2 x)/x$  attains its minimum value  $1 - 1/(e \ln 2)$  for  $x = e$ .

Noticing that  $1/(2e \log 2) < 1 - 1/(e \log 2)$  we have obtained a contradiction, and the proof is complete (compare with [8]).

There remains to consider the case when the function  $f_\varphi$  vanishes over  $\mathbb{R}_+^n$  and is  $> 0$  over  $(\mathbb{R}_+^*)^n \setminus \mathbb{R}_+^n$ . Suppose that  $F$  is an open face of  $(\mathbb{R}_+^*)^n \setminus \mathbb{R}_+^n$  contained in the hyperplane  $x_i = \infty$  and that there exists  $\mathbf{q} \in F$  such that  $f_\varphi(\mathbf{q}) > 0$ . For simplicity, suppose that  $f$  is not contained in any hyperplane  $x_j = \infty$  with  $j \neq i$ . By Lemma 2.3.8, for every  $\mathbf{x} \in F$ ,  $f_\varphi(\mathbf{x}) = f_\vartheta(\mathbf{x}^i)$ , where  $\text{var}(\vartheta) \subseteq \text{var}(\varphi) \setminus \{X_i\}$  and  $\#(\vartheta) \leq \#(\varphi)$ . Hence  $f_\vartheta(\mathbf{q}^i) > 0$ . Since  $\mathbf{q}^i \in \mathbb{R}_+^{n-1}$ , we can now apply to  $\vartheta$  the result of the first part of this proof. We obtain a point  $\mathbf{p} = (p_1, \dots, p_{n-1}) \in \mathbb{R}_+^{n-1}$  such that  $f_\vartheta(\mathbf{p}) > 0$  and  $\text{den}(\mathbf{p})$  divides  $2^{\#(\vartheta)-1}$ . The point  $\mathbf{p}' = (p_1, \dots, p_{i-1}, \infty, p_i, \dots, p_{n-1}) \in (\mathbb{R}_+^*)^n$  is such that  $\text{den}(\mathbf{p}')$  divides  $2^{\#(\vartheta)-1}$ , whence  $2^{\#(\varphi)-1}$  and, further,

$$f_\varphi(\mathbf{p}') = f_\vartheta(\mathbf{p}) > 0.$$

■

### 3.3.3 Finite-valued approximations of Product Logic

As already remarked, it is not possible to define a finite-valued Product Logic by restricting the set of truth values to a finite subset of  $[0, 1]$ . However, once a specific function  $f_\varphi$  is considered, one can conceivably define a (finite) set of critical points for  $f_\varphi$  having the desired closure properties. To this purpose, we first introduce a new unary operator as follows.

Let  $\mathbb{R}_+^* = [0, \infty]$  and

$$S_k^* = S_k \cup \{\infty\} = \left\{ 0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1, \infty \right\}.$$

Let  $\varphi$  be a formula of  $\Sigma$  and let  $m, l > 0$  be integer numbers.

**Definition 3.3.13** For every  $x \in \mathbb{R}_+^*$  we define the unary operator  $\diamond_m^l : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  such that, for all  $x < \infty$

$$\diamond_m^l(x) = \frac{\lfloor lx \rfloor}{lm}$$

and  $\diamond_m^l(\infty) = \infty$ .

Let us consider the logic  $\mathcal{S}_m^l = (S_{lm}^*, \{0\}, \{\oplus, \neg_\Sigma, \rightarrow_\Sigma, \diamond\})$ , where  $\neg_\Sigma$  and  $\rightarrow_\Sigma$  are obtained by restricting to  $S_{lm}^*$  the operations given by the connectives in  $\Sigma$ . Assume further that, for every  $x, y \in S_{lm}^*$ ,

- $x \oplus y$  is the truncated sum  $\min(1, x + y)$  in case  $x, y < \infty$ , and  $x \oplus \infty = \infty \oplus y = \infty \oplus \infty = \infty$ ;
- $\diamond x = \diamond_m^l(x)$ .

For every subformula  $\psi$  of  $\varphi$  we shall denote by  $\tilde{\psi}$  the formula obtained by replacing every occurrence of a variable  $X_i$  in  $\psi$  by  $\diamond X_i$  and replacing every occurrence of  $+$  by  $\oplus$ . Obviously,  $\tilde{\psi}$  is a formula of  $\mathcal{S}_m^l$ .

For our current purposes it is convenient to separately study the restriction of  $f_{\tilde{\varphi}}$  to  $(S_{lm}^*)^n$  and to  $(S_{lm}^*)^n \setminus (S_{lm})^n$ .

For every variable  $X_i \in \text{var}(\varphi)$  and  $(x_1, \dots, x_n) \in (S_{lm})^n$  we have

$$f_{\diamond X_i}(x_1, \dots, x_n) = \frac{\lfloor lx_i \rfloor}{lm} \leq \frac{1}{m}.$$

For any formula  $\varphi$ , we let  $m \geq \#(\varphi)$ ; then it is easy to see that, if  $\varphi$  is such that the restriction of  $f_\varphi$  to  $(\mathbb{R}_+)^n$  is different from  $\infty$ , then  $m \geq \max_{\mathbf{x} \in [0, 1]^n} f_\varphi(\mathbf{x})$ .

Further, if  $\psi_1, \psi_2 \preceq \varphi$ , and  $\widetilde{\psi}_1$  and  $\widetilde{\psi}_2$  are the corresponding formulas of  $\mathcal{S}_m^l$ , then, whenever  $m \geq \#(\varphi) \geq \#(\psi_1) + \#(\psi_2)$ , we have

$$f_{\widetilde{\psi}_1}(x_1, \dots, x_n) \leq \sum_{i=1}^n \#(X_i, \psi_1) \cdot \frac{\lfloor lx_i \rfloor}{lm} \leq \sum_{i=1}^n \#(X_i, \psi_1) \cdot \frac{1}{m} = \frac{\#(\psi_1)}{m}$$

$$f_{\widetilde{\psi}_2}(x_1, \dots, x_n) \leq \sum_{i=1}^n \#(X_i, \psi_2) \cdot \frac{\lfloor lx_i \rfloor}{lm} \leq \sum_{i=1}^n \#(X_i, \psi_2) \cdot \frac{1}{m} = \frac{\#(\psi_2)}{m}$$

and so

$$f_{\widetilde{\psi_1 + \psi_2}} = f_{\widetilde{\psi_1} \oplus \widetilde{\psi_2}} \leq \frac{\#(\psi_1) + \#(\psi_2)}{m} \leq 1$$

hence

$$f_{\widetilde{\psi_1 + \psi_2}} = f_{\widetilde{\psi_1}} + f_{\widetilde{\psi_2}}. \quad (3.2)$$

Trivially,

$$f_{\widetilde{\psi_1 \rightarrow \psi_2}} = f_{\widetilde{\psi_1}} \rightarrow f_{\widetilde{\psi_2}}. \quad (3.3)$$

**Lemma 3.3.14** *For every  $\mathbf{x} \in (S_l)^n$ ,  $m \geq \#(\varphi)$  and  $\varepsilon \in \{0, 1/(lm), 2/(lm), \dots, (m-1)/(lm)\}^n$  such that  $\mathbf{x} + \varepsilon \leq \mathbf{1}$ ,*

$$f_{\widetilde{\varphi}}(\mathbf{x} + \varepsilon) = \frac{1}{m} f_{\varphi}(\mathbf{x})$$

**Proof.** If the restriction of  $f_{\varphi}$  to  $(\mathbb{R}_+)^n$  is equal to  $\infty$  the Lemma trivially holds. Otherwise, we proceed by induction on the complexity of  $\varphi$ . Let  $\mathbf{x} = \mathbf{h}/l$  with  $\mathbf{h} \in \{0, \dots, l\}^n$ .

- Let  $\varphi = X_i$ . Then  $\widetilde{\varphi} = \diamond X_i$  and

$$f_{\diamond X_i}(\frac{\mathbf{h}}{l} + \varepsilon) = \frac{\lfloor l(\frac{h_i}{l} + \varepsilon) \rfloor}{lm} = \frac{h_i}{lm} = \frac{1}{m} f_{X_i}(\frac{\mathbf{h}}{l}).$$

- Let  $\varphi = \psi_1 + \psi_2$ . Then, by (3.2),

$$f_{\widetilde{\psi_1 + \psi_2}}(\frac{\mathbf{h}}{l} + \varepsilon) = f_{\widetilde{\psi_1}}(\frac{\mathbf{h}}{l} + \varepsilon) + f_{\widetilde{\psi_2}}(\frac{\mathbf{h}}{l} + \varepsilon) = \frac{1}{m} f_{\psi_1 + \psi_2}(\frac{\mathbf{h}}{l}).$$

- Let  $\varphi = \psi_1 \rightarrow \psi_2$ . Then, by (3.3),

$$f_{\widetilde{\psi_1 \rightarrow \psi_2}}(\frac{\mathbf{h}}{l} + \varepsilon) = f_{\widetilde{\psi_2}}(\frac{\mathbf{h}}{l} + \varepsilon) - f_{\widetilde{\psi_1}}(\frac{\mathbf{h}}{l} + \varepsilon) = \frac{1}{m} f_{\psi_1 \rightarrow \psi_2}(\frac{\mathbf{h}}{l}).$$

■

The above Lemma can be easily generalized to  $\mathbf{x} \in (S_l^*)^n$ . As a matter of fact, by Proposition 2.3.9, the restriction of  $f_\varphi$  to  $(\mathbb{R}_+^*)^n$  is either equal to  $\infty$ , or it coincides with the function associated with a formula  $\vartheta$  such that all the variables taking  $\infty$  do not occur in  $\vartheta$ .

**Corollary 3.3.15** *For every  $\mathbf{x} \in (S_l^*)^n$  and  $\varepsilon \in \{0, 1/(lm), 2/(lm), \dots, (m-1)/(lm)\}^n$  such that  $\mathbf{x} + \varepsilon \in (S_{lm}^*)^n$ , we have  $f_\varphi(\mathbf{x}) = 0$  if and only if  $f_{\tilde{\varphi}}(\mathbf{x} + \varepsilon) = 0$ .* ■

In agreement with our presentation of the logics  $L_n$  and  $G_n$ , we will introduce the following map from  $S_n^* = S_n \cup \{\infty\}$  to  $S_n \cup \{\perp\}$ :

$$\hat{\cdot} : x \in S_n \cup \{\infty\} \mapsto \begin{cases} 1 - x & \text{if } x \in S_n \\ \perp & \text{if } x = \infty, \end{cases}$$

where  $\perp$  is a new symbol for a distinguished element strictly less than all other elements of  $S_n$ . The map  $\hat{\cdot}$  induces an isomorphism between  $\mathcal{S}_m^l = (S_{lm}^*, \{0\}, \{\oplus, \neg_\Sigma, \rightarrow_\Sigma, \diamond\})$  and  $\mathfrak{S}_m^l = (S_{lm} \cup \{\perp\}, \{1\}, \{\odot, \neg_\mathfrak{S}, \rightarrow_\mathfrak{S}, \natural\})$ , where

$$\begin{aligned} x \odot y &= \begin{cases} \max\{0, x + y - 1\} & \text{if } x, y > \perp \\ \perp & \text{otherwise} \end{cases} \\ x \rightarrow_\mathfrak{S} y &= \begin{cases} \min\{1, 1 + y - x\} & \text{if } x, y > \perp \\ \perp & \text{if } y = \perp \text{ and } x \neq \perp \\ 1 & \text{if } x = \perp. \end{cases} \\ \neg_\mathfrak{S} x &= x \rightarrow_\Sigma \perp = \begin{cases} 1 & \text{if } x = \perp \\ \perp & \text{if } x > \perp. \end{cases} \\ \natural x &= 1 - \frac{\lfloor l(1 - x) \rfloor}{lm}. \end{aligned}$$

The operations  $\odot$  and  $\rightarrow_\mathfrak{S}$  coincide with suitable extensions to  $\perp$  of Lukasiewicz conjunction and implication, respectively. A formula  $\vartheta$  is a tautology of  $\mathfrak{S}_m^l$  if and only if  $f_\vartheta(\mathbf{x}) = 1$  for every  $\mathbf{x} \in (S_{lm} \cup \{\perp\})^n$ .

In the following we will consider  $\hat{\varphi}$  as the syntactic translation of a formula  $\varphi$  of  $\Sigma$  (or, equivalently, of Product logic), obtained by substituting each occurrence of a variable  $X_i$  with  $\natural X_i$ , each occurrence of  $+$  with  $\odot$ , each occurrence of  $\rightarrow_\Sigma$  with  $\rightarrow_\mathfrak{S}$  and each occurrence of  $\neg_\Sigma$  with  $\neg_\mathfrak{S}$ . Further, if  $\mathbf{x} = (x_1, \dots, x_n) \in (S_{lm}^*)^n$  we shall denote by  $\hat{\mathbf{x}}$  the vector  $(\hat{x}_1, \dots, \hat{x}_n)$ . From the isomorphism between logics  $\mathcal{S}_m^l$  and  $\mathfrak{S}_m^l$  and from Corollary 3.3.15 it follows that

**Corollary 3.3.16** *Assume  $m \geq \#(\varphi)$ . Then for every  $\mathbf{x} \in (S_l^*)^n$  and  $\varepsilon \in \{0, 1/(lm), 2/(lm), \dots, (m-1)/(lm)\}^n$  such that  $\widehat{\mathbf{x}} - \varepsilon \in (S_{lm} \cup \{\perp\})^n$ , we have  $f_\varphi^\Sigma(\mathbf{x}) = 0$  if and only if  $f_{\widehat{\varphi}}^\Theta(\widehat{\mathbf{x}} - \varepsilon) = 1$ . ■*

**Theorem 3.3.17** *For every formula  $\varphi$  of Product logic we have*

$$\models_{\Pi_\infty} \varphi \quad \text{if and only if} \quad \models_{\mathfrak{S}_m^l} \widehat{\varphi},$$

where  $m = \#(\varphi)$  and  $l = 2^{m-1}$ .

**Proof.** By Theorem 3.3.12,  $\varphi$  is a tautology of Product logic if and only if for every  $\mathbf{x} \in (S_l^*)^n$ ,  $f_\varphi^\Sigma(\mathbf{x}) = 0$ . By Corollary 3.3.16 this is equivalent to saying that  $f_{\widehat{\varphi}}(\widehat{\mathbf{x}} - \varepsilon) = 1$ , for every  $\varepsilon$  such that  $\widehat{\mathbf{x}} - \varepsilon \in (S_{lm} \cup \{\perp\})^n$ . Since for every  $\mathbf{y} \in (S_{lm} \cup \{\perp\})^n$  there exists  $\widehat{\mathbf{x}} \in (S_l \cup \{\perp\})^n$  and  $\varepsilon \leq (m-1)/(lm)$  such that  $\mathbf{y} = \widehat{\mathbf{x}} - \varepsilon$ , then  $f_{\widehat{\varphi}}(\widehat{\mathbf{x}} - \varepsilon) = 1$  holds if and only if, for every  $\mathbf{y} \in (S_{lm} \cup \{\perp\})^n$ ,  $f_{\widehat{\varphi}}(\mathbf{y}) = 1$ . This yields the desired conclusion. ■

### 3.4 Logical consequence in Gödel and Product Logic

When dealing with the lattice connectives  $\wedge$  and  $\vee$ , some extra care must be taken, since both  $\#(\varphi \wedge \psi)$  and  $\#(\varphi \vee \psi)$  are in general strictly greater than  $\#(\varphi) + \#(\psi)$ . Since, on the other hand, these two connectives play an important role in the treatment of logical consequence, we are interested in adding them as primitive connectives, without increasing the denominator upper bound of the logic. That this is indeed feasible follows from the observation that the function associated with  $\vee$  is given by

$$f_\vee(x, y) = \begin{cases} x & \text{if } y \leq x \\ y & \text{if } x < y \end{cases}$$

in all logics considered in this paper, since their sets of truth values are linearly ordered. Since  $f_\vee$  has the same boundary polynomial  $(x-y)$  as  $f_{\rightarrow_\Sigma}$  and since  $f_\vee$  does not introduce new polynomials, we can safely incorporate  $\vee$  among the primitive connectives of our logics. The same applies to  $\wedge$ .

The method used to reduce the infinite-valued tautology problem to its finite-valued counterparts, cannot in general be extended to the problem of logical consequence. Indeed, in order to prove that a formula of  $\mathcal{C}$  (with  $\mathcal{C} \in \{G, \Sigma\}$ ) is not a tautology we have used a continuity argument to find a point where the associated formula takes a value not belonging to

the set of designated values  $D(\mathcal{C})$ . On the other hand, for  $\Gamma \models_{\mathcal{C}} \Delta$  to be falsified, one must typically exhibit a point  $\mathbf{p}$  such that  $f_{\bigwedge \Gamma}(\mathbf{p}) \in D(\mathcal{C})$  and  $f_{\bigvee \Delta}(\mathbf{p}) \notin D(\mathcal{C})$ . Suppose that for any other  $\mathbf{q} \neq \mathbf{p}$ ,  $f_{\bigwedge \Gamma}(\mathbf{q}) \notin D(\mathcal{C})$ . In this case, the only finite logic that can be used must have as denominator the denominator of  $\mathbf{p}$ , because continuity arguments yield nothing here. As expected, the methods of the previous sections cannot in general be applied to reduce infinite-valued consequence to its finite-valued counterpart.

Still, our geometrical analysis and our estimates on denominators of critical points yield the following:

**Lemma 3.4.1** *Let  $\Gamma$  and  $\Delta$  be two finite sets of Gödel formulas and let  $n$  be the cardinality of  $\text{var}(\Gamma \cup \Delta)$ . Then we have:*

$$\Gamma \models_{G_{\infty}} \Delta \text{ if and only if } \Gamma \models_{G_{m+1}} \Delta \text{ for all integers } 1 \leq m \leq n.$$

**Proof.** Let  $\varphi = \bigwedge \Gamma$  and  $\psi = \bigvee \Delta$ . Since, as remarked above, we cannot directly reduce to a unique finite-valued logic, we shall examine the denominator of all points arising as intersections of boundaries of cells. The polyhedral complexes  $C(\varphi)$  and  $C(\psi)$  can be jointly refined by

$$C(\varphi, \psi) = \{A \cap B \mid A \in C(\varphi), B \in C(\psi)\}.$$

Vertices of  $C(\varphi, \psi)$  are solutions of systems  $\begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix} \mathbf{x} = \mathbf{b}$ , where every  $\mathbf{a}_i$  either belongs to  $D(\varphi, \sigma_1)$  or to  $D(\psi, \sigma_2)$ . Thus, in order to calculate denominators of vertices, we can use the same computations as for formulas  $\varphi * \psi$ , where  $*$  is any arbitrary binary connective. Theorem 3.2.2 now yields the desired conclusion. ■

Since, as is well known, the Deduction Theorem holds in Gödel logic (see [67]), we have

**Theorem 3.4.2** *Let  $\Gamma$  and  $\Delta$  be two finite sets of Gödel formulas and let  $n$  be the cardinality of  $\text{var}(\Gamma \cup \Delta)$ . Then*

$$\begin{aligned} \Gamma \models_{G_{\infty}} \Delta & \text{ if and only if } \models_{G_{\infty}} \bigwedge \Gamma \rightarrow \bigvee \Delta \\ & \text{ if and only if } \models_{G_{n+1}} \bigwedge \Gamma \rightarrow \bigvee \Delta \\ & \text{ if and only if } \Gamma \models_{G_{n+1}} \Delta. \end{aligned}$$

■

Geometrically, the above result states that, if  $\bar{\mathbf{p}}$  is a point such that  $f_{\wedge\Gamma}(\bar{\mathbf{p}}) = 1$ , then there exists a rational point  $\bar{\mathbf{q}}$  with  $\text{den}(\bar{\mathbf{q}})$  a divisor of  $n + 1$ , such that  $f_{\wedge\Gamma}(\bar{\mathbf{q}}) = 1$ . Indeed, if  $\bar{\mathbf{p}}$  belongs to the relative interior  $K$  of some cell of dimension  $k \leq n$  and  $f_{\wedge\Gamma}(\bar{\mathbf{p}}) = 1$  then, by Corollary 5.4,  $f_{\wedge\Gamma}(\mathbf{p}) = 1$  for every point  $\mathbf{p} \in K$ , whence in particular, for some point whose denominator divides  $n + 1$ .

We now turn to Product logic. If  $\Gamma = \{\gamma_1, \dots, \gamma_k\}$  is a set of formulas of Product logic, let us denote by  $\hat{\Gamma}$  the set  $\{\hat{\gamma}_1, \dots, \hat{\gamma}_k\}$ .

**Theorem 3.4.3** *Let  $\Gamma$  and  $\Delta$  be two finite sets of formulas of  $\Sigma$  and let  $n$  be the cardinality of  $\text{var}(\Gamma \cup \Delta)$ . Then we have:*

$$\Gamma \models_{\Pi_\infty} \Delta \text{ if and only if } \hat{\Gamma} \models_{\mathfrak{S}_m^l} \hat{\Delta}$$

for all  $l \leq \left(\frac{\#(\varphi) + \#(\psi) - 1}{n-1}\right)^{n-1}$  and  $m = \#(\Delta) + \#(\Gamma)$ .

**Proof.** We argue as in the proof of Lemma 3.4.1, with some extra care needed to repeat the “normalization” argument therein, in order to get a finite-valued logic. To this purpose, let  $\varphi = \wedge\Gamma$  and  $\psi = \vee\Delta$ . Suppose  $\Gamma \not\models_{\Pi_\infty} \Delta$ . Then there exists a point  $\mathbf{p}$  such that  $f_\varphi(\mathbf{p}) = 0$  and  $f_\psi(\mathbf{p}) > 0$ . We can safely suppose that  $\mathbf{p}$  is a vertex of  $C(\varphi, \psi)$  and so, by Lemma 3.3.11,

$$\text{den}(\mathbf{p}) < \left(\frac{\#(\varphi) + \#(\psi) - 1}{n-1}\right)^{n-1}.$$

By Corollary 3.3.16, there exists  $l = \text{den}(\mathbf{p})$  such that in the logic  $\mathfrak{S}_m^l$ ,  $f_{\hat{\varphi}}(\mathbf{p}) = 1$  and  $f_{\hat{\psi}}(\mathbf{p}) < 1$ , whence  $\hat{\Gamma} \not\models_{\mathfrak{S}_m^l} \hat{\Delta}$ .

Conversely, let us suppose that there exists  $l \leq \left(\frac{\#(\varphi) + \#(\psi) - 1}{n-1}\right)^{n-1}$  such that  $\hat{\Gamma} \not\models_{\mathfrak{S}_m^l} \hat{\Delta}$ . Then there exists a point  $\mathbf{p}$  such that  $\text{den}(\mathbf{p}) = lm$  and  $f_{\hat{\varphi}}(\mathbf{p}) = 1$  and  $f_{\hat{\psi}}(\mathbf{p}) < 1$ . Since  $\mathbf{p} = \mathbf{q}/l - \varepsilon$  for some  $\mathbf{q} \in \{0, \dots, l\}^n$  and  $\varepsilon \in \{0, \frac{1}{lm}, \dots, \frac{m-1}{lm}\}^n$ , then by Corollary 3.3.16,  $f_\varphi(\mathbf{q}/l) = 0$  and  $f_\psi(\mathbf{q}/l) > 0$ . In conclusion,  $\Gamma \not\models_{\Pi_\infty} \Delta$  as desired. ■

### 3.5 Logics combining Product, Gödel and Łukasiewicz connectives

The methods of the previous sections can be generalized to find upper bounds for denominators of critical points in logics obtained by combining Product, Gödel and Łukasiewicz connectives, as sketched below. In the

following we shall denote by  $n$  the number of variables of a generic formula  $\varphi$ .

- We let  $\Pi + G = ([0, 1], \{1\}, \{\cdot, \neg_G, \rightarrow_\Pi, \rightarrow_G\})$  denote the logic obtained by combining the connectives of Product and of Gödel Logic. Let us consider the logic  $\Sigma + G = ([0, \infty], \{0\}, \{+, \neg_\Sigma, \rightarrow_\Sigma, \rightarrow_{\Sigma G}\})$ , where  $\rightarrow_{\Sigma G}$  is defined by

$$x \rightarrow_{\Sigma G} y = \begin{cases} 0 & \text{if } x \geq y \\ y & \text{otherwise.} \end{cases}$$

$\Sigma + G$  and  $\Pi + G$  are clearly isomorphic. Moreover  $\Sigma + G$  satisfies all the piecewise linear requirements needed to apply our method as introduced in Section 4.

The construction of the polyhedral complex  $C_{\Sigma G}(\varphi)$  associated in  $\Sigma + G$  with a formula  $\varphi$  as defined in Section 4, proceeds as for the construction of the polyhedral complex  $C_\Sigma(\varphi)$  associated with  $\varphi$  in  $\Sigma$ , with the only additional stipulation that  $C_{\Sigma G}(\psi_1 \rightarrow_{\Sigma G} \psi_2) = C_{\Sigma G}(\psi_1 \rightarrow_\Sigma \psi_2)$ . Since  $f_{\rightarrow_{\Sigma G}}$  has the same boundary polynomial as  $f_{\rightarrow_\Sigma}$ , Propositions and Lemmas of Sections 6.1 and 6.2 can be repeated to describe the function associated with  $\varphi$ , with the only exception of Lemma 2.3.7. Indeed  $f_\varphi$  is in general not continuous, but is linear and homogeneous over every open face of any cell of  $C_{\Sigma G}(\varphi)$ . Repeating the argument done for Product Logic, we get that this function must be tested also in the relative interior of each cell. For any formula  $\varphi$  with  $n$  variables, a denominator upper bound is hence given by  $n \cdot \left(\frac{\#(\varphi)-1}{n-1}\right)^{n-1}$ . This is so because a point in the relative interior of any given cell can be chosen as the Farey mediant of vertices of simplexes of  $C_{\Sigma G}(\varphi) \cap [0, 1]^n$ . Then one can apply the same trick used to reduce Product Logic to a finite-valued logic.

- We let  $\mathbb{L} + G = ([0, 1], \{1\}, \{\oplus, \rightarrow_{\mathbb{L}}, \neg_{\mathbb{L}}, \neg_G, \rightarrow_G\})$  denote the logic obtained by combining the connectives of Łukasiewicz and Gödel Logic. Analogously to the logic  $\Sigma + G$ , since  $f_{\rightarrow_G}$  has the same boundary polynomial as  $f_{\rightarrow_{\mathbb{L}}}$ , the construction of the polyhedral complex  $C_{\mathbb{L}G}(\varphi)$  associated in  $\mathbb{L} + G$  with a formula  $\varphi$ , proceeds as for the construction of the polyhedral complex  $C_{\mathbb{L}}(\varphi)$  associated with  $\varphi$  in  $\mathbb{L}$ , with the only additional stipulation that  $C_{\mathbb{L}G}(\psi_1 \rightarrow_G \psi_2) = C_{\mathbb{L}G}(\psi_1 \rightarrow_{\mathbb{L}} \psi_2)$ .

The functions associated with formulas are (possibly discontinuous) piecewise linear with integer coefficients. These functions must be



tested also in the relative interior of each cell. A denominator upper bound is  $(n + 1) \cdot \left(\frac{\#(\varphi)}{n}\right)^n$ .

- We let  $G + \Delta = ([0, 1], \{1\}, \{\wedge, \neg_G, \rightarrow_G, \Delta\})$  denote the logic obtained by adding to connectives of Gödel Logic the connective  $\Delta$  interpreted by the function  $\Delta(x) = 1$  if only if  $x = 1$  and  $\Delta(x) = 0$  otherwise (axiomatized in [14]). Even if the class of functions associated with  $G + \Delta$  strictly contains the class of functions associated with  $G$  (see [52]), Corollary 3.2.1 and Theorem 3.2.2 can be repeated exactly as for Gödel logic. Hence  $n + 1$  is a denominator upper bound.
- When Łukasiewicz connectives are combined with Product connectives, piecewise linearity is lost and our present approach is inadequate.

## Chapter 4

# Calculi

In this section we will extend to Gödel and Product Logics methods and results of [8] and [9]. We refer to [112] for further background on proof theory.

In the literature one can find deductive systems for a multitude of finite-valued logics ([109, 25, 58]). In particular in [64] a signed tableaux system is presented, where signs are sets of values. The calculi for finite-valued logics presented in this section, are in a sense a translation of signed tableau: we shall present them in a form that makes it easy to extend them to calculi for infinite-valued logics. Indeed, also if some calculi have been defined for infinite-valued Łukasiewicz logic, they present some drawback, as giving no hints for proof search ([108, 100]) and involving geometrical and algebraic computations that shed no light on the real logical structure of formulas [65]. For an overview see for example [30].

In [8] a calculus for infinite-valued Łukasiewicz logic is described, based on the fact that the decidability of a Łukasiewicz infinite-valued formula  $\varphi$  can be reduced to deciding  $\varphi$  in a suitable finite-valued Łukasiewicz logic.

Our calculi depend on the rules reducing the infinite-valued tautology problem to its finite-valued counterpart. As an example, we shall provide sequent calculi building on the proof-theoretic machinery of [9]. As in [8], our calculi in this chapter will stem from very simple operations, such as doubling the places in a sequent, shifting formulas from one place to another, and similar “typographical” operations. We shall obtain *elementary* calculi whose rules are easy to apply. Since the operations needed to build one premise of a rule of our calculi only take constant time, as an immediate consequence we shall give new proofs that the tautology problem for Gödel and Product logics is in co-NP. In the case of Product logic such proof is

alternative to the one in [15].

For the sake of simplicity in our exposition, sometimes the definition of our rules will involve arithmetical operations. However, in all cases it will always be possible to rewrite such rules without resorting to any arithmetical operations.

## 4.1 Preliminaries

If  $\mathcal{L}_n \in \{\mathcal{L}_n, \mathcal{G}_n, \mathfrak{S}_m^l \text{ (with } lm = n-1)\}$  is an  $(n+1)$ -valued logic, we denote by  $\mathcal{LC}_n$  a sequent calculus in which every sequent consists of  $n$  parts. More precisely, *sequents*  $\Upsilon$  of  $\mathcal{LC}_n$  have the form

$$\Gamma_1 \vdash \Delta_1 \mid \dots \mid \Gamma_n \vdash \Delta_n,$$

where, for every  $i = 1, \dots, n$ ,  $\Gamma_i$  and  $\Delta_i$  are finite sets of formulas of  $\mathcal{L}_n$ .  $\Gamma_i \vdash \Delta_i$  is called the *i*th *component* of  $\Upsilon$ .  $\Gamma_i$  is the *premise of the component*  $\Gamma_i \vdash \Delta_i$ ,  $\Delta_i$  is the *conclusion (of the component*  $\Gamma_i \vdash \Delta_i$ ).

We shall adopt the following notation.  $[\Gamma \vdash \Delta]_i^n$  denotes the  $n$ -component sequent whose *i*th component is  $\Gamma \vdash \Delta$ , and the remaining ones are empty. Whenever the number of components of a sequent is clear from the context, we write  $[\Gamma \vdash \Delta]_i$ . For instance,  $[\vdash \varphi]_1^1$  denotes the sequent  $\vdash \varphi$ .

Sequents are to be interpreted in the following way:

**Definition 4.1.1** A sequent  $\Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid \dots \mid \Gamma_n \vdash \Delta_n$  is *valid* in  $\mathcal{L}_n$  if for all interpretations  $v$  for  $\mathcal{L}_n$ , there exists  $h \in \{1, \dots, n\}$  such that

$$v\left(\bigwedge_{\gamma \in \Gamma_h} \gamma\right) \leq \frac{n-h}{n} \quad \text{or} \quad v\left(\bigvee_{\delta \in \Delta_h} \delta\right) \geq \frac{n-h+1}{n}. \quad (4.1)$$

Clearly,  $\Gamma \models_{\mathcal{L}_n} \Delta$  if and only if  $[\Gamma \vdash \Delta]_1^n$  is valid in  $\mathcal{L}_n$ . Intuitively, the position of a formula in the *i*-th sequent is a statement about its possible truth values, in accordance with (4.1). Namely, we represent the boolean statement "the value of  $\varphi$  is  $\geq \frac{n-i+1}{n}$ " by putting  $\varphi$  in the conclusion of the *i*-th component of the  $n$ -part sequent. Analogously, the boolean statement "the value of  $\varphi$  is  $\leq \frac{n-i}{n}$ " is represented by an occurrence of  $\varphi$  in the premise of the *i*-th component of the  $n$ -part sequent.

**Notation:** As usual in sequent calculi notation, we shall always write  $\Gamma, \varphi$  for  $\Gamma \cup \{\varphi\}$ .

Further, given  $m$ -component sequents  $\Upsilon_1, \Upsilon_2, \dots, \Upsilon_h$  we let  $[\Upsilon_1 \Upsilon_2 \dots \Upsilon_h]$  denote the  $m$ -component sequent  $\Upsilon$  obtained by the componentwise

*merging* of  $\Upsilon_1, \dots, \Upsilon_h$ : that is, for any  $i \in \{1, \dots, m\}$ , the  $i$ th component of  $\Upsilon$  will contain as premise the union of premises of  $i$ th components of all  $\Upsilon_j$  and as conclusion the union of conclusions of  $i$ th components of all  $\Upsilon_j$ , for each  $j \in \{1, \dots, h\}$ .

During the construction of a proof, the set of possible truth values for  $\varphi$  (in our notation, the position of  $\varphi$ ) is modified in accordance with a set of rules that we will introduce later on.

Any (instance of a) rule is determined by a set of sequents called *premises of the rule* and a sequent called *conclusion of the rule*. For every connective there are  $n$  rules, each rule dealing with a fixed component of the sequent. As in standard sequent calculi, each rule comes in a left and in a right version, according as the rule modifies the left, or the right hand side of the sequent. Rules can be logical or structural. Structural rules are divided into *weakening rules* and *migration rules*. A rule is *valid* when the validity of all the premises implies the validity of the conclusion. A valid rule is *bivalent* when the validity of the conclusion implies the validity of all the premises.

Take a generic sequent  $\Upsilon = \Gamma_1 \vdash \Delta_1 \mid \dots \mid \Gamma_n \vdash \Delta_n$ . Suppose we are applying a rule of the calculus to an occurrence of the formula  $\varphi$  in the premise  $\Gamma_i$  (resp., conclusion  $\Delta_i$ ) of the  $i$ -th sequent  $\Gamma_i \vdash \Delta_i$ . Then this occurrence of  $\varphi$  is called the *active (occurrence of) formula*, and we call *context sequent* the sequent  $\Upsilon'$  obtained from  $\Upsilon$  by replacing the  $i$ -th component with  $\Gamma_i \setminus \{\varphi\} \vdash \Delta_i$  (resp.,  $\Gamma_i \vdash \Delta_i \setminus \{\varphi\}$ ). By using componentwise merging notation introduced above, we can write down  $\Upsilon$  as

$$\Upsilon = [\Upsilon'[\varphi \vdash]_i] \quad (\text{resp., } \Upsilon = [\Upsilon'[\vdash \varphi]_i]).$$

In this way we can easily formulate the context part in any sequent under consideration. Suppose that for instance  $\Theta$  contains the active formula. Then a generic sequent containing  $\Theta$  can be written down as  $[\Theta\Upsilon]$ , where  $\Upsilon$  is a generic context sequent.

For the sake of readability, we shall define rules for  $\mathcal{LC}_n$  calculi in a uniform way for every finite-valued logic. For each integer  $m > 0$  the sequent calculus  $\mathcal{LC}_m$  is given by:

**Axioms.** For every  $0 < j \leq m$ ,

$$[\varphi \vdash \varphi]_j.$$

**Structural Rules.** For each  $i \in \{1, \dots, m\}$ ,  $j \in \{1, \dots, m-1\}$  and for every context sequent  $\Upsilon$ ,

$$\begin{aligned} (w, l)_i &= \frac{\Upsilon}{[\Upsilon [\varphi \vdash]_i]} & (w, r)_i &= \frac{\Upsilon}{[\Upsilon [\vdash \varphi]_i]} \\ (m, l)_j &= \frac{[\Upsilon [\varphi \vdash]_{j+1}]}{[\Upsilon [\varphi \vdash]_j]} & (m, r)_j &= \frac{[\Upsilon [\vdash \varphi]_j]}{[\Upsilon [\vdash \varphi]_{j+1}]} \end{aligned}$$

**Cut Rules.** For each  $i, j \in \{1, \dots, m\}$  such that  $i \leq j$  and for every context sequent  $\Upsilon$ ,

$$(cut)_{ij} = \frac{[\Upsilon [\vdash \varphi]_i] \quad [\Upsilon [\varphi \vdash]_j]}{\Upsilon}.$$

We are now in a position to define logical rules for several finite-valued logics.

As usual, a sequent  $\Gamma$  is said to be *provable* in the  $\mathcal{LC}_m$  calculus, if there is a tree of sequents, rooted in  $\Gamma$ , such that every leaf is an axiom and every inner node is obtained from its parent nodes by an application of a rule. Such proof tree is said to be *closed*.

**Lemma 4.1.2** *If  $\mathcal{LC}_n$  is a sequent calculus for an  $(n+1)$ -valued logic  $\mathcal{L}_n$  such that every logical rule is bivalid, then every sequent valid in  $\mathcal{L}_n$  is provable in  $\mathcal{LC}_n$  without using the cut rule.*

**Proof.** Let  $\Upsilon$  be a sequent valid in  $\mathcal{L}_n$ . Starting from  $\Upsilon$  we construct a tree by applying logical rules of  $\mathcal{LC}_n$  in the inverse direction. Since each premise of a logical rule contains fewer connectives than the conclusion, the process must terminate. After a finite number of steps we obtain a tree whose leaves only contain sequents where every formula is a variable. Since, by hypothesis, all the logical rules are bivalid, all these leaves must be valid in  $\mathcal{L}_n$ . It is easy to see that a sequent whose components contain only variables is valid in  $\mathcal{L}_n$  if and only if it has the form  $[\Upsilon' [X \vdash]_j [\vdash X]_k]$  for some  $j, k$  such that  $0 < j \leq k < n$ . Thus, using the weakening and the migration rules we get a proof of  $\Upsilon$  in  $\mathcal{LC}_n$ . ■

For all logics  $G_n$ ,  $L_n$  and  $\mathfrak{S}_m^l$  (with  $n = lm$ ) we have the following rules for  $\vee$  and  $\wedge$ , where the sequents are supposed to have  $n$  components and  $\Upsilon$  is a context sequent:

$$(\vee, l)_i = \frac{[\Upsilon [\varphi \vdash]_i] \quad [\Upsilon [\psi \vdash]_i]}{[\Upsilon [\varphi \vee \psi \vdash]_i]} \quad (\vee, r)_i = \frac{[\Upsilon [\vdash \varphi, \psi]_i]}{[\Upsilon [\vdash \varphi \vee \psi]_i]}$$

$$(\wedge, l)_i = \frac{[\Upsilon [\varphi, \psi \vdash]_i]}{[\Upsilon [\varphi \wedge \psi \vdash]_i]} \quad (\wedge, r)_i = \frac{[\Upsilon [\vdash \varphi]_i] \quad [\Upsilon [\vdash \psi]_i]}{[\Upsilon [\vdash \varphi \wedge \psi]_i]}.$$

## 4.2 Logical rules for Gödel Logics $G_n$

The logical rules of  $GC_n$  for the connective  $\wedge$  are as above. The others are as follows, where all sequents must be considered as having  $n$  components and  $\Upsilon$  is, as usual, a context sequent.

$$\begin{aligned} (\neg_G, r)_i &= \frac{[\Upsilon [\varphi \vdash]_n]}{[\Upsilon [\vdash \neg_G \varphi]_i]} & (\neg_G, l)_i &= \frac{[\Upsilon [\vdash \varphi]_n]}{[\Upsilon [\neg_G \varphi \vdash]_i]} \\ (\rightarrow_G, r)_i &= \frac{\{[\Upsilon [\varphi \vdash \psi]_j] : j = i, \dots, n\}}{[\Upsilon [\vdash \varphi \rightarrow_G \psi]_i]} \\ (\rightarrow_G, l)_i &= \frac{[\Upsilon [\vdash \varphi]_n] \quad [\Upsilon [\psi \vdash]_i] \quad \{[\Upsilon [\vdash \varphi]_j [\psi \vdash]_{j+1}] : j = i, \dots, n-1\}}{[\Upsilon [\varphi \rightarrow_G \psi \vdash]_i]}. \end{aligned}$$

**Remark.** Consider the rule

$$(\rightarrow_G, r)'_1 = \frac{[\Upsilon [\varphi \vdash \psi]_1]}{[\Upsilon [\vdash \varphi \rightarrow \psi]_1]}.$$

From the Deduction theorem it follows that, when  $\Upsilon = \emptyset$  (i.e., when the context is empty), this rule holds. Indeed we are going to show that if a closed proof tree for  $[\varphi \vdash \psi]_1$  exists, then there exist closed proof trees of  $[\varphi \vdash \psi]_i$ , for every  $i = 1, \dots, n$ .

As a matter of fact, let us suppose that there exists a proof tree  $P$  for  $[\varphi \vdash \psi]_1$  such that every leaf is an axiom and every internal node is obtained from its parent nodes by an application of a rule. For every  $[\varphi \vdash \psi]_i$  we shall construct a proof tree  $P_i$  in the following way:

Let  $\rho_1$  be the first rule applied to  $[\varphi \vdash \psi]_1$  in  $P$ . We apply  $\rho_i$  to each of  $[\varphi \vdash \psi]_i$  in such a way that

- if either  $\rho = (\wedge, r)$  or  $\rho = (\wedge, l)$  then we add the corresponding sequents in position  $i$  in the proof tree  $P_i$ ;
- if either  $\rho = (\neg, r)$  or  $\rho = (\neg, l)$  then we add the corresponding sequents in position  $n$  in the proof tree  $P_i$ ;
- if either  $\rho = (\rightarrow, r)$  or  $\rho = (\rightarrow, l)$  then we add the corresponding sequents in position  $i, i+1, \dots, n$  in the proof tree  $P_i$ .

This completes the description of the first step in our proof. In the second step we apply the second rule of  $P$  with obvious modifications, and so on. In each of the proof trees  $P_i$  thus obtained, none of the sequents is in position  $j < i$ . The leaves of  $P_i$  are axioms. Indeed, every  $P_i$  is a copy of  $P$ , with the possible exception given by branches occurring in  $P$  and missing in  $P_i$ , because the rules for implication applied in  $P_i$  generally introduce less sequents than the corresponding rules when applied in  $P$ .

So, when  $\Upsilon = \emptyset$  the rule  $(\rightarrow, r)'_1$  can be applied instead of  $(\rightarrow, r)_1$ . On the other hand, if the context  $\Upsilon$  is non-empty, the rule  $(\rightarrow, r)'_1$  is no longer sound and hence the rule  $(\rightarrow, r)$  must be considered. For example, consider the formula  $(X \rightarrow Y) \vee (X \vee \neg Y)$ . This is not a tautology in 4-valued Gödel logic  $G_3$ . A proof tree for  $(X \rightarrow Y) \vee (X \vee \neg Y)$  is

$$\frac{\frac{\frac{X \vdash X \mid \vdash \mid \vdash}{X \vdash Y, X \mid \vdash \mid Y \vdash}}{X \vdash Y, X, \neg Y \mid \vdash \mid \vdash} \quad \frac{\vdash X \mid X \vdash Y \mid Y \vdash}{\vdash X, \neg Y \mid X \vdash Y \mid \vdash} \quad \frac{\vdash \mid \vdash \mid Y \vdash Y}{\vdash X \mid \vdash \mid X, Y \vdash Y}}{\vdash X, \neg Y \mid \vdash \mid X \vdash Y} (\rightarrow, r)_1}{\vdash (X \rightarrow Y), X, \neg Y \mid \vdash \mid \vdash} (\rightarrow, r)_1}{\vdash (X \rightarrow Y) \vee (X \vee \neg Y) \mid \vdash \mid \vdash}$$

and this is not closed. On the other hand, if we used the rule  $(\rightarrow, r)'_1$  we would get

$$\frac{\frac{\frac{X \vdash X \mid \vdash \mid \vdash}{X \vdash Y, X \mid \vdash \mid Y \vdash}}{X \vdash Y, X, \neg Y \mid \vdash \mid \vdash}}{\vdash (X \rightarrow Y), X, \neg Y \mid \vdash \mid \vdash} (\rightarrow, r)'_1}{\vdash (X \rightarrow Y) \vee (X \vee \neg Y) \mid \vdash \mid \vdash}$$

and this is closed.

**Lemma 4.2.1** *For each integer  $n > 0$ , all logical rules of  $GC_n$  are bivald.*

**Proof.** By direct inspection, as an immediate consequence of the definition of validity for the sequents under consideration. ■

The following theorem is a consequence of Lemma 4.2.1.

**Theorem 4.2.2** *A sequent is valid in  $G_n$  if and only if it is provable in  $GC_n$ .* ■

### 4.3 Logical rules for $\mathfrak{S}_m^l$ Logics

In order to write rules for  $\mathfrak{S}_m^l$ , where  $ml = n$ , first of all let us label by the symbol  $\perp$  the  $(n + 1)$ th position in any sequent. Thus the sequent  $\Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid \dots \mid \Gamma_n \vdash \Delta_n \mid \Gamma_{\perp} \vdash \Delta_{\perp}$  is *valid* in  $\mathfrak{S}_m^l$  if for all interpretations  $v$ , either there exists  $h \in \{1, \dots, n\}$  such that (4.1) holds, or else

$$v \left( \bigwedge_{\gamma \in \Gamma_{\perp}} \gamma \right) = \perp \quad \text{or} \quad v \left( \bigvee_{\delta \in \Delta_{\perp}} \delta \right) \geq 0.$$

The logical rules of  $\Pi C_n$  are as follows, where all sequents are considered as having  $n$  components and  $\Upsilon$  is intended as the context sequent:

$$\begin{aligned} (\odot, r)_{\perp} &= \frac{[\Upsilon \vdash \varphi]_{\perp} \quad [\Upsilon \vdash \psi]_{\perp}}{[\Upsilon \vdash \varphi \odot \psi]_{\perp}} & (\odot, l)_{\perp} &= \frac{[\Upsilon [\varphi, \psi \vdash]_{\perp}]}{[\Upsilon [\varphi \odot \psi \vdash]_{\perp}]} \\ (\odot, r)_i &= \frac{[\Upsilon \vdash \varphi]_i \quad [\Upsilon \vdash \psi]_i \quad \{[\Upsilon \vdash \varphi]_j [\vdash \psi]_k : j + k = i\}}{[\Upsilon \vdash \varphi \odot \psi]_i} \\ (\odot, l)_i &= \frac{\{[\Upsilon [\varphi \vdash]_j [\psi \vdash]_k : j + k = i + 1\}}{[\Upsilon [\varphi \odot \psi \vdash]_i]} \\ (\rightarrow_{\Sigma}, r)_{\perp} &= \frac{[\Upsilon [\varphi \vdash \psi]_{\perp}]}{[\Upsilon \vdash \varphi \rightarrow_{\Sigma} \psi]_{\perp}} & (\rightarrow_{\Sigma}, l)_{\perp} &= \frac{[\Upsilon \vdash \varphi]_{\perp} \quad [\Upsilon [\psi \vdash]_{\perp}]}{[\Upsilon [\varphi \rightarrow_{\Sigma} \psi \vdash]_{\perp}]} \\ (\rightarrow_{\Sigma}, r)_i &= \frac{\{[\Upsilon [\varphi \vdash]_j [\vdash \psi]_k : k - j = i - 1\} \quad [\Upsilon [\varphi \vdash \psi]_{\perp}]}{[\Upsilon \vdash \varphi \rightarrow_{\Sigma} \psi]_i} \\ (\rightarrow_{\Sigma}, l)_i &= \frac{[\Upsilon \vdash \varphi]_{\perp} \quad [\Upsilon [\psi \vdash]_i] \quad \{[\Upsilon \vdash \varphi]_j [\psi \vdash]_k : k - j = i\} \quad [\Upsilon \vdash \varphi]_{n-i+1} [\psi \vdash]_{\perp}}{[\Upsilon [\varphi \rightarrow_{\Sigma} \psi \vdash]_i]} \\ (\neg_{\Sigma}, r)_i &= \frac{[\Upsilon [\varphi \vdash]_{\perp}]}{[\Upsilon \vdash \neg_{\Sigma} \varphi]_i} & (\neg_{\Sigma}, l)_i &= \frac{[\Upsilon \vdash \varphi]_{\perp}}{[\Upsilon [\neg_{\Sigma} \varphi \vdash]_i]} & \text{also for } i = \perp. \\ (\natural, r)_{\perp} &= \frac{[\Upsilon \vdash \varphi]_{\perp}}{[\Upsilon \vdash \natural \varphi]_{\perp}} & (\natural, l)_{\perp} &= \frac{[\Upsilon [\varphi \vdash]_{\perp}]}{[\Upsilon [\natural \varphi \vdash]_{\perp}]} \\ (\natural, r)_{i \leq l} &= \frac{[\Upsilon \vdash \varphi]_{mi}}{[\Upsilon \vdash \natural \varphi]_i} & (\natural, l)_{i \leq l} &= \frac{[\Upsilon [\varphi \vdash]_{mi}}{[\Upsilon [\natural \varphi \vdash]_i]} \\ (\natural, r)_{i \geq l+1} &= \frac{[\Upsilon \vdash \varphi]_{\perp}}{[\Upsilon \vdash \natural \varphi]_i} & (\natural, l)_{i \geq l+1} &= \frac{[\Upsilon [\varphi \vdash]_{\perp}]}{[\Upsilon [\natural \varphi \vdash]_i]} \end{aligned}$$

**Lemma 4.3.1** *For each integer  $n > 0$ , all logical rules of  $\Pi C_n$  are bivald.*



**Proof.** By direct inspection, as an immediate consequence of the definition of validity for the sequents under consideration. ■

**Theorem 4.3.2** *A sequent is valid in  $\mathfrak{S}_m^l$  if and only if it is provable in  $\Pi C_{lm}$ .* ■

## 4.4 Calculi for infinite-valued logics

Using the results of Sections 3.2 and 3.3 one can transform every formula  $\varphi$  of the infinite-valued calculus of  $\mathcal{L}_\infty \in \{G_\infty, \Pi_\infty\}$ , into a formula  $\widehat{\varphi}$  of a suitable finite-valued logic  $\mathcal{L}'_n$ , such that  $\models_{\mathcal{L}_\infty} \varphi$  if and only if  $\models_{\mathcal{L}'_n} \widehat{\varphi}$ , and then check the tautologousness of  $\varphi$  working on  $\widehat{\varphi}$  in  $\mathcal{L}'_n$ .

We must still provide rules enabling us to automatically associate to the formula  $\varphi$  the integer  $n$  such that  $\models_{\mathcal{L}_\infty} \varphi$  if and only if  $\models_{\mathcal{L}'_n} \widehat{\varphi}$ .

To this purpose we consider *labelled sequents* of the form

$$(\Sigma) : \Upsilon^m$$

where  $\Upsilon$  is a sequent of the  $\mathcal{L}C_m$  calculus, for some integer  $m > 0$ , and  $\Sigma$  is a (possibly empty) finite set or multiset of formulas of  $\mathcal{L}_\infty$  called *label*. The empty label is denoted by  $\epsilon$ .

We shall first define a “partial” calculus  $\mathcal{L}T_\infty$ , that will only find use in proving that a formula is a tautology. We shall then give the rules for a sequent calculus  $\mathcal{L}C_\infty$ , taking into account the fact that infinite-valued logical consequence cannot generally be reduced to a single finite-valued logic (Theorem 3.4.3). Rules are divided into *sequent* and *label* rules. Axioms and sequent rules are the same as for  $\mathcal{L}C_m$  calculi, for every  $m$ , with the proviso that the label  $\epsilon$  is added to every sequent. As an example, for every integer  $m > 0$  and  $0 < j \leq m$ , our axioms are given by

$$(\epsilon) : [A \vdash A]_j^m.$$

Label rules of  $\mathcal{L}T_\infty$  are used to find, for every formula  $\varphi$ , the integer  $n$  allowing us to apply to  $\varphi$  a calculus for  $\mathcal{L}C_n$ . Accordingly, using the abbreviations  $b_G(\varphi) = |\text{var}(\varphi)| + 1$  and by  $b_\Pi(\varphi) = 2^{\#(\varphi)-1} \#(\varphi)$ , the label rules of  $GT_\infty$  must transform the initial sequent  $\vdash \varphi$  into a sequent having  $b_G(\varphi)$  components, and the label rules of  $\Pi T_\infty$  must transform the initial sequent  $\vdash \varphi$  into a sequent having  $b_\Pi(\varphi)$  components. On the other hand, the label rules of  $GC_\infty$  (respectively, the label rules of  $\Pi C_\infty$ ) must reduce the calculus for  $\Gamma \models \Delta$  to its counterpart for all  $m$ -valued logics, where  $m \leq b_G(\wedge \Gamma \vee \vee \Delta)$  (respectively,  $m \leq b_\Pi(\wedge \Gamma \vee \vee \Delta)$ ).

### 4.4.1 Gödel label rules

Labels are pairs whose first component is a multiset (which is used to disassemble the formula) and whose second component is a set (used to count variables). More precisely, for any set  $\Omega$  of formulas, multiset  $\Theta$  of formulas and connective  $\star \in \{\wedge, \rightarrow_G\}$ , the label rules of  $GT_\infty$  are given by

$$\frac{((\Theta, \varphi, \psi), \Omega) : [\Gamma \vdash \Delta]_1^m}{((\Theta, \varphi \star \psi), \Omega) : [\Gamma \vdash \Delta]_1^m} \quad \frac{((\Theta, \psi), \Omega) : [\Gamma \vdash \Delta]_1^m}{((\Theta, \neg_G \psi), \Omega) : [\Gamma \vdash \Delta]_1^m}$$

$$\frac{((\Theta), \Omega \cup \{X\}) : [\Gamma \vdash \Delta]_1^m}{((\Theta, X), \Omega) : [\Gamma \vdash \Delta]_1^m} \quad \frac{((\epsilon), \Omega \setminus \{X\}) : [\Gamma \vdash \Delta]_1^{m+1}}{((\epsilon), \Omega \cup \{X\}) : [\Gamma \vdash \Delta]_1^m}$$

where  $X$  is a variable. Label rules are devised to keep track of the number of different variables occurring in the sequent to be proved. It is clear that, if  $\varphi$  is a formula of Gödel logic, starting from  $((\varphi), \emptyset) \vdash \varphi$  and applying Gödel label rules in the inverse direction, we obtain the sequent  $(\epsilon) : [\vdash \varphi]_1^{n+1}$  where  $n = |\text{var}(\varphi)|$ . In particular, the multiset  $\Theta$  will contain the subformulas of  $\varphi$ , while the set  $\Omega$  will eventually coincide with the set  $\text{var}(\varphi)$ . For each element of  $\Omega$  we increase by one the number of components of sequents in the proof. We can then apply sequent rules  $GC_{n+1}$  for the finite  $(n+2)$ -valued Gödel logic.

**Definition 4.4.1** A formula  $\varphi$  is *provable* in  $GT_\infty$  if there is a tree of labelled sequents, rooted in  $((\varphi), \emptyset) \vdash \varphi$ , such that every leaf is an axiom and every internal node is obtained from its parent nodes by an application of a rule.

**Theorem 4.4.2** *A formula is a tautology in  $G_\infty$  if and only if it is provable in  $GT_\infty$ .*

**Proof.** If  $\varphi$  is a tautology of  $G_\infty$  then  $\varphi$  is a tautology of  $G_{b_G}$  where  $b_G = |\text{var}(\varphi)| + 1$ . By Theorem 4.2.2, the sequent  $[\vdash \varphi]_1^{b_G}$  is provable in  $GC_{b_G}$ . It follows that each leaf of the tree rooted in  $((\varphi), \emptyset) \vdash \varphi$  such that every internal node is obtained from its parent nodes by an application of a rule of  $GT_\infty$ , must be an axiom.

Vice-versa, if  $\varphi$  is provable in  $GT_\infty$ , then all the sequents in the proof tree with the empty label are a proof of  $\varphi$  in  $GC_{b_G}$ , and so, by Theorem 4.2.2,  $\varphi$  is a tautology in  $G_{b_G}$  whence it is a tautology in  $G_\infty$ . ■

Since the deduction theorem holds for Gödel logic, from the remark before Theorem 4.2.2, we see that the calculus  $GT_\infty$  can be also considered a calculus for  $G_\infty$

**Theorem 4.4.3** *Let  $\Gamma$  and  $\Delta$  be finite sets of formulas of  $G_\infty$ . Then  $\Gamma \models_{G_\infty} \Delta$  if and only if the sequent  $((\Gamma \cup \Delta), \emptyset) : \Gamma \vdash \Delta$  is provable in  $GT_\infty$ . ■*

#### 4.4.2 Product label rules

For each formula  $\varphi$  of  $\Pi_\infty$  let  $\widehat{\varphi}$  be the formula obtained by substituting every occurrence of  $\cdot$  with  $\odot$ , every occurrence of  $\rightarrow_\Pi$  with  $\rightarrow'_\Sigma$ , every occurrence of  $\neg_\Pi$  with  $\neg_\Sigma$  and every variable  $X_i$  with  $\natural X_i$ . Let us agree to say that  $\widehat{\varphi}$  is the *translation* of  $\varphi$ . Let  $\Gamma$  and  $\Delta$  be sets of translated formulas,  $\Theta$  be a multiset of translated formulas and  $\star \in \{\odot, \rightarrow'_\Sigma\}$ . Our labels shall consist of two distinct parts: the first one will be used to disassemble the formula  $\varphi$ , the second one will count the number of binary connectives. They are defined by:

$$\frac{((\Theta, \widehat{\psi}, \widehat{\vartheta}), 2j) : [\Gamma \vdash \Delta]_1^{i+1}}{((\Theta, \widehat{\psi} \star \widehat{\vartheta}), j) : [\Gamma \vdash \Delta]_1^i} \quad \frac{((\Theta, \widehat{\psi}), j) : [\Gamma \vdash \Delta]_1^i}{((\Theta, \neg_\Sigma \widehat{\psi}), j) : [\Gamma \vdash \Delta]_1^i}$$

$$\frac{((\Theta), j) : [\Gamma \vdash \Delta]_1^i}{((\Theta, \natural X), j) : [\Gamma \vdash \Delta]_1^i} \quad \frac{[\Gamma \vdash \Delta]_1^{ij}}{((\epsilon), j) : [\Gamma \vdash \Delta]_1^i}$$

For every formula  $\varphi$  of Product logic, starting from  $((\widehat{\varphi}), 1) : \vdash \widehat{\varphi}$  and applying Product label rules in the inverse direction, we obtain the sequent  $[\vdash \widehat{\varphi}]_1^{\#(\varphi)2^{\#(\varphi)-1}}$ . Then, we can apply rules of  $\Sigma_l^m$  to this sequent.

**Definition 4.4.4** A formula  $\varphi$  is *provable* in the  $\Pi T_\infty$  calculus if there is a tree of labelled sequents, rooted in  $((\widehat{\varphi}), 1) : \vdash \widehat{\varphi}$ , such that every leaf is an axiom and every internal node is obtained from its parent nodes by application either of a product label rule, or of a  $\Sigma_l^m$  rule.

**Theorem 4.4.5** *A formula is a tautology in  $\Pi_\infty$  if and only if it is provable in  $\Pi T_\infty$ .*

**Proof.** Similar to the Proof of Theorem 4.4.2. ■

By slightly modifying the  $\Pi T_\infty$  calculus in the light of Theorem 3.4.3 we obtain a calculus  $\Pi C_\infty$  for  $\Pi_\infty$  as follows.

Axioms and rules are those of  $\Pi T_\infty$ . Let  $\widehat{\Gamma}$  and  $\widehat{\Delta}$  be sets of translated formulas,  $\Theta$  be any multiset of such formulas and  $\star \in \{\odot, \rightarrow'_\Sigma\}$ . Then:

$$\frac{((\Theta, \widehat{\psi}), j) : [\widehat{\Gamma} \vdash \widehat{\Delta}]_1^m}{((\Theta, \neg_\Sigma \widehat{\psi}), j) : [\widehat{\Gamma} \vdash \widehat{\Delta}]_1^m} \quad \frac{((\Theta), j) : [\widehat{\Gamma} \vdash \widehat{\Delta}]_1^m}{((\Theta, \natural X), j) : [\widehat{\Gamma} \vdash \widehat{\Delta}]_1^m}$$

$$\frac{((\Theta, \widehat{\psi}, \widehat{\vartheta}), 2j-1) : [\widehat{\Gamma} \vdash \widehat{\Delta}]_1^{i+1} \quad ((\Theta, \widehat{\psi}, \widehat{\vartheta}), 2j) : [\widehat{\Gamma} \vdash \widehat{\Delta}]_1^{i+1}}{((\Theta, \widehat{\psi} \star \widehat{\vartheta}), j) : [\widehat{\Gamma} \vdash \widehat{\Delta}]_1^i}$$

$$\frac{[\widehat{\Gamma} \vdash \widehat{\Delta}]_1^{ij}}{(\epsilon, j) : [\widehat{\Gamma} \vdash \widehat{\Delta}]_1^i}$$

A sequent  $\Upsilon = (\Theta, k) : \widehat{\Gamma} \vdash \widehat{\Delta}$  is said to be *provable* in the  $\Pi C_\infty$  calculus if there is a tree of labelled sequents, rooted in  $\Upsilon$ , such that every leaf is an axiom and every internal node is obtained from its parent nodes by an application of a rule.

**Theorem 4.4.6** *Let  $\Gamma$  and  $\Delta$  be finite sets of formulas of  $\Pi_\infty$ . Then  $\Gamma \models_{\Pi_\infty} \Delta$  if and only if the sequent  $((\bigwedge \widehat{\Gamma}, \bigvee \widehat{\Delta}), 1) : \widehat{\Gamma} \vdash \widehat{\Delta}$  is provable in  $\Pi C_\infty$ .*

**Proof.** An easy induction on  $k$  shows that  $((\bigwedge \widehat{\Gamma}, \bigvee \widehat{\Delta}), 1) : \widehat{\Gamma} \vdash \widehat{\Delta}$  is provable in  $\Pi C_\infty$  if and only if all the sequents  $[\widehat{\Gamma} \vdash \widehat{\Delta}]_1^{lm}$  are provable in  $\Sigma_k^m$ , for  $m = \#(\Gamma) + \#(\Delta)$  and for every  $1 \leq l \leq 2^{\#(\Gamma) + \#(\Delta) - 1}$ . Now apply Theorems 4.3.2 and 3.4.3. ■

Similar calculi can be described for all logics considered in Section 3.5.

**Theorem 4.4.7** *The tautology problem for logics  $\Pi + G$ ,  $L + G$  and  $G + \Delta$  is in co-NP.*

**Proof.** Let us consider a branch of a proof tree for  $\varphi$ . It consists of polynomially many occurrences of subformulas of  $\varphi$ . Its length is linearly bounded by  $\#(\varphi)$ . Checking whether any two adjacent sequents  $\Upsilon_i, \Upsilon_{i+1}$  in the branch are such that  $\Upsilon_i$  is a premise of a rule yielding  $\Upsilon_{i+1}$  just takes constant time in the length of  $\Upsilon_i$  and  $\Upsilon_{i+1}$ . By keeping track only of the positions of nonempty components of sequents in the branch, we immediately get the desired conclusion. ■

## Chapter 5

# Rational Łukasiewicz Logic and DMV-algebras

By McNaughton theorem [82], the functions associated with formulas of Łukasiewicz logic are the totality of continuous, piecewise linear functions in which every piece has integer coefficients: it seems natural to weaken the restriction of integer coefficients, and consider instead rational coefficients.

To this purpose different approaches have been proposed in the literature. The authors of [13] introduced Łukasiewicz propositional logic with one quantified propositional variable  $\exists L$ . In [45] Riesz MV-algebras are defined as a special class of MV-algebras with a family of unary operators, and are shown to be the MV-algebraic counterpart of vector lattices over real numbers. In [71] root (in fact, division) operators are introduced and in [16] Łukasiewicz logic plus root operators is shown to correspond to continuous piecewise linear functions with rational coefficients and to have the interpolation property.

In [55] we collected all these results and we gave an equational definition of root operators, defining the variety of *DMV-algebras* (divisible MV-algebras). Such structures maintain some basic properties of MV-algebras, and are *intervals* of lattice-ordered vector spaces over the rationals just as MV-algebras are intervals of lattice-ordered abelian groups [32].

In this chapter we extend to DMV-algebras some results holding for MV-algebras, like the representation theorem and the correspondence with divisible  $\ell$ -groups. We further give a direct proof that the variety of DMV-algebras is generated by  $[0, 1]$ . Rational Łukasiewicz logic is then introduced and is shown to be an extension of Rational Pavelka logic: the tautology problem for Rational Łukasiewicz logic is shown to be co-NP-complete.

The last section is devoted to the introduction of rational numbers into MV-algebras, via the construction of weakly divisible hulls.

## 5.1 DMV-algebras

In Section 1.2.1 we cited some results on MV-algebras. It turns out that both the completeness theorem in [29] and the representation theorem in [41] are based on results for the theory of  $\ell$ -groups. In particular the authors use the result that every totally ordered group can be embedded in a totally ordered divisible group, and that quantifier elimination holds for totally ordered divisible groups. We shall study here algebraic structures that are more directly connected with divisible groups.

**Notation:** If  $A$  is an MV-algebra,  $x \in A$  and  $n \in \mathbb{N}$  we denote by  $n.x$  the element of  $A$  inductively defined by  $0.x = 0$ ,  $(n-1).x = n.x \oplus x$ . Further, we denote by  $nx$  the element of  $G_A$  defined by  $0x = 0$  and  $(n-1)x = nx + x$ . If  $u$  is a strong unit of  $G_A$  such that  $\Gamma(G, u) = A$ , it follows that  $n.x = nx \wedge u$ .

**Definition 5.1.1 (Mundici)** A DMV-algebra  $A = (A, \oplus, \neg, \{\delta_n\}_{n \in \mathbb{N}}, 0, 1)$  is an algebraic structure such that  $A^* = (A, \oplus, \neg, 0, 1)$  is an MV-algebra and the following hold for every  $x \in A$  and  $n \in \mathbb{N}$ :

$$(D1n) \quad n.\delta_n x = x$$

$$(D2n) \quad \delta_n x \odot (n-1).\delta_n x = 0$$

The MV-algebra  $A^*$  is the *MV-reduct* of the DMV-algebra  $A$ . If  $A$  is a DMV-algebra, then the  $\ell$ -group  $G$  with strong unit  $u$  such that  $A^* = \Gamma(G, u)$  satisfies the condition of divisibility, i.e., for every  $n \in \mathbb{N}$  and for every  $x \in G$  there exists  $y \in G$  such that  $ny = x$ .

**Example 5.1.2** For each  $k = 1, 2, \dots$ , the set

$$L_{k+1} = \left\{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\right\},$$

equipped with the operations

$$x \oplus y = \min\{1, x + y\}, \quad x \odot y = \max\{0, x + y - 1\}, \quad \neg x = 1 - x$$

is a linearly ordered MV-algebra (also called *MV-chain*), but cannot be enriched to a DMV-algebra. The set of all rationals between 0 and 1 where each  $\delta_n$  is interpreted as division by  $n$ , is a DMV-algebra that we shall denote by  $(\Gamma(\mathbb{Q}, 1), \delta_n)$ . In this case, Axioms (D1n) and (D2n) state that the sum of  $n$  copies of  $x/n$  coincides with  $x$ .

**Proposition 5.1.3** *Let  $A$  be a DMV-algebra, let  $A^*$  be its MV-reduct and let  $(G, u)$  be the unique  $\ell$ -group with strong unit  $u$  such that  $\Gamma(G, u) = A^*$ . Then, for every  $x \in A$ ,*

(i)  $m\delta_m x = x$

(ii)  $\delta_m x$  is the unique element of  $A$  satisfying axioms (D1) and (D2).

**Proof.**

(i) If  $x \in A$ , equations (D1) and (D2) become

$$m\delta_m x \wedge u = x \tag{5.1}$$

$$(\delta_m x + ((m-1)\delta_m x \wedge u) - u) \vee 0 = 0 \tag{5.2}$$

whence  $\delta_m x + ((m-1)\delta_m x \wedge u) \leq u$  and, by definition of  $\ell$ -group,  $m\delta_m x \wedge (u + \delta_m x) \leq u$ . Since  $\delta_n x \geq 0$ , then  $u + \delta_n x \geq u$  and

$$m\delta_m x \wedge u \leq m\delta_m x \wedge (u + \delta_m x) \leq u$$

and hence, from (5.1),  $m\delta_m x = x$ .

(ii) For every  $y \in A \subseteq G$  satisfying (D1) and (D2),  $my \wedge u = x$  and  $(y + ((m-1)y \wedge u) - u) \vee 0 = 0$ . Repeating the same argument as above,  $my = x = m\delta_m x$  and then  $y = \delta_m x$ . ■

Chang's distance function  $d : A \times A \rightarrow A$  is defined by

$$d(x, y) = (x \odot \neg y) \oplus (y \odot \neg x).$$

**Proposition 5.1.4** *Let  $A$  be a DMV-algebra and let  $x, y \in A$ .*

(i) *If  $x \odot y = 0$  then  $\delta_n(x \oplus y) = \delta_n x \oplus \delta_n y$*

(ii)  $\delta_n d(x, y) = d(\delta_n x, \delta_n y)$

**Proof.** We shall give the proof for the case  $n = 2$ . This can be generalized to every  $n > 0$ .

(i) If  $x, y \in A$ ,

$$(\delta_2x \oplus \delta_2y) \oplus (\delta_2x \oplus \delta_2y) = (\delta_2x \oplus \delta_2x) \oplus (\delta_2y \oplus \delta_2y) = x \oplus y.$$

Further note that in every MV-algebra, if  $a \odot b = 0$ ,  $a \odot a = 0$  and  $b \odot b = 0$  then  $(a \oplus b) \odot (a \oplus b) = (a \oplus a) \odot (b \oplus b)$ . Thus, since  $\delta_2x \odot \delta_2y \leq x \odot y = 0$ ,

$$(\delta_2x \oplus \delta_2y) \odot (\delta_2x \oplus \delta_2y) = x \odot y = 0.$$

Therefore,  $\delta_2(x \oplus y) = \delta_2x \oplus \delta_2y$

(ii) Let  $a, b$  elements of  $[0, 1]$  such that  $a \odot b = 0$ . In  $[0, 1]$  we have  $d(a, b) = |a - b|$  and hence

$$d(a, b) \odot d(a, b) = |a - b| \odot |a - b| = 2|a - b| - 1 \vee 0.$$

If  $a \leq b$  then  $2|a - b| - 1 \vee 0 = 2(a - b) - 1 \vee 0 = 2a - 2b - 1 \vee 0$  and since  $2a - 1 = 0$  then  $d(a, b) \odot d(a, b) = 0$ . Analogously the same conclusion can be drawn in case  $a \geq b$ .

By the Chang representation theorem it follows that if  $\delta_2x \odot \delta_2x = 0$ ,

$$d(\delta_2x, \delta_2y) \odot d(\delta_2x, \delta_2y) = 0.$$

Further,

$$d(\delta_2x, \delta_2y) \oplus d(\delta_2x, \delta_2y) = d(2\delta_2x, 2\delta_2y) = d(x, y)$$

whence the claim follows. ■

Note that by Condition (ii) of Proposition 5.1.4 and using notation and results of [84] and [93], we can say that  $\delta_n$  operators fit Łukasiewicz equivalence.

**Definition 5.1.5** *If  $A$  and  $B$  are DMV-algebras, a function  $f : A \rightarrow B$  is a homomorphism of DMV-algebras if  $f$  is a MV-homomorphism from  $A^*$  to  $B^*$  and for every  $x \in A$ ,*

$$f(\delta_n x) = \delta_n f(x)$$

**Definition 5.1.6** *A subset  $J$  of a DMV-algebra  $A$  is said to be a DMV-ideal of  $A$  if it is an ideal of the MV-reduct  $A^*$ , that is:*



- $0 \in J$
- For every  $x \in J$  and  $y \leq x$  then  $y \in J$
- If  $x, y \in J$  then  $x \oplus y \in J$

Note that if  $J$  is an ideal and  $x \in J$  then also  $\delta_n x \in J$  for every  $n$ . A DMV-ideal  $J$  is a *prime ideal* iff it is not trivial and for every  $x, y \in A$ , either  $x \odot \neg y \in J$  or  $y \odot \neg x \in J$ .

**Proposition 5.1.7** *Let  $I$  be an ideal of  $A$ . The binary relation  $\equiv_I$  on  $A$  defined by  $x \equiv_I y$  if and only if  $d(x, y) \in I$  is a congruence relation.*

**Proof.** Indeed  $\equiv_I$  is a congruence on the MV-reduct  $A^*$ . Further, if  $x, y \in A$  and  $x \equiv_I y$  then, by Proposition 5.1.4,  $d(\delta_n x, \delta_n y) = \delta_n d(x, y) \leq d(x, y)$ , hence  $d(\delta_n x, \delta_n y) \in I$  and  $\delta_n x = \delta_n y$ . ■

Let  $I$  be a DMV-ideal of  $A$  and let  $\pi : x \in A \mapsto [x]_I \in A/I$ . Then  $\text{Ker}(\pi) = \{x \in A \mid [x]_I = [0]_I\} = \{x \in A \mid d(x, 0) \in I\} = I$ . Vice-versa, if  $f$  is a DMV-homomorphism, then  $\text{Ker}(f) = \{x \in A \mid f(x) = 0\}$  is a DMV-ideal. We get

**Proposition 5.1.8**  *$I$  is a DMV-ideal of  $A$  if and only if there exists a DMV-homomorphism  $f$  such that  $I = \text{Ker}(f)$ .*

By Proposition 5.1.7, if  $I$  is a DMV-ideal of  $A$ , then setting

- $[x]_I \oplus [y]_I = [x \oplus y]_I$
- $\neg[x]_I = [\neg x]_I$
- $\delta_n [x]_I = [\delta_n x]_I$ ,

the structure  $(A/I, \oplus, \neg, \{\delta_n\}, [0]_I)$  is a DMV-algebra. Further, the quotient  $A/I$  is totally ordered iff  $I$  is a prime ideal. The proof of the following Proposition is the same as for MV-algebras:

**Proposition 5.1.9** *Let  $A$  be a DMV-algebra and  $I$  an ideal of  $A$ . If  $z \notin I$  then there exists a prime ideal  $P$  of  $A$  such that  $I \subseteq P$  and  $z \notin P$ .*

Then we can extend to DMV-algebras the Chang Representation theorem.

**Theorem 5.1.10** *Every DMV-algebra is the subdirect product of linear DMV-algebras.*

The functor  $\Gamma$  induces a correspondence between DMV-algebras and divisible  $\ell$ -groups:

**Definition 5.1.11 ([85])** A good sequence of a DMV-algebra  $A$  is a sequence  $(a_1, \dots, a_n)$  of elements of  $A$  such that for every  $i = 1, \dots, n-1$ ,  $a_i \oplus a_{i+1} = a_i$ .

If  $A$  is linear then every good sequence has the form  $(1^p, a) = (\underbrace{1, \dots, 1}_p, a)$

with  $a \in A$ . Further, if  $(a_1, \dots, a_n)$  is a good sequence of  $A$  then also  $(a_1, \dots, a_n, 0)$  is a good sequence.

**Proposition 5.1.12** Let  $A$  be a totally ordered DMV-algebra. Then there exists a totally ordered divisible group  $G$  together with a strong unit  $u$  such that  $A = \{x \in G \mid 0 \leq x \leq u\}$ .

**Proof.** If  $A$  is a totally ordered DMV-algebra, then the MV-reduct  $A^*$  is a MV-chain and  $A^*$  is isomorphic with  $\Gamma(G_{A^*}, (1, 0))$  (see [85]). Let  $u = (1, 0)$ . For every  $n \in \mathbb{N}$  and for every  $x \in [0, u]$  there exists  $y \in [0, u]$  such that  $ny = x$ . Since  $u$  is a strong unit and  $G$  is linear, for every  $x \in G$  there exists an integer number  $n_x$  such that  $n_x u \leq x < (n_x + 1)u$ . Let  $x' = x - n_x u \in [0, u]$ . Then let  $y'$  such that  $ny' = x'$  and let  $u'$  such that  $nu' = u$ . Then the element  $n_x u' + y'$  is such that  $n(n_x u' + y') = x$ , hence the totally ordered group  $G_{A^*}$  is divisible. ■

**Theorem 5.1.13** Let  $A$  be a DMV-algebra. Then there exists a unique divisible  $\ell$ -group  $G$  together with a strong unit  $u$  for  $G$  such that  $A = \{x \in G \mid 0 \leq x \leq u\}$ .

**Proof.** By Theorem 5.1.10,  $A$  is a subdirect product of totally ordered DMV-algebras  $(A_i)_{i \in I}$  and every  $A_i$  is equal to  $\Gamma(G_i, u_i)$  with  $G_i$  totally ordered divisible group (Proposition 5.1.12), hence

$$A \subseteq \prod_{j \in J} A_j \subseteq \prod_{j \in J} G_j. \quad (5.3)$$

Let  $u = (u_j)_{j \in J}$ . By [85], if  $G$  is the group generated by  $A$  in  $\prod_{j \in J} G_j$  and if

$$G^+ = \{a_1 + \dots + a_n \mid (a_1, \dots, a_n) \text{ good sequence of } A\}$$

then  $G = G^+ - G^+$ ,  $u$  is a strong unit of  $G$  and  $\Gamma(G, u) = A^*$ . There remains to show that  $G$  is divisible, i.e. if  $m \in \mathbb{N}$ , for every  $x \in G$  there exists  $y \in G$  such that  $my = x$ . It is enough to restrict to  $x \in G^+$ .

Let  $a_1, \dots, a_r$  be a good sequence of  $A$  ( $a_r \neq 0$ ) and  $x = a_1 + \dots + a_r \in G^+$ . Then  $\delta_n a_i \in A \subseteq G$  for every  $i = 1, \dots, r$ . Let

$$y = \delta_n a_1 + \dots + \delta_n a_r \in G.$$

By Proposition 5.1.3(i) we have  $ny = x$ . ■

Since for divisible, totally ordered abelian groups the quantifier elimination theorem holds, then a universal sentence  $\chi$  is satisfied by every divisible totally ordered abelian group if and only if it is satisfied by  $\mathbb{Q}$ .

**Definition 5.1.14** *A DMV-equation in the variables  $X_1, \dots, X_n$  is an expression  $\tau = \sigma$ , where  $\tau$  and  $\sigma$  are terms over the alphabet  $\{\oplus, \neg, 0, 1\} \cup \{\delta_n\}_{n \in \mathbb{N}}$  with variables among  $X_1, \dots, X_n$  (DMV-terms). A DMV-equation  $\tau = \sigma$  is satisfied by a DMV-algebra  $A$  if, for every  $n$ -tuple  $(a_1, \dots, a_n) \in A^n$ ,  $\tau(a_1, \dots, a_n) = \sigma(a_1, \dots, a_n)$ , where  $\tau(a_1, \dots, a_n)$  and  $\sigma(a_1, \dots, a_n)$  are elements of  $A$  obtained by substituting  $X_1, \dots, X_n$  by  $a_1, \dots, a_n$  in  $\tau$  and  $\sigma$ .*

Repeating the same argument as for MV-algebras ([29]), we have

**Theorem 5.1.15** *A DMV-equation is satisfied by every MV-algebra if and only if it is satisfied by the DMV-algebra  $(\Gamma(\mathbb{Q}, 1), \delta_n)$ .*

### 5.1.1 Varieties and quasi-varieties of DMV-algebras

Since DMV-algebras have an equational definition, the class of all DMV-algebras is a variety. By Theorem 5.1.15 we have

**Theorem 5.1.16** *The variety of DMV-algebras is generated by  $[0, 1] \cap \mathbb{Q}$ .*

It is possible to give an alternative proof of this theorem, by translating any equation of DMV-algebras in a quasi-equation of MV-algebras.

Indeed, suppose that  $\tau = 1$  is a DMV-equation and let  $\mathcal{T}$  be the parsing tree of  $\tau$ , that is,  $\mathcal{T}$  is a tree which nodes are subformulas of  $\tau$  and such that each node has as children its direct subformulas. Leaves of  $\mathcal{T}$  are all the occurrences of variables occurring in  $\tau$ .

Let us display the occurrences of variables  $x_1, \dots, x_n$  in any formula  $\varphi$  by writing  $\varphi(x_1, \dots, x_n)$ . Suppose that  $\delta_{i_1} \tau_{i_1}, \dots, \delta_{i_m} \tau_{i_m}$  is an enumeration of all nodes of  $\mathcal{T}$  that begin with a symbol  $\delta$ . Each of these nodes  $\delta_{i_j} \tau_{i_j}$  has a unique child  $\tau_{i_j}$ . Let us introduce  $m$  new variables in order to eliminate

the occurrences of  $\delta$ : if  $\tau = \tau(x_1, \dots, x_n)$ , let  $\tau^*(x_1, \dots, x_n, z_1, \dots, z_m)$  be the formula obtained by substituting every subformula  $\delta_{i_j} \tau_{i_j}$  by  $z_j$ .

If  $\tau_{i_j}$  is disjoint from any other subformula in  $\{\tau_{i_1}, \dots, \tau_{i_m}\} \setminus \{\tau_{i_j}\}$  then we denote by  $\sigma_1(z_j)$  the MV-equation  $\{i_j \cdot z_j = \tau_{i_j}\}$  and by  $\sigma_2(z_j)$  the MV-equation  $\{z_j \odot (i_j - 1)z_j = 0\}$ . Otherwise, suppose that there exists  $h_1, \dots, h_l \in \{i_1, \dots, i_m\}$  such that  $\tau_{h_1}, \dots, \tau_{h_l}$  are subformulas of  $\tau_{i_j}$ . By induction, let  $\tau_{i_j}^*$  be obtained by substituting each  $\tau_{h_k}$  by  $z_{h_k}$  and let  $\sigma_1(\tau_{i_j})$  be the MV-equation  $\{i_j \cdot z_{i_j} = \tau_{i_j}^*\}$  and  $\sigma_2(\tau_{i_j})$  be the MV-equation  $\{z_{i_j} \odot (i_j - 1) \cdot z_{i_j} = 0\}$ .

The equation  $\tau = 1$  holds in a DMV-algebra  $A$  if and only if the quasi-equation

$$\text{IF } \left. \begin{array}{l} \sigma_1 \tau_{i_1} = 1 \text{ AND } \sigma_2 \tau_{i_1} = 1 \\ \vdots \\ \text{AND } \sigma_1 \tau_{i_m} = 1 \text{ AND } \sigma_2 \tau_{i_m} = 1 \end{array} \right\} \text{ THEN } \tau^* = 1 \quad (5.4)$$

holds in the MV-reduct  $A^*$ . Since the quasi-varieties of MV-algebras is generated by  $\mathbb{Q} \cap [0, 1]$ , quasi-equation (5.4) fails in an MV-algebra if and only if it fails in  $\mathbb{Q} \cap [0, 1]$ .

In [41] it is shown that every MV-algebra is an algebra of functions over an ultrapower of  $[0, 1]$ . This is equivalent to saying that the quasi-variety generated by  $[0, 1]$  is the whole variety of MV-algebras. The proof of this theorem can be adapted to DMV-algebras in the following way:

Let  $A$  be a DMV-algebra. Then  $A$  is a subdirect product of totally ordered DMV-algebras  $A_i = \Gamma(G_i, u_i)$ . Since each  $G_i$  is a totally ordered divisible group, then it is elementarily equivalent to the additive group  $\mathbb{R}$  of real numbers with natural order. Then  $\Gamma(G_i, u_i)$  is elementarily equivalent to the MV-algebra  $[0, 1]$  and hence, by Frayne's theorem (see for example [26]), it is elementarily embeddable in a suitable ultrapower  $[0, 1]^{*i}$  of  $[0, 1]$ . Therefore, since

$$A \subseteq \prod_{j \in J} A_j \subseteq \prod_{j \in J} \Gamma(G_j, u_j) \subseteq \prod_{j \in J} [0, 1]^{*j}$$

and applying the joint embedding property of first-order logic, there exists an ultrapower of  $[0, 1]^{*i}$  of  $[0, 1]$  only depending on  $A$  such that  $A \subseteq [0, 1]^*$ .

## 5.2 Rational Łukasiewicz logic

Formulas of Rational Łukasiewicz calculus are built from the connectives of negation ( $\neg$ ), implication ( $\rightarrow$ ), and division ( $\delta_n$ ) in the usual way. An axiom

is a formula that can be written in any one of the following ways, where  $\varphi$ ,  $\psi$  and  $\gamma$  denote arbitrary formulas:

- A1)  $\varphi \rightarrow (\psi \rightarrow \varphi)$   
 A2)  $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \gamma) \rightarrow (\varphi \rightarrow \gamma))$   
 A3)  $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$   
 A4)  $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$

plus, writing  $\varphi \oplus \psi$  as an abbreviation of  $\neg\varphi \rightarrow \psi$ ,

- A5)  $\underbrace{\delta_n\varphi \oplus \dots \oplus \delta_n\varphi}_{n \text{ times}} \rightarrow \varphi$   
 A6)  $\varphi \rightarrow \underbrace{\delta_n\varphi \oplus \dots \oplus \delta_n\varphi}_{n \text{ times}}$   
 A7)  $\neg\delta_n\varphi \oplus \neg\underbrace{(\delta_n\varphi \oplus \dots \oplus \delta_n\varphi)}_{n-1 \text{ times}}.$

We shall denote by  $\mathbf{1}$  the formula  $X \rightarrow (X \rightarrow X)$  where the variable  $X$  is fixed once and for all. Proofs and provability are as usual; if  $\Gamma$  is a set of formulas,  $\Gamma \vdash \varphi$  means that  $\Gamma$  proves  $\varphi$  (or  $\varphi$  is provable from  $\Gamma$ ), that is there exists a sequence of formulas  $\gamma_1, \dots, \gamma_u$  such that  $\gamma_u = \varphi$  and every  $\gamma_i$  either is an axiom of rational Łukasiewicz logic, or belongs to  $\Gamma$  or is obtained from  $\gamma_{i_1}, \gamma_{i_2}$  ( $i_1, i_2 < i$ ) by modus ponens.  $\varphi$  is *provable* ( $\vdash \varphi$ ) if is provable from the emptyset.

Let *Form* be the set of Rational Łukasiewicz formulas and let  $\equiv$  be the binary relation over *Form* defined by  $\varphi \equiv \psi$  if and only if  $\varphi \rightarrow \psi$  and  $\psi \rightarrow \varphi$  are provable. Then  $\equiv$  is an equivalence relation and if  $\varphi$  and  $\psi$  are provable formulas then  $\varphi \equiv \psi$ .

**Proposition 5.2.1 (Lindenbaum algebra)** *The set  $\mathcal{L} = \text{Form}/\equiv$  equipped with the operations*

$$\neg[\varphi]_{\equiv} = [\neg\varphi]_{\equiv}; \quad [\varphi]_{\equiv} \oplus [\psi]_{\equiv} = [\varphi \oplus \psi]_{\equiv}; \quad \delta_n[\varphi]_{\equiv} = [\delta_n\varphi]_{\equiv}$$

*is a DMV-algebra where  $1 = \{[\varphi]_{\equiv} \mid \varphi \text{ is provable}\} = [\mathbf{1}]_{\equiv}$ .*

**Proof.** Since a similar result holds for Łukasiewicz logic, we have to prove that  $\mathcal{L}$  satisfies *D1n* and *D2n*. Indeed for every  $[\varphi]_{\equiv} \in \mathcal{L}$ , by Axioms A5 and A6,

$$n.\delta_n[\varphi]_{\equiv} = [n.\delta_n\varphi]_{\equiv} = [\varphi]_{\equiv}$$

and by Axiom A7,

$$\begin{aligned} \delta_n[\varphi]_{\equiv} \odot (n-1) \cdot \delta_n[\varphi]_{\equiv} &= \neg(\neg\delta_n[\varphi]_{\equiv} \oplus \neg(n-1) \cdot \delta_n[\varphi]_{\equiv}) = \\ &= \neg[\neg\delta_n\varphi \oplus \neg(n-1)\delta_n\varphi]_{\equiv} = \neg 1. \end{aligned}$$

■

Interpretation of connectives of Rational Łukasiewicz logic is given by

**Definition 5.2.2** An assignment is a function  $v : \text{Form} \rightarrow [0, 1]$  such that

- $v(\neg\varphi) = 1 - v(\varphi)$
- $v(\varphi \rightarrow \psi) = \min(1 - v(\varphi) + v(\psi), 1)$
- $v(\delta_n\varphi) = \frac{v(\varphi)}{n}$ .

Every function  $\iota$  from the set of variables to  $[0, 1]$  is uniquely extendible to an assignment  $v^\iota$ . For each point  $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n$  let  $\iota_{\mathbf{x}}$  be the function mapping each variable  $X_j$  into  $x_j$ . Fix  $n$ . Then each formula  $\varphi$  with  $|\text{var}(\varphi)| \leq n$  is associated with the function

$$f_\varphi : \mathbf{x} \in [0, 1]^n \mapsto v^{\iota_{\mathbf{x}}}(\varphi) \in [0, 1]$$

by means of the following stipulations:

- $f_{X_i}(x_1, \dots, x_n) = x_i$  = the  $i$ th projection.
- $f_{\neg\varphi} = 1 - f_\varphi$ .
- $f_{(\varphi \rightarrow \psi)} = \min(1, 1 - f_\varphi + f_\psi)$
- $f_{(\delta_n\varphi)} = \frac{f_\varphi}{n}$ .

A formula  $\varphi$  with  $|\text{var}(\varphi)| < n$  is *satisfiable* iff there exists  $\mathbf{x} \in [0, 1]^n$  such that  $f_\varphi(\mathbf{x}) = 1$ .  $\varphi$  is a *tautology* iff for every  $\mathbf{x} \in [0, 1]^n$ ,  $f_\varphi(\mathbf{x}) = 1$ . An assignment  $v$  is a *model* of a set of formulas  $\Gamma$  if for every  $\tau \in \Gamma$ ,  $v(\tau) = 1$ .

**Theorem 5.2.3 (Completeness)**  $\varphi$  is a tautology of Rational Łukasiewicz calculus if and only if  $\varphi$  is provable.

**Proof.** Axioms A1-A7 are tautologies and modus ponens preserves tautologicity, so every provable formula is a tautology.

Suppose that  $\varphi$  is not provable. Then the equation  $\varphi = 1$  is not true in the Lindenbaum DMV-algebra  $\mathcal{L}$  of Proposition 5.2.1, and so by Theorem 5.1.15,  $\varphi \neq 1$  in  $(\Gamma(\mathbb{Q}, 1), \delta_n)$ . This means that there exists  $\mathbf{y} \in [0, 1]^n$  such that  $f_\varphi(\mathbf{y}) < 1$ , hence  $\varphi$  is not a tautology. ■

Then,  $\varphi \equiv \psi$  if and only if  $\vdash \psi \rightarrow \varphi$  and  $\vdash \varphi \rightarrow \psi$ , if and only if, for every assignment  $v$ ,  $v(\varphi) = v(\psi)$ , if and only if  $f_\varphi = f_\psi$ .

### 5.2.1 Free DMV-algebras

The Lindenbaum algebra of Proposition 5.2.1 is the free DMV-algebra  $Free_\omega$  over a denumerable set of generators. In this section we shall describe the free DMV-algebra over a finite number of generators in terms of continuous piecewise linear functions.

A direct inspection shows that every function  $f_\varphi$  is a continuous piecewise linear function, where each piece has rational coefficients.

McNaughton theorem ([82] and [89] for a constructive proof) states that a function is associated with a Łukasiewicz formula if and only if it is a continuous piecewise linear function, each piece having integer coefficients. In [16], the authors showed that for every continuous piecewise linear function  $f$  with rational coefficients there exists a Łukasiewicz formula  $\tau$  with division operators such that  $f = f_\tau$ . The proof can be summarized as follows:

Let  $f : [0, 1]^n \rightarrow [0, 1]$  be a continuous piecewise linear function, such that each piece has rational coefficients. Further, let  $s$  be an integer such that  $s \cdot f : \mathbf{x} \in [0, 1]^n \mapsto s \cdot f(\mathbf{x}) \in [0, s]$  is a continuous function with integer coefficients (for example  $s$  is the least common multiple of the denominators of the coefficients of pieces of  $f$ ).

For every  $i = 0, \dots, s - 1$ , let

$$f_i : \mathbf{x} \in [0, 1]^n \mapsto ((s \cdot f(\mathbf{x}) - i) \wedge 1) \vee 0 \in [0, 1].$$

For every  $\mathbf{x} \in [0, 1]^n$  such that  $f(\mathbf{x}) \in [i, i + 1]$ , we have  $s \cdot f(\mathbf{x}) = i + f_i(x)$ . Since  $f_i$  are continuous functions with integer coefficients there exist MV-terms  $\psi_i$  such that  $f_i = f_{\psi_i}$ . If  $g : [0, 1]^n \rightarrow [0, 1]$  is any function, let us define

$$\begin{aligned} Supp(g) &= \{\mathbf{x} \in [0, 1]^n \mid g(\mathbf{x}) > 0\} \\ Supp^{<1}(g) &= \{\mathbf{x} \in [0, 1]^n \mid 0 < g(\mathbf{x}) < 1\}. \end{aligned}$$

We have, for every  $i = 1, \dots, s - 1$ ,

$$Supp^{<1}(f_i) \subseteq Supp(f_i) \subseteq Supp(f_{i-1}).$$

Indeed

$$Supp(f_i) = \{\mathbf{x} \in [0, 1]^n \mid s \cdot f(\mathbf{x}) > i\} \subseteq \{\mathbf{x} \in [0, 1]^n \mid s \cdot f(\mathbf{x}) > i - 1\}.$$

Further, for any  $i \neq j$ ,  $Supp^{<1}(f_i) \cap Supp^{<1}(f_j) = \emptyset$ .

**Proposition 5.2.4** *In accordance with the previous notation, if  $f : [0, 1]^n \rightarrow [0, 1]$  is a continuous piecewise linear function with rational coefficients, then for every  $\mathbf{x} \in [0, 1]^n$ ,*

$$f(\mathbf{x}) = f_\varphi(\mathbf{x}) \quad \text{where} \quad \varphi = \bigoplus_{i=1}^{s-1} \delta_s \psi_i.$$

**Proof.** Suppose that  $\mathbf{x} \in [0, 1]^n$  and  $f(\mathbf{x}) = 0$ . Then for every  $i = 0, \dots, s$ ,  $f_i(\mathbf{x}) = 0$  whence  $f_{\psi_i} = 0$  and  $f_\varphi = 0$ .

If  $f(\mathbf{x}) = 1$  then for every  $i = 0, \dots, s-1$   $f_{\psi_i} = 1$  whence  $f_{\delta_s \psi_i} = 1/s$  and  $f_\varphi = 1$ .

Suppose now that there exists  $i \in \{0, \dots, s-1\}$  such that  $i < s \cdot f(\mathbf{x}) < i + 1$ . Then  $\mathbf{x} \in \text{Supp}(f_i)$  and  $f(\mathbf{x}) = f_i(\mathbf{x}) + i/s$ . For every  $j \geq i + 1$ , we have  $s \cdot f(\mathbf{x}) - j \leq s \cdot f(\mathbf{x}) - i - 1 < 0$  whence  $f_j(\mathbf{x}) = 0$ . Further, for every  $j \leq i - 1$ , we have  $s \cdot f(\mathbf{x}) - j \geq s \cdot f(\mathbf{x}) - i + 1 > 1$  whence  $f_j(\mathbf{x}) = 1$ .

The last case to consider is when  $0 < s \cdot f(\mathbf{x}) = i < 1$ . Then for every  $j \leq i$ ,  $f_j(\mathbf{x}) = 0$  and for every  $k > i$ ,  $f_k(\mathbf{x}) = 1$ . ■

**Theorem 5.2.5** *The free DMV-algebra over  $n$  generators is the algebra of all functions from  $[0, 1]^n$  to  $[0, 1]$  that are continuous, piecewise linear and such that each linear piece has rational coefficients.*

**Proof.** Let  $\mathcal{RM}_n$  denote the set of continuous piecewise linear function with rational coefficients over  $n$  variables and let  $\mathcal{X} = \{x_1, \dots, x_n\}$  the set of variables. By identifying each variable  $x_i$  with the  $i$ -th projection,  $\mathcal{X}$  is included in  $\mathcal{RM}_n$ . If  $A$  is any DMV-algebra and  $h$  is a map from  $\mathcal{X}$  to  $A$ , then, for every  $f_\varphi \in \mathcal{RM}_n$ , the map

$$\beta_h(f_\varphi(x_1, \dots, x_n)) = f_\varphi(h(x_1), \dots, h(x_n))$$

is a DMV-homomorphism such that  $\beta_h(x_i) = h(x_i)$  for every  $x_i \in \mathcal{X}$ . If  $\gamma : \mathcal{RM}_n \rightarrow A$  is any DMV-homomorphism such that  $\gamma(x_i) = h(x_i)$ , then

$$\begin{aligned} \gamma(f_\varphi(x_1, \dots, x_n)) &= f_\varphi(\gamma(x_1), \dots, \gamma(x_n)) = \\ &= f_\varphi(h(x_1), \dots, h(x_n)) = \beta_h(f_\varphi(x_1, \dots, x_n)) \end{aligned}$$

hence  $\gamma = \beta_h$ . ■



### 5.2.2 Pavelka-style Completeness

In [98] the author, starting from the notion of many valued rules of inference, defined a class of complete residuated lattice-valued propositional calculi and introduced degrees of provability and degrees of validity. Then he proved that in Lukasiewicz propositional calculus, enriched by a denumerable set of rational constants (what in [67] is called Rational Pavelka Logic), the degree of provability of each formula coincides with the degree of validity (*Pavelka-style completeness*).

We shall show that Rational Lukasiewicz logic is a proper extension of Rational Pavelka logic. Indeed every formula of Rational Pavelka logic can be expressed in Rational Lukasiewicz language, and, after defining the degree of provability and the degree of truth, we shall prove that the completeness with respect to this degrees still holds. We shall adapt to our context the arguments in [32, 67].

**Definition 5.2.6** *An RL-theory  $T$  is a set of Rational Lukasiewicz formulas such that*

- *All axioms belong to  $T$ ;*
- *If  $\varphi \rightarrow \psi \in T$  and  $\varphi \in T$  then  $\psi \in T$ .*

If  $T$  is an RL-theory, let us denote by  $[T]$  the set  $\{[\varphi]_{\equiv} \mid \varphi \in T\}$ . Then  $T$  is an RL-theory if and only if  $\neg[T] = \{[\neg\varphi]_{\equiv} \mid \varphi \in T\}$  is an ideal of the Lindenbaum algebra in Proposition 5.2.1. If  $X$  is any set of formulas, then the RL-theory  $\mathbf{Th}(X)$  generated by  $X$  is the smallest RL-theory containing  $X$ .

An RL-theory  $T$  is *consistent* if there exists a formula  $\varphi$  such that  $\varphi \notin T$ . Following [91], an RL-theory is *prime* if it is consistent and for every pair of formulas  $\varphi$  and  $\psi$ , either  $\varphi \rightarrow \psi \in T$  or  $\psi \rightarrow \varphi \in T$ .

By Proposition 5.1.9, if  $T$  is a consistent RL-theory then there exists a prime RL-theory  $T'$  such that  $T' \supseteq T$ .

**Definition 5.2.7** *Let  $\Gamma$  be an RL-theory and  $\varphi$  a Rational Lukasiewicz formula. For every  $r \leq s \in \mathbb{N} \setminus \{0\}$ , the formula  $r.(\delta_s \mathbf{1})$  will be denoted by the rational number  $r/s$ . Then,*

- *the truth degree of  $\varphi$  over  $\Gamma$  is  $\|\varphi\|_{\Gamma} = \inf\{v(\varphi) \mid v \text{ is a model of } \Gamma\}$ ;*
- *the provability degree of  $\varphi$  over  $\Gamma$  is  $|\varphi|_{\Gamma} = \sup\{r \mid r \rightarrow \varphi \in \Gamma\}$ .*

Note that if  $\varphi \in \Gamma$  then by Axiom A1,  $\mathbf{1} \rightarrow \varphi \in \Gamma$ . Hence  $|\varphi|_{\Gamma} = 1$ .

In order to prove the completeness theorem, we note that the following results holding for Rational Pavelka logic can be easily generalized for Rational Łukasiewicz logic.

**Lemma 5.2.8** *Let  $T$  be an RL-theory.*

(a) *If  $T$  does not contain  $(r \rightarrow \varphi)$  then the RL-theory  $\mathbf{Th}(T \cup \{\varphi \rightarrow r\})$  generated by  $T \cup \{\varphi \rightarrow r\}$  is consistent.*

(b) *If  $T$  is prime, for each  $\varphi$*

$$|\varphi|_T = \sup\{r \mid r \rightarrow \varphi \in T\} = \inf\{s \mid \varphi \rightarrow s \in T\}.$$

**Theorem 5.2.9** *If  $T$  is a prime RL-theory, the function  $e : \varphi \in \text{Form} \rightarrow |\varphi|_T \in [0, 1]$  is an assignment. That is,*

$$|\neg\varphi|_T = 1 - |\varphi|_T, \quad |\varphi \rightarrow \psi|_T = |\varphi|_T \rightarrow |\psi|_T, \quad |\delta_n\varphi|_T = \frac{|\varphi|_T}{n}$$

hence  $e$  is a model of  $T$ .

**Proof.** Since the theorem holds for Rational Pavelka logic, we only have to prove  $|\delta_n\varphi|_T = \frac{1}{n} \cdot |\varphi|_T$ .

Since  $\vdash (t \rightarrow \delta_n\varphi) \rightarrow (nt \rightarrow \varphi)$ ,

$$\begin{aligned} \frac{1}{n} \cdot |\varphi|_T &= \frac{\inf\{s \mid \varphi \rightarrow s \in T\}}{n} \\ &= \inf\left\{\frac{s}{n} \mid \varphi \rightarrow s \in T\right\} = \inf\{t \mid \varphi \rightarrow nt \in T\} \leq \\ &\leq \inf\{t \mid \delta_n\varphi \rightarrow t \in T\} = |\delta_n\varphi|. \end{aligned}$$

Conversely,

$$\begin{aligned} |\delta_n\varphi| &= \sup\{t \mid t \rightarrow \delta_n\varphi \in T\} \leq \sup\{t \mid nt \rightarrow \varphi \in T\} \\ &= \sup\left\{\frac{s}{n} \mid s \rightarrow \varphi \in T\right\} \\ &= \frac{\sup\{s \mid s \rightarrow \varphi \in T\}}{n} = \delta_n|\varphi|_T. \end{aligned}$$

■

**Theorem 5.2.10 (Pavelka-style Completeness)** *For any RL-theory  $T$*

$$|\varphi|_T = \|\varphi\|_T.$$

**Proof.** Soundness (i.e.,  $|\varphi|_T \leq \|\varphi\|_T$ ) easily follows from definition:

$$\begin{aligned} |\varphi|_T &= \sup\{r \mid r \leq e(\varphi) \text{ with } e \text{ model of } T\} \\ &\leq \inf\{e(\varphi) \mid e \text{ model of } T\} = \|\varphi\|_T. \end{aligned}$$

Suppose without loss of generality that  $T$  is a consistent RL-theory. Then there exists a prime extension  $T' \supseteq T$ . By Theorem 5.2.9, the function  $e : \varphi \in Form \rightarrow |\varphi|_T \in [0, 1]$  is a model of  $T'$ , and  $|\varphi|_T = e(\varphi) \geq \|\varphi\|_T$ . ■

### 5.3 Complexity Issues

In [86] the SAT problem for Łukasiewicz logic is proved to be NP-complete. In this section we shall prove that the tautology problem for Rational Łukasiewicz logic is in co-NP and since tautology problem of Łukasiewicz formulas can be reduced to tautology problem of Rational Łukasiewicz formulas as a subset, then the latter is co-NP-complete. Such result will be a byproduct of the fact that if  $\Gamma$  is a finite set of Łukasiewicz formulas and  $\varphi$  is a Łukasiewicz formula, then the problem to establish if  $\Gamma \vdash \varphi$  is in co-NP (see, for example, [3], [8]). By [114], in this case  $\Gamma \vdash \varphi$  if and only if for every assignment  $v$  satisfying every formula of  $\Gamma$ ,  $v(\varphi) = 1$ .

Let us consider an alphabet containing  $\delta$  and a symbol  $|$  in such a way that  $\delta \underbrace{|| \dots |}_{n \text{ times}}$  stands for  $\delta_n$ .

Let  $\tau$  be a formula of Rational Łukasiewicz logic, with variables among  $\{X_1, \dots, X_n\}$ . Using the same notation as in Subsection 5.1.1, let  $\delta_{i_1}\tau_{i_1}, \dots, \delta_{i_m}\tau_{i_m}$  denote all nodes of the parsing tree of subformulas of  $\tau$  that begin with the symbol  $\delta$ .

Let  $\tau^*(X_1, \dots, X_n, Z_1, \dots, Z_m)$  be the formula obtained by substituting every subformula  $\delta_{i_j}\tau_{i_j}$  by the new variable  $Z_j$ .

Let  $\Gamma$  be the set of Łukasiewicz formulas defined by

$$\Gamma = \bigcup_{j=1}^m \{i_j.Z_{i_j} \leftrightarrow \tau_{i_j}^*, \neg Z_{i_j} \odot (i_j - 1).Z_{i_j}\},$$

where  $\tau_{i_j}^*$  has been obtained as  $\tau^*$ , accordingly substituting occurrences of  $\delta_{h_k}$  by new variables  $Z_{h_k}$ . Then the formula  $\tau$  is satisfiable in Rational Łukasiewicz logic if and only if  $\Gamma \vdash \tau^*$  holds. Since this last problem is in co-NP, we have to give an estimation of lengths of  $\Gamma$  and  $\Delta$  in terms of the length of  $\tau$ .

**Definition 5.3.1** *The length of a formula of Rational Lukasiewicz logic is inductively defined as follows:*

- (i) For every variable  $X_i$ ,  $\#X_i = 1$
- (ii)  $\#(\varphi \oplus \psi) = \#\varphi + \#\psi$
- (iii)  $\#(\neg\varphi) = \#\varphi$
- (iv)  $\#(\delta_n\varphi) = n + \#\varphi$

Since this definition is an extension of the definition of length for Lukasiewicz formulas, we shall use the notation  $\#\varphi$  also when  $\varphi$  is a Lukasiewicz formula. We set, without loss of generality,  $\#(\varphi \leftrightarrow \psi) = 2(\#\varphi + \#\psi)$ . If  $\Lambda$  is a finite set of formulas then

$$\#\Lambda = \sum_{\lambda \in \Lambda} \#\lambda.$$

If  $\delta_{i_1}\tau_{i_1}, \dots, \delta_{i_m}\tau_{i_m}$  are all subformulas of  $\tau$  involving a connective  $\delta$ , we have

$$\#\tau^* \leq \#\tau - \sum_{j=1}^m i_j + m \leq \#\tau, \quad (5.5)$$

because  $\tau^*$  is obtained from  $\tau$  by removing all occurrences of  $\delta_{i_j}$ . The first inequality in (5.5) holds because not every new variable  $Z_{i_j}$  appears in  $\tau^*$ . The second inequality holds since  $\sum_{j=1}^m i_j \geq m$ .

In  $\Gamma$  there is a pair of formulas  $i_j.Z_{i_j} \leftrightarrow \tau_{i_j}^*$ ,  $\neg Z_{i_j} \odot (i_j - 1).Z_{i_j}$  for every  $\delta_{i_j}\tau_{i_j}$  occurring in  $\tau$ . Since

$$\#(\delta_{i_j}\tau_{i_j}) = i_j + \#\tau_{i_j}$$

and

$$\begin{aligned} \#(i_j.Z_{i_j} \leftrightarrow \tau_{i_j}^*) &= 2(i_j + \#(\tau_{i_j}^*)) \leq 2(i_j + \#(\tau_{i_j})) \\ \#(\neg Z_{i_j} \odot (i_j - 1).Z_{i_j}) &= 1 + i_j - 1, \end{aligned}$$

then

$$\begin{aligned} \#\Gamma &= \sum_{j=1}^m \left( \#(i_j.Z_{i_j} \leftrightarrow \tau_{i_j}^*) + \#(\neg Z_{i_j} \odot (i_j - 1).Z_{i_j}) \right) \leq \\ &\leq \sum_{j=1}^m (2(i_j + \#(\tau_{i_j})) + i_j) \leq \sum_{j=1}^m 3\#(\delta_{i_j}\tau_{i_j}) \\ &\leq 3\#\tau. \end{aligned}$$

Putting together Equations (5.6) and (5.5) we get the desired conclusion.

We shall now show that the complexity of the tautology problem for Rational Łukasiewicz logic does not change if the index  $n$  of  $\delta_n$  is written in binary notation.

Then let, in Definition 5.3.1,  $\#\delta_n\varphi = \log_2 n + \#\varphi$ . If  $\delta_n\tau_n$  occur in  $\tau$  let  $m_1 > \dots > m_h \geq 0$  be integer numbers (depending on  $n$ ) such that

$$n = 2^{m_1} + \dots + 2^{m_h}.$$

We introduce  $m_1$  new variables  $Y_1, \dots, Y_{m_1}$  and new formulas

$$\begin{aligned} \sigma(1, n) &= Y_1 \oplus Y_1 \leftrightarrow Y_2 \\ &\dots \\ \sigma(m_1 - 1, n) &= Y_{m_1-1} \oplus Y_{m_1-1} \leftrightarrow Y_{m_1} \\ \sigma'(1, n) &= Y_1 \odot Y_1 \leftrightarrow 0 \\ &\dots \\ \sigma'(m_1 - 1, n) &= Y_{m_1-1} \odot Y_{m_1-1} \leftrightarrow 0 \\ \tau^* &= \tau_n \leftrightarrow \left( 2Y_{m_1} \oplus 2Y_{m_2} \oplus \dots \oplus \begin{cases} 2Y_{m_h} & \text{if } m_h > 0 \\ Y_{m_h} & \text{if } m_h = 0 \end{cases} \right) \end{aligned}$$

We have  $2(m_1 - 1) \leq 2\log_2 n$  formulas  $\sigma(i, n)$  and  $\sigma'(i, n)$  of constant length and further  $\#\tau^* = 2(\#\tau + 2(m_1 + \dots + m_h)) \leq 2(\#\tau + 2\log_2 n)$ . Since  $\#(\delta_n\tau) = \#\tau + \log_2 n$ , then  $\#\tau^* \leq 4\#(\delta_n\tau)$ .

If  $\delta_{i_1}\tau_{i_1}, \dots, \delta_{i_m}\tau_{i_m}$  are all subformulas of  $\tau$  that begin with a symbol  $\delta$ , then for any  $\delta_{i_j}\tau_{i_j}$  we suitably introduce formulas  $\sigma, \sigma', \tau^*$  and thus reduce the problem of tautology to the problem of deciding if a Łukasiewicz formula is consequence of a finite set of formulas. The latter is co-NP in the length of  $\tau$ .

## 5.4 Weakly divisible MV-algebras

Divisible MV-algebra are reducts of DMV-algebras. Divisibility implies that an algebra contain the  $n$ th divisor of every of its elements. This notion can be weakened by requiring that the algebra contains just the  $n$ th divisor of 1 [49]. In [31] we shown how to construct the weakly divisible hull of any MV-algebra.

Let  $A$  be an MV-algebra and let  $(G, u)$  be the  $\ell$ -group associated with  $A$  ( $A = \Gamma(G, u)$ ).

**Definition 5.4.1** *A is said to be  $n$ -weakly divisible if there exists  $x \in A$  such that*

$$nx = \underbrace{x + \dots + x}_{n \text{ times}} = 1.$$

*An MV-algebra  $A$  is weakly divisible if it is  $n$ -weakly divisible for every integer  $n \geq 1$ .*

Note that if  $\xi$  exists in  $A$  such that  $\xi + \xi = 1$ , then trivially  $\neg\xi = \xi$  (such  $\xi$  is said a *selfcomplemented* element).

The set  $nA = \{nx \mid x \in A\}$  is an MV-subalgebra of  $\Gamma(G, nu)$  equipped with the operations of  $\Gamma(G, nu)$ , i.e.,

$$\begin{aligned} x \oplus_n y &= \inf\{x + y, nu\} \\ \neg_n x &= nu - x. \end{aligned}$$

Every element  $x + \dots + x \in nA$  will be denoted by  $nx$ . Further, consider the operation

$$x \ominus_n y = \sup\{x - y, 0\} = \neg_n(\neg_n x \oplus_n y).$$

**Proposition 5.4.2** *The subalgebra  $\langle nA, u \rangle$  of  $\Gamma(G, nu)$  generated by  $nA$  and  $u$  is given by*

$$\{nx, ru \oplus_n nx, ru \ominus_n nx \mid x \in A, r = 1, \dots, n\} \subseteq \Gamma(G, nu)$$

*equipped with the operations of  $\Gamma(G, nu)$ .*

**Proof.** Since every abelian  $\ell$ -group is a subdirect power of linearly ordered abelian groups, arguing componentwise we get that equations holding for linearly ordered abelian groups are still valid for any abelian  $\ell$ -group. Let  $T = \{kx, ru \oplus_k kx, ru \ominus_k kx \mid x \in A, r = 1, \dots, k-1\}$ .  $T$  equipped with the operation of  $\Gamma(G, ku)$  is an MV-algebra. Indeed the following holds:

- $kx_1 \oplus_k (r_1u \oplus_k kx_2) = r_1u \oplus_k k(x_1 \oplus_A x_2) \in T$ , since, if  $x_1 + x_2 \geq u$  then  $k(x_1 \oplus_A x_2) \geq ku$ ;
- $kx_1 \oplus_k (r_1u \ominus_k kx_2) = \begin{cases} kx_1 & \text{if } ru_1 - kx_2 \leq 0 \\ r_1u \oplus_k k(x_1 \ominus_k x_2) & \text{otherwise.} \end{cases}$
- $(r_1u \oplus_k kx_1) \oplus_k (r_2u \ominus_k kx_2) = \begin{cases} r_1u \oplus_k kx_1 & \text{if } r_2u - kx_2 \leq 0 \\ ku & \text{if } r_1u + kx_1 \geq ku \\ (r_1 \oplus r_2)u \oplus_k k(x_1 \ominus_k x_2) & \text{otherwise.} \end{cases}$

$$\bullet (r_1u \ominus_k kx_1) \oplus_k (r_2u \ominus_k kx_2) = \begin{cases} r_1u \ominus_k kx_1 & \text{if } r_2u - kx_2 \leq 0 \\ r_2u \ominus_k kx_2 & \text{if } r_1u - kx_1 \leq 0 \\ (r_1 \oplus_k r_2)u \ominus_k k(x_1 \oplus_k x_2) & \text{otherwise.} \end{cases}$$

Hence  $T$  is the MV-subalgebra of  $\Gamma(G, ku)$  generated by  $kA = \{kx \mid x \in A\}$  and  $\{ru \mid r = 1, \dots, k-1\}$ .  $\blacksquare$

**Definition 5.4.3** Given MV-algebras  $A$ ,  $B$  and  $C$  and homomorphism  $f : A \rightarrow C$  and  $g : B \rightarrow C$ ,  $C$  is the direct sum or coproduct  $A \amalg B$  of  $A$  and  $B$  if for every MV-algebra  $N$  and homomorphism  $f^* : A \rightarrow N$  and  $g^* : B \rightarrow N$  there exists a unique homomorphism  $h : C \rightarrow N$  such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & C & \xleftarrow{g} & B \\ & \searrow^{f^*} & \downarrow h & \swarrow_{g^*} & \\ & & N & & \end{array}$$

**Proposition 5.4.4** Let  $A$  be an MV-algebra,  $S_n = \{0, 1/n, \dots, 1\}$ ,  $\varphi : x \in A \mapsto nx \in nA$  and  $\theta : S_n \rightarrow \{0, u, 2u, \dots, nu\}$  such that  $\theta(0) = 0, \theta(1/n) = u, \dots, \theta(1) = nu$ . Then  $\langle nA, u \rangle$  is the coproduct of  $(A, \varphi)$  and  $(S_n, \theta)$ .

**Proof.** We must show that for every MV-algebra  $(D, \oplus_D, \neg_D, 0_D)$  and homomorphism  $v_1 : A \rightarrow D$  and  $v_2 : L_2 \rightarrow D$ , there exists a unique homomorphism  $h : \langle nA, u \rangle \rightarrow D$  such that  $v_1 = \varphi h$  and  $v_2 = \theta h$ .

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & \langle nA, u \rangle & \xleftarrow{\theta} & L_2 \\ & \searrow^{v_1} & \downarrow h & \swarrow_{v_2} & \\ & & D & & \end{array}$$

Indeed, the function  $h$  can be defined on elements of  $\langle nA, u \rangle$  in the following way:

$$\begin{aligned} h(nx) &= v_1(x) \\ h(ru \oplus_n nx) &= v_2(r/n) \oplus_D v_1(x) \\ h(ru \ominus_n nx) &= v_2(r/n) \ominus_D (v_1(x)). \end{aligned}$$

$\blacksquare$

**Proposition 5.4.5** If  $A$  is not  $n$ -weakly divisible (i.e., there does not exist  $x \in A$  such that  $nx = 1$ ), then, if  $r < n, r' < n, x, x' \in A$  and  $ru \oplus_n nx < nu$ ,

$$ru \oplus_n nx = r'u \oplus_n nx' \Leftrightarrow r = r' \text{ and } x = x'.$$

**Proof.** Suppose that  $ru \oplus_n nx = r'u \oplus_n nx'$  and  $r \neq r'$ . Note that  $ru \oplus_n nx < nu$  if and only if  $x < u$ . Indeed, if  $ru + nx \geq nu$  then  $ru \geq n(u - x)$  and since  $r < n$  then it must be  $u = x$ . So, let  $x, x' < u$  so that  $ru \oplus_n nx = r'u \oplus_n nx' < nu$  and hence  $ru + nx = r'u + nx'$ . If  $A = \Gamma(G, u)$ , in the group  $G$  such equation is equivalent to

$$(r - r')u = n(x' - x).$$

Note that  $u > y = x - x' \in A$ , and, since  $r - r' \neq 0$ , in  $A$

$$ny = \underbrace{y \oplus_1 \dots \oplus_1 y}_n = u$$

This is a contradiction since  $A$  is not  $n$ -weakly divisible.  $\blacksquare$

**Proposition 5.4.6** *If  $A$  is not  $n$ -weakly divisible then, if  $r < n$ ,  $r' < n$ ,  $x, x' \in A$  and  $ru \ominus_n nx > 0$ ,*

$$ru \ominus_n nx = r'u \ominus_n nx' \Leftrightarrow r = r' \text{ and } x = x'.$$

**Proof.** The proof is analogous to the previous one. Note that  $ru \ominus_n nx = 0$  if and only if  $ru - nx = 0$  if and only if  $nx = ru$  if and only if  $x = 0$  ( $A$  is not n-w.d.). Then  $ru - nx = r'u - nx'$  if and only if  $(r - r')u = n(x - x')$  if and only if  $r = r'$  and  $x = x'$ .  $\blacksquare$

Suppose now that the MV-algebra  $A$  is not weakly divisible.

**Definition 5.4.7** *A direct family of algebras  $\mathcal{A}$  is defined to be a triplet of the following objects:*

- *A directed partially ordered set  $\langle I; \leq \rangle$ ;*
- *algebras  $A_i$  of the same type, for each  $i \in I$ ;*
- *homomorphisms  $\varphi_{ij}$  for all  $i \leq j$ , between  $A_i$  and  $A_j$ , such that*

$$\varphi_{ij}\varphi_{jk} = \varphi_{ik} \quad \text{if } i \leq j \leq k$$

*and  $\varphi_{ii}$  is the identity mapping for all  $i \in I$ .*

For a direct family of algebras  $\mathcal{A}$  consider the set  $\cup\{A_i \mid i \in I\}$  (or, if  $A_i$  are not pairwise disjoint, consider the disjoint union). Define on it a binary relation  $\equiv$  by  $x \equiv y$  if and only if  $x \in A_i$ ,  $y \in A_j$ , for some  $i, j \in I$  and



there exists a  $z \in A_k$  such that  $i \leq k$ ,  $j \leq k$ ,  $\varphi_{ik}(x) = z = \varphi_{jk}(y)$ . It is an equivalence relation.

The set of equivalence classes is called *direct limit* of the direct family of algebras  $\mathcal{A}$ , and it is denoted by  $\lim_{i \in I} A_i$ . Operations over  $\lim_{i \in I} A_i$  are suitably defined in such a way that  $\lim_{i \in I} A_i$  is an MV-algebra.

**Example of direct family of algebras.** Consider the partially ordered set  $\langle \mathbb{N}, \preceq \rangle$  of natural numbers, where  $n \preceq m$  if and only if there exists  $k \in \mathbb{N}$  such that  $m = nk$ . Then the family  $\{S_n \mid n \in \mathbb{N}\}$  with homomorphism  $\varphi_{nm}$  where  $n \preceq m$  defined by  $\varphi_{nm}(i/n) = (ik/m)$  is a direct family. Further, the direct limit of this family is constituted by the equivalence classes  $[i/n]$  where  $j/m \in [i/n]$  if and only if  $nj = mi$ . Such direct limit can be shown to be isomorphic to  $\mathbb{Q} \cap [0, 1]$ .

Denote by  $u_n$  the element  $u \in A \coprod S_n$ . We want now to describe the limit of the direct family  $(\langle \mathbb{N}, \preceq \rangle, A \coprod S_n, \varphi_{ij})$ , where, for every  $i \preceq j$  ( $j = li$ ),  $\varphi_{ij} : A \coprod L_i \rightarrow A \coprod L_j$  is an homomorphism such that

$$\begin{aligned}\varphi_{ij}(ix) &= jx \\ \varphi_{ij}(ru_i \oplus_i ix) &= lru_j \oplus_j jx \\ \varphi_{ij}(ru_i \ominus_i ix) &= lru_j \ominus_j jx\end{aligned}$$

It is a direct family since if  $h = mj = mli$  then

$$\begin{aligned}\varphi_{jh}(\varphi_{ij}(ix)) &= \varphi_{jh}(jx) = hx = \varphi_{jh}(jx) \\ \varphi_{jh}(\varphi_{ij}(ru_i \oplus_i ix)) &= \varphi_{jh}(lru_j \oplus_j jx) = mlru_h \oplus_h hx = \varphi_{ih}(ru_i \oplus_i ix) \\ \varphi_{jh}(\varphi_{ij}(ru_i \ominus_i ix)) &= \varphi_{jh}(lru_j \ominus_j jx) = mlru_h \ominus_h hx = \varphi_{ih}(ru_i \ominus_i ix)\end{aligned}$$

Over such direct family, let us see what the relation  $\equiv$  becomes for  $v \in A \coprod L_i$  and  $w \in A \coprod L_j$ : it must exist  $z \in A \coprod L_k$  such that  $i \preceq k$ ,  $j \preceq k$  and  $\varphi_{ik}(v) = z = \varphi_{jk}(w)$ . If we take  $k = ij$ , then  $v \equiv w$  if and only if one of the following cases hold:

- if  $v = ix_1$  and  $w = jx_2$  then  $ijx_1 = ijx_2$ , and hence  $x_1 = x_2$ ;
- if  $v = r_1u_i \oplus_i ix_1$  and  $w = jx_2$  then  $r_1ju_{ij} \oplus_{ij} ijx_1 = ijx_2$ . Since  $A$  is not weakly divisible then it holds if and only if  $r_1 = 0$  and  $x_1 = x_2$ . The case is analogous for  $v = ix_1$  and  $w = r_2u_j \oplus_j jx_2$ ;
- if  $v = r_1u_i \oplus_i ix_1$  and  $w = r_2u_j \oplus_j jx_2$  then  $r_1ju_{ij} \oplus_{ij} ijx_1 = r_2iu_{ij} \oplus_{ij} ijx_2$ . Hence

$$[r_1u_i \oplus_i ix_1] = \{r_2u_j \oplus_j jx_2 \mid r_2 = r_1 \text{ and } x_2 = x_1\}$$

- $[r_1 u_i \ominus_i i x_1] = \{r_2 u_j \ominus_j j x_2 \mid r_2 = r_1 \text{ and } x_2 = x_1\}$

Suppose now to consider maps

$$\varphi : x \in A \rightarrow [kx] \in \lim_{n \in \mathbb{N}} A \coprod S_n$$

and

$$i : \frac{r}{q} \in \mathbb{Q} \cap [0, 1] \rightarrow [r u_q] \in \lim_{n \in \mathbb{N}} A \coprod S_n$$

**Proposition 5.4.8**  $\lim_{n \in \mathbb{N}} A \coprod S_n$  is the coproduct of  $A$  and  $\mathbb{Q} \cap [0, 1]$ .

**Proof.** Let  $D$  be any MV-algebra and  $h_1 : A \rightarrow D$  and  $h_2 : \mathbb{Q} \cap [0, 1] \rightarrow D$  homomorphism. We must define a homomorphism  $h : \lim_{n \in \mathbb{N}} A \coprod S_n \rightarrow D$  such that the diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & \lim_{n \in \mathbb{N}} A \coprod S_n & \xleftarrow{i} & \mathbb{Q} \cap [0, 1] \\ & \searrow v_1 & \downarrow h & \swarrow v_2 & \\ & & D & & \end{array}$$

It is enough to define  $h$  over elements:

$$[kx] = \{hx \mid h \in \mathbb{N}\}$$

$$[r u_i \oplus_i i x] = \{r_1 u_j \oplus_j j x_1 \mid r_j = r_1 i \text{ and } x = x_1\}$$

$$[r u_i \ominus_i i x] = \{r_1 u_j \ominus_j j x_1 \mid r_j = r_1 i \text{ and } x = x_1\}$$

Let

$$\begin{aligned} h([kx]) &= h_1(x) \\ h([r u_1 \oplus_i i x]) &= h_2\left(\frac{r}{i}\right) \oplus_D h_1(x) \\ h([r u_1 \ominus_i i x]) &= h_2\left(\frac{r}{i}\right) \ominus_D h_1(x) \end{aligned}$$

In order to prove that  $h$  is well defined, let  $r' u_j \oplus_j j x' \in [r u_1 \oplus_i i x]$ . Then  $r_j = r' i$  and  $x = x'$ . So

$$\begin{aligned} h([r' u_j \oplus_j j x']) &= h_2(r'/j) \oplus_D h_1(x') \\ &= h_2(r/i) \oplus_D h_1(x) = h([r u_1 \oplus_i i x]) \end{aligned}$$

■

**Definition 5.4.9** *The weakly divisible hull of an MV-algebra  $(A, \oplus_A, \neg_A, 0, u)$  is the smallest (w.r.t. the set inclusion) MV-algebra containing  $A$  that is weakly divisible.*

**Theorem 5.4.10** *Given a not weakly divisible MV-algebra  $A$ ,  $A \coprod (\mathbb{Q} \cap [0, 1])$  is the weakly divisible hull of  $A$ .*

**Proof.** We must show that

- (i)  $A \coprod (\mathbb{Q} \cap [0, 1])$  is weakly divisible;
- (ii)  $A \coprod (\mathbb{Q} \cap [0, 1])$  contains  $A$ ;
- (iii) if  $(B, \oplus_B, \neg_B, 0, u)$  is a weakly divisible MV-algebra such that  $A$  is a subalgebra of  $B$ , then  $A \coprod (\mathbb{Q} \cap [0, 1]) \subseteq B$ .

i) By Proposition 5.4.8 we know that  $A \coprod (\mathbb{Q} \cap [0, 1]) = \lim_{k \in \mathbb{N}} A \coprod L_k$ .

The unit element of  $\lim_{k \in \mathbb{N}} A \coprod L_k$  is  $[u]$ . For every  $n > 0$  the equation  $n[x] = [u]$  is equivalent to  $[nx] = [u]$  and so it has always solution in  $A \coprod (\mathbb{Q} \cap [0, 1])$ .

ii)  $A$  can be embedded in  $\lim_{k \in \mathbb{N}} A \coprod L_k$  considering the immersion  $i : x \in A \mapsto [x] \in \lim_{k \in \mathbb{N}} A \coprod L_k$ .

iii) if  $A \leq B$  and  $B$  is weakly divisible, then  $B$  contains solutions  $x$  of equation  $nx = u$  for every  $n > 0$ . Let us denote by  $u/n$  such a solution. Consider the map  $i : \lim_{k \in \mathbb{N}} A \coprod L_k \rightarrow B$  such that  $i([hx]) = x$ ,  $i([ru \oplus_h hx]) = r(u/h) \oplus_B x$  and  $i([ru \ominus_h hx]) = r(u/h) \ominus_B x$ . It is an embedding of  $\lim_{k \in \mathbb{N}} A \coprod L_k$  into  $B$ . ■

## Chapter 6

# Probability of Fuzzy events

In this chapter we shall study probabilities over MV-algebras, as introduced in [90] under the name of *states*. In the first part we shall use the game of multiple bets [54] in order to give a subjective interpretation of states on MV-algebras, following the work in [39]. In the second part we shall investigate conditional states corresponding to conditional probability and we shall describe a probabilistic approach to Ulam game [53, 113, 87, 88].

### 6.1 States and conditional states

Let  $L$  be a finite MV-chain:

$$L = S_k = \{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\}.$$

For any set  $X$ , an  $L$ -subset of  $X$  is a function  $d : X \rightarrow L$ . The class  $L^X$  of all  $L$ -subsets inherits from  $L$  the structure of an MV-algebra: operations are obtained by pointwise application of the Łukasiewicz operations and are called *Łukasiewicz union*, *intersection* and *complement*. Identifying subsets of  $X$  with their characteristic functions, the powerset  $2^X$  of  $X$  then coincides with the boolean skeleton of  $L^X$ .

We say that two  $L$ -subsets  $\mu$  and  $\nu$  are  $\odot$ -disjoint if  $\mu \odot \nu = 0$ , that they are  $\wedge$ -disjoint if  $\mu \wedge \nu = 0$ .

An  $L$ -singleton is an  $L$ -subset  $\mu$  whose support  $Supp(\mu) = \{x \in X \mid \mu(x) \neq 0\}$  is a singleton. If  $\lambda \in L - \{0\}$  and  $x \in X$ , then we denote by  $\langle x, \lambda \rangle$  the  $L$ -singleton defined by

$$\langle x, \lambda \rangle(y) = \begin{cases} \lambda, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases} \quad (6.1)$$

Every  $g \in L^X - \{0\}$  can be written as the sum

$$\bigoplus_{\substack{x \in X \\ g(x) \neq 0}} \langle x, g(x) \rangle,$$

of pairwise  $\wedge$ -disjoint  $L$ -singletons  $\langle x, g(x) \rangle$ .

Let  $L = S_{k+1}$ . For each  $x \in X$  let  $\delta_x : X \rightarrow L$  be the  $L$ -singleton  $\langle x, 1/(k+1) \rangle$ . It is immediate that these  $L$ -singletons are pairwise  $\odot$ -disjoint and that every  $g \in L^X - \{0\}$  is a linear combination of the  $\delta_x$ 's with non-negative integer coefficients  $g(x)(k+1)$

$$\bigoplus_{\substack{x \in X \\ g(x) \neq 0}} (k+1)g(x)\delta_x.$$

The classical notion of (finitely additive) probability measure on boolean algebras was generalized to MV-algebras in [90] as follows:

**Definition 6.1.1** *By a state of an MV-algebra  $A$  we mean a function  $s : A \rightarrow [0, 1]$  satisfying the following conditions:*

- (i)  $s(0) = 0$ ;
- (ii)  $s(1) = 1$ ;
- (iii) whenever  $a, b \in A$  and  $a \odot b = 0$ , then  $s(a) + s(b) = s(a \oplus b)$ .

A state is called faithful if for every nonzero  $a \in A$ ,  $s(a) > 0$ .

As observed in [90], a state is a monotone functions and a faithful state  $s$  is a valuation, i.e.,  $s(a \oplus b) + s(a \odot b) = s(a) + s(b)$ .

The following natural example of state in  $L^X$  was furnished by Zadeh in [116]. As usual, given a set  $X$ , a probability  $p : 2^X \rightarrow [0, 1]$  and  $x \in X$ , we write  $p(x)$  instead of  $p(\{x\})$ .

**Proposition 6.1.2** *Let  $X$  be a finite set and  $p : 2^X \rightarrow [0, 1]$  an arbitrary probability measure. Let the function  $p^\natural : L^X \rightarrow [0, 1]$  be defined by stipulating that, for every  $\mu \in L^X$ ,*

$$p^\natural(\mu) = \sum_{x \in X} \mu(x)p(x). \tag{6.2}$$

*Then  $p^\natural$  is a state of  $L^X$ .*

**Proof.** Trivially,  $p^{\natural}(0) = 0$ . Further,  $p^{\natural}(1) = \sum_{x \in X} p(x) = 1$ . Now assume  $\mu, \nu \in L^X$  and  $\mu \odot \nu = 0$ ; it follows that  $\mu(x) \oplus \nu(x) = \mu(x) + \nu(x) \leq 1$ , whence

$$\begin{aligned} p^{\natural}(\nu \oplus \mu) &= \sum_{x \in X} (\nu \oplus \mu)(x)p(x) = \sum_{x \in X} (\nu + \mu)(x)p(x) \\ &= \sum_{x \in X} \nu(x)p(x) + \sum_{x \in X} \mu(x)p(x) = p^{\natural}(\nu) + p^{\natural}(\mu). \end{aligned}$$

■

The following proposition is a consequence of a general result in [42]. For reader's convenience we give here the proof.

**Proposition 6.1.3** *The map  $p \mapsto p^{\natural}$ , where  $p^{\natural}$  is given by (6.2), is a one-one correspondence between probability measures on the boolean algebra  $2^X$  and states on the MV-algebra  $L^X$ . The inverse of this map is obtained by restricting each state to the boolean skeleton  $2^X$ .*

**Proof.** Let  $L = S_k$ . Skipping all trivialities, we have only to prove that the state  $p^{\natural}$  is the *unique* state on  $L^X$  extending  $p$ . Now, it is immediate that, by definition, every state is determined by its values on the atoms  $\delta_x$ . Suppose that  $q$  is a state such that  $q \neq p^{\natural}$ . Then for some  $x \in X$  we have  $p^{\natural}(\delta_x) \neq q(\delta_x)$ . Since  $k\delta_x \in 2^X$ , and  $p^{\natural}$  extends  $p$ , then  $p^{\natural}(k\delta_x) = kp^{\natural}(\delta_x) = p(x)$ , whence,  $p^{\natural}(\delta_x) = p(x)/k$ . In case  $q(\delta_x) < p(x)/k$  then  $q(k\delta_x) < p(x)$ , and  $q$  does not extend  $p$ . On the other hand, in the case  $q(\delta_x) > p(x)/k$ , we have  $q(k\delta_x) = p^{\natural}(k\delta_x) > p(x)$ , and again,  $q$  does not extend  $p$ . ■

Notice that, in particular, given any state  $s$ , for every  $\mu$  in  $L^X$ ,

$$s(\mu) = \sum_{x \in X} s(x)\mu(x).$$

## 6.2 Multiple bets and subjective states

In this section we shall use MV-algebra operations to describe *multiple bets*. Two players, A and B, agree on a finite set  $\Omega$  of elementary events. A subset  $X \subseteq \Omega$  will be called an event. They also fix an integer  $k > 0$ . Player A buys from Player B (the Bank) a sequence of events  $u = X_1 \dots X_n$  (that we will call *multiple bet*), for a price, say  $s(u)$ €, fixed by Player B. Then, an

elementary event  $x \in \Omega$  is extracted and Player B pays 1\$ to Player A for each distinct  $X_i$  among  $X_1, \dots, X_n$  containing  $x$ . Further, we suppose that Player B cannot give to Player A more than  $k$ \$. By a suitable normalization we can suppose that the maximum winning is 1\$ and that Player A wins  $\frac{h}{k}$ \$ if  $x$  belongs to  $h$  many distinct elements among  $X_1, \dots, X_n$ .

Different sequences of events can be considered equivalent whenever they lead to the same winnings for Player A.

The set of equivalence classes of multiple bets can be equipped with a structure of MV-algebras and turns out to be isomorphic to the boolean power of the MV-chain of  $k$  elements.

As we shall see in Section 6.2.3, states over the MV-algebras of equivalence classes of multiple bets are rates given by Player A in such a way that the betting system is fair.

In Section 6.2.4 such results are extended to DMV-algebras.

### 6.2.1 Identifying bets

Let  $\mathcal{B} = (B, \vee, \wedge, 0, 1)$  be a boolean algebra. We will denote by  $B^+$  the free semigroup on the domain  $B$  of  $\mathcal{B}$ . Then,  $B^+$  is the set of words  $X_1 \cdots X_n$  with  $X_i \in B$ , equipped with the operation of juxtaposition. In the sequel we shall not make distinctions between  $\mathcal{B}$  and  $B$ .

For any boolean algebra  $B$  and for any  $n$ -tuple  $u = X_1 \cdots X_n \in B^+$ , let us denote by  $B_u$  the (finite) boolean subalgebra of  $B$  generated by  $X_1, \dots, X_n$ . The set of atoms of  $B_u$  will be denoted by  $at(B_u)$ .

Let  $k > 0$  be a natural number. For every  $u = X_1 \cdots X_n \in B^+$  and  $X \in B$  the quantity

$$c_k(X, u) = \frac{\min\{k, \text{card}(\{i \mid X \leq X_i\})\}}{k} \quad (6.3)$$

is called the *frequency (up to  $k$ )* of  $X$  in  $u$ . Note  $c_k(X, u) \in S_{k+1}$ .

**Lemma 6.2.1** *Let  $u, v \in B^+$ ,  $X \in at(B_{uv})$  and  $Z \in at(B_u)$ ,  $Z' \in at(B_v)$  such that  $Z \wedge Z' \neq 0$ . Then:*

- (i)  $c_k(X, uv) = c_k(X, u) \oplus c_k(X, v)$ .
- (ii)  $c_k(Z \wedge Z', uv) = c_k(Z \wedge Z', u) \oplus c_k(Z \wedge Z', v) = c_k(Z, u) \oplus c_k(Z', v)$ .
- (iii)  $c_k(Y, u) = c_k(Z, u)$  for every  $Y \in B$  such that  $Y \leq Z$ .
- (iv)  $at(B_{uv}) = \{\bar{Z} \wedge \bar{Z}' \mid \bar{Z} \in at(B_u), \bar{Z}' \in at(B_v), \bar{Z} \wedge \bar{Z}' \neq 0\}$ .

■

**Definition 6.2.2** Given  $u = X_1 \cdots X_n$  and  $v = Y_1 \cdots Y_m$  in  $B^+$ , we set  $u \preceq_k v$  if and only if, for every  $X \in at(B_{uv})$ ,

$$c_k(X, u) \leq c_k(X, v). \quad (6.4)$$

Intuitively,  $u \preceq_k v$  means that the winning obtainable by playing  $u$  is less than the winning of playing  $v$ .

The relation  $\preceq_k$  is a pre-order, i.e., a reflexive and transitive relation. Then we can associate  $\preceq_k$  with an equivalence relation as usual:

**Definition 6.2.3** Two elements  $u, v \in B^+$  are  $k$ -equivalent, and we write  $u \equiv_k v$ , if  $u \preceq_k v$  and  $v \preceq_k u$ .

Consider the quotient  $B^+ / \equiv_k$  of  $B^+$  modulo  $\equiv_k$ . Then it is easy to see that the relation  $\leq_k$  given by

$$[u] \leq_k [v] \Leftrightarrow u \preceq_k v \quad (6.5)$$

is a partial order relation over  $B^+ / \equiv_k$ . Further,  $\equiv_k$  is a congruence in the semigroup  $B^+$ . We can then consider the operation  $\oplus$  in  $B^+ / \equiv_k$  induced by the operation in  $B^+$  as follows:

$$[X_1 \cdots X_n] \oplus [Y_1 \cdots Y_m] = [X_1 \cdots X_n Y_1 \cdots Y_m]. \quad (6.6)$$

Further let us denote by  $\mathbf{0}$  the element  $[0^k] = [0^{k-1}] = \dots = [0]$ . The resulting structure  $(B^+ / \equiv_k, \oplus, \mathbf{0})$  is a monoid. We shall denote by  $\mathbf{1}$  the element  $[1^k]$ , where 1 is the unit element of  $B$ .

**Proposition 6.2.4** If  $X, Y \in B$ , then

- (i)  $XY \equiv_k YX$
- (ii)  $XY \equiv_k (X \vee Y)(X \wedge Y)$
- (iii)  $X^{k+1} \equiv_k X^k$
- (iv)  $X0 \equiv_k X$
- (v)  $XY \equiv_k (X - Y)(Y - X)(X \wedge Y)^2$ .



**Proof.** We will prove (ii). Proofs of the other conditions are similar.

We must compare the words  $u = XY$  and  $v = (X \vee Y)(X \wedge Y)$ . If  $k = 1, 2$  then the result trivially holds. Suppose that  $k > 2$ . For every  $Z \in at(B_{uv})$  possible cases are:

- $c_k(Z, u) = 0$ ; then  $Z \not\leq X$  and  $Z \not\leq Y$ , so  $Z \not\leq X \vee Y$ ,  $Z \not\leq X \wedge Y$  and  $c_k(Z, v) = 0$ ;
- $c_k(Z, u) = 1$ ; then we can assume that  $Z \leq X$  and  $Z \not\leq Y$  so that  $Z \leq X \vee Y$ ,  $Z \not\leq X \wedge Y$  and  $c_k(Z, v) = 1$ ;
- $c_k(Z, u) = 2$ ; then  $Z \leq X$  and  $Z \leq Y$ , so that  $Z \leq X \wedge Y$ ,  $Z \leq X \vee Y$  and  $c_k(Z, v) = 2$ .

So  $c_k(Z, u) = c_k(Z, v)$  and  $u \equiv_k v$ . ■

Proposition 6.2.4 suggests a rewriting system that enables us to choose in an effective way a particular element as the representative of an equivalence class (for an overview of rewriting systems see [40]).

**Definition 6.2.5** *A word on  $B$  can be transformed into another element of  $B^+$  applying the following rewriting rules:*

- (a) *If  $X \wedge Y \neq 0$  then  $X^n Y^m \rightarrow (X - Y)^n (Y - X)^m (X \wedge Y)^{n+m}$ ;*
- (b) *if  $X \wedge Y = 0$  then  $X^n Y^n \rightarrow (X \vee Y)^n$ ;*
- (c) *if  $n \leq m$  then  $Y^m X^n \rightarrow X^n Y^m$ ;*
- (d) *if  $h > k$  then  $X^h \rightarrow X^k$ .*

This rule system is *terminating*, i.e., after a finite number of applications of rules to a word  $w$  over  $B$  it is not possible to apply other rules. The expression resulting from such a *derivation* is called *normal form* of  $w$  and will be denoted by  $N(w)$ .

**Proposition 6.2.6** *A word  $w$  is in normal form if and only if it has the form  $X_1^{m_1} \cdots X_n^{m_n}$  where  $X_i$  are non-zero pairwise disjoint elements of  $B$  and  $(m_i)_{i=1, \dots, n}$  is a strictly increasing sequence of positive integers  $\leq k$ . ■*

**Proposition 6.2.7** *For every  $w \in B^+$ ,  $N(w)$  is unique.*

**Proof.** We give here only a sketch of the proof. Using the above notations, we have to prove that our rule system is *locally confluent*, i.e, if two expressions  $w_1$  and  $w_2$  are deducible from the expression  $w$ , then  $w_1$  and  $w_2$  have the same normal form. Since the system is terminating, this property assures the uniqueness of the normal form (See [40]). We will prove that if  $w_1$  and  $w_2$  are different words deducible from  $w$ , then a finite number of application of rules to  $w_1$  and  $w_2$  yields to the same word (i.e., there is  $w'$  such that  $w_1 \rightarrow^* w'$  and  $w_2 \rightarrow^* w'$  where  $\rightarrow^*$  is the transitive closure of  $\rightarrow$ ).

- If  $w_1$  and  $w_2$  are obtained from  $w$  applying rules to two different disjoint sub-words of  $w$ , then applying the rules again in the opposite order, we obtain the same word  $w'$  and then the same normal form.
- Otherwise, suppose that  $w_1$  and  $w_2$  are obtained from  $w$  using respectively rules  $h$  and  $k$  (where  $h, k \in \{(a), (b), (c), (d)\}$ ), applied to the same sub-word  $X^n Y^m$  of  $w$ . Then surely  $\{h, k\} \neq \{(a), (b)\}$  and applying  $h$  and  $k$  respectively to  $w_2$  and  $w_1$ , we obtain the same word  $w'$ . In symbols:

$$w \xrightarrow{h} w_2 \xrightarrow{k} w' \tag{6.7}$$

$$w \xrightarrow{k} w_1 \xrightarrow{h} w'. \tag{6.8}$$

■

**Proposition 6.2.8** *If  $u$  is obtained from  $v$  using rules (a), (b), (c), (d), then  $u \equiv_k v$ .* ■

An immediate consequence is that every word over  $B$  is equivalent to its normal form.

We will refer to an element  $\alpha = [X_1^{m_1} \cdots X_n^{m_n}]$  written in normal form as the generic element of the quotient  $B^+/\equiv_k$ .

Using the operation  $\oplus$  and the order relation  $\leq_k$ , we can introduce the complement  $\neg[u]$  of an element  $[u]$  of  $B^+/\equiv_k$  as the least element  $[z] \in B^+/\equiv_k$  such that  $[u] \oplus [z] = [1^k]$ .

Taking normal forms as representatives of equivalence classes, we are able to describe the complement of an element of  $B^+/\equiv_k$  in a simple way and, at the same time, to prove its existence. Indeed if  $[w]$  is in normal form, say

$[w] = [X_1^{m_1} \cdots X_n^{m_n}]$ , it is easy to prove that

$$\neg[w] = \neg[X_1^{m_1} \cdots X_n^{m_n}] = \begin{cases} [X_0^k X_1^{k-m_1} \cdots X_n^{k-m_n}], & \text{if } m_n \neq k; \\ [X_0^k X_1^{k-m_1} \cdots X_{n-1}^{k-m_{n-1}}], & \text{if } m_n = k \end{cases} \quad (6.9)$$

where  $X_0 = 1 - \bigvee_{i=1}^n X_i$ .

**Lemma 6.2.9** *For any  $[u] \in B^+/\equiv_k$ , the identity  $[v] = \neg[u]$  holds if and only if for every  $X \in at(B_u)$ ,  $c_k(X, v) = \neg c_k(X, u)$ . ■*

Within the above algebraic context, we can formalize multiple bets described in the introduction. To this purpose let us fix a finite set  $\Omega$  (space of events) and let

$$\mathcal{B}_{(k)} = ((2^\Omega)^+/\equiv_k, \oplus, \neg, [\emptyset^k], [\Omega^k]). \quad (6.10)$$

Then an element of  $(2^\Omega)^+/\equiv_k$  will be called *multiple bet*. If Player A buys from the Bank (Player B) the multiple bet  $\alpha = [u] = [X_1 \cdots X_n]$  paying  $s(\alpha)$ , the *winning* given by elementary event  $x \in \Omega$  is the frequency  $c_k(x, u)$  of  $x$  in  $u$ . The total *gain* is given by the difference  $c_k(x, u) - s(\alpha)$ .

The relation  $\preceq_k$  over  $(2^\Omega)^+$  is such that  $X_1 \cdots X_n \preceq_k Y_1 \cdots Y_m$  if and only if for every  $x \in \Omega$ ,

$$\min\{k, \text{card}\{i \mid x \in X_i\}\} \leq \min\{k, \text{card}\{j \mid x \in Y_j\}\}. \quad (6.11)$$

We say that a multiple bet  $X_1 \cdots X_n$  is smaller than  $Y_1 \cdots Y_m$  provided that, whenever the elementary event  $x$  happens, the winning resulting from the first is less than the winning resulting from the second. Consequently, two multiple bets are equivalent if they led to the same winning.

A normal form for an element of  $(2^\Omega)^+$  has the form  $X_1^{m_1} \cdots X_n^{m_n}$  with  $X_i \subseteq \Omega$  and it represents a multiple bet such that if an elementary event  $x \in \Omega$  happens then Player A wins  $m_i$  if  $x \in X_i$ , for a suitable  $i$ , otherwise he wins 0. The complement of a multiple bet  $\alpha$  is the least bet  $\beta$  such that if a player plays on  $\alpha$  and  $\beta$  then the total winning is exactly 1.

More generally, for any boolean algebra  $(B, \vee, \wedge, \neg, 0, 1)$ , the structure  $\mathcal{B}_{(k)} = (B^+/\equiv_k, \oplus, \neg, \mathbf{0}, \mathbf{1})$  will be called the *algebra of  $k$ -bets*. We will denote by  $B_{(k)} = B^+/\equiv_k$  the underlying set of  $\mathcal{B}_{(k)}$ . The boolean algebra

$\mathcal{B} = (B, \vee, \wedge, \neg, 0, 1)$  can be easily embedded in such a structure by considering the function

$$i : X \in B \rightarrow [X^k] \in B^+ / \equiv_k . \quad (6.12)$$

Moreover for  $k = 1$ ,  $\mathcal{B}_{(k)}$  is isomorphic to  $\mathcal{B}$ .

**Definition 6.2.10** For every  $u, v \in B^+$  the conjunction  $\odot$  is defined by

$$[u] \odot [v] = \neg(\neg[u] \oplus \neg[v]). \quad (6.13)$$

From Lemmas 6.2.1(i) and 6.2.9 it follows that if  $[u] \odot [v] = [w]$  then for every  $X \in B_{uv}$ ,

$$c_k(X, w) = c_k(X, u) \odot c_k(X, v). \quad (6.14)$$

In the next subsection, using boolean powers, we will demonstrate that  $\mathcal{B}_{(k)}$  is an MV-algebra.

### 6.2.2 Boolean powers

Let us recall the definition of boolean power of an MV-algebra (see also [23], [46] and references therein):

**Definition 6.2.11** Let  $B$  be a boolean algebra and  $A$  a finite MV-algebra. The boolean power  $\mathbf{A}[B] = (A[B], \oplus, \neg, \mathbf{0}, \mathbf{1})$  is defined in the following way:

$$\begin{aligned} A[B] &= \{f \in B^A \mid f(a_1) \wedge f(a_2) = 0 \text{ if } a_1 \neq a_2 \text{ and } \bigvee_{a \in A} f(a) = 1\} \\ (f \oplus g)(x) &= \bigvee_{h \oplus k = x} f(h) \wedge g(k) \\ \neg f(x) &= f(\neg x) \end{aligned} \quad (6.15)$$

and where  $\mathbf{0}$  is the characteristic function of  $\{0\}$  and  $\mathbf{1}$  is the characteristic function of  $\{1\}$ , i.e.,

$$\mathbf{0}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \quad \mathbf{1}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1. \end{cases} \quad (6.16)$$

The boolean power of an MV-algebra is an MV-algebra. Further, if the boolean algebra is supposed to be complete then it is possible to define boolean powers for infinite MV-algebras.

Using the following theorem we will show that every algebra  $\mathcal{B}_{(k)}$  of  $k$ -bets is an MV-algebra and that every boolean power of the form  $\mathbf{S}_{k+1}[\mathbf{B}]$  where  $S_{k+1} = \{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\}$ , can be interpreted as an algebra of  $k$ -bets.

**Theorem 6.2.12** *The algebra of  $k$ -bets  $\mathcal{B}_{(k)}$  is isomorphic to the MV-algebra  $(S_{k+1}[B], \oplus, \neg, \mathbf{0}, \mathbf{1})$ . Thus, in particular,  $\mathcal{B}_{(k)}$  is an MV-algebra.*

**Proof.** We will construct an isomorphism

$$F : \mathcal{B}_{(k)} \rightarrow \mathbf{S}_{k+1}[\mathbf{B}] \quad (6.17)$$

starting from a homomorphism of semigroups  $G : B^+ \rightarrow S_{k+1}[B]$ .

For every  $u \in B^+$  we define the function  $\psi_u \in S_{k+1}[B]$  such that for every  $r \in S_{k+1}$ :

$$\psi_u(r) = \bigvee \{X \in at(B_u) \mid c_k(X, u) = r\}. \quad (6.18)$$

In case  $u = X_1^{m_1} \cdots X_n^{m_n}$  is in normal form, then  $\psi_u$  becomes

$$\psi_u(r) = \begin{cases} X_i, & \text{if } r = \frac{m_i}{k} \\ X_0 = 1 - \bigvee_{i=1}^n X_i, & \text{if } r = 0 \\ 0, & \text{otherwise.} \end{cases} \quad (6.19)$$

In the interpretation of multiple bets,  $\psi_u(r)$  is the set of elementary events for which Player  $A$  wins  $r$ .

**Fact 1.**  $\psi_u \in S_{k+1}[B]$ .

Indeed, if  $r \neq s \in S_{k+1}$  we have

$$\psi_u(r) \wedge \psi_u(s) = \quad (6.20)$$

$$\bigvee \{X \in at(B_u) \mid c_k(X, u) = r\} \wedge \bigvee \{Y \in at(B_u) \mid c_k(Y, u) = s\} = \quad (6.21)$$

$$\bigvee (X \wedge Y \mid X, Y \in at(B_u), c_k(X, u) = r, c_k(Y, u) = s) = 0. \quad (6.22)$$

since different atoms of  $B_u$  are always disjoint.

Further,  $\bigvee_{r \in S_{k+1}} \psi_u(r) = 1$ , because

$$\bigvee_{r \in S_{k+1}} \psi_u(r) = \bigvee_{r \in S_{k+1}} \left( \bigvee \{X \in at(B_u) \mid c_k(X, u) = r\} \right) = \bigvee_{X \in at(B_u)} X = 1. \quad (6.23)$$

**Fact 2.** The function  $G : u \in B^+ \rightarrow \psi_u \in S_{k+1}[B]$  is an epimorphism of semigroups.

Indeed, if  $u, v \in B^+$  then for every  $r \in S_{k+1}$ ,

$$(\psi_u \oplus \psi_v)(r) = \bigvee_{i \oplus j = r} (\psi_u(i) \wedge \psi_v(j)) = \quad (6.24)$$

$$\bigvee_{i \oplus j = r} \left( \bigvee \{X \mid X \in at(B_u) \text{ and } c_k(X, u) = i\} \wedge \quad (6.25)$$

$$\bigvee \{Y \mid Y \in at(B_v) \text{ and } c_k(Y, v) = j\} \right) = \quad (6.26)$$

$$\bigvee_{i \oplus j = r} \bigvee \{X \wedge Y \mid X \in at(B_u), Y \in at(B_v), c_k(X, u) = i, c_k(Y, v) = j\}. \quad (6.27)$$

If  $i$  and  $j$  are integers such that  $i \oplus j = r$ , and  $X \in at(B_u)$  and  $Y \in at(B_v)$  are such that  $X \wedge Y \neq 0$ ,  $c(X, u) = i$ ,  $c(Y, v) = j$ , then  $Z = X \wedge Y$  is an atom of  $B_{uv}$ , such that (by Lemma 6.2.1(ii)),  $c_k(Z, uv) = c_k(Z, u) \oplus c_k(Z, v) = r$ .

Conversely, let  $Z$  be an atom in  $B_{uv}$ , such that  $c_k(Z, uv) = r$ . Then (by Lemma 6.2.1(iv)), there exist  $X \in at(B_u), Y \in at(B_v)$  such that  $X \wedge Y = Z$ . By setting  $i = c(X, u)$  and  $j = c(Y, v)$  we have

$$r = c_k(Z, uv) = c_k(Z, u) \oplus c_k(Z, v) = c_k(X, u) \oplus c_k(Y, v) = i \oplus j. \quad (6.28)$$

Thus,

$$(\psi_u \oplus \psi_v)(r) = \bigvee \{Z \in at(B_{uv}) \mid c_k(Z, uv) = r\} = \psi_{uv}(r). \quad (6.29)$$

In order to show that  $G$  is surjective, let  $f : S_{k+1} \rightarrow B$  be an element of the boolean power  $\mathbf{S}_{k+1}[\mathbf{B}]$ . Then the element

$$u = f\left(\frac{1}{k}\right)f\left(\frac{2}{k}\right)^2 \dots f\left(\frac{k-1}{k}\right)^{(k-1)} f(1)^k \in B^+ \quad (6.30)$$

satisfies the identity  $f = \psi_u = G(u)$ .

**Fact 3.** The congruence  $\equiv_G$  associated to  $G$ , defined by  $u \equiv_G v$  if and only if  $G(u) = G(v)$ , coincides with the congruence  $\equiv_k$ .

Indeed, let  $u \equiv_G v$ , i.e., for every  $r \in S_{k+1}$ ,  $\psi_u(r) = \psi_v(r)$ . Then

$$\bigvee \{X \in at(B_u) \mid c_k(X, u) = r\} = \bigvee \{Y \in at(B_v) \mid c_k(Y, v) = r\}. \quad (6.31)$$

Let  $Z$  be an atom of  $B_{uv}$  such that  $c_k(Z, u) = r$ . Then there exists  $X \in at(B_u)$  such that  $Z \leq X$  and  $c_k(X, u) = r$ . By (6.31)

$$Z \leq \bigvee \{Y \in at(B_v) \mid c_k(Y, v) = r\} \quad (6.32)$$

whence there exists  $Y \in at(B_v)$  such that  $Z \leq Y$  and  $c_k(Y, v) = r$ . Consequently,  $c_k(Z, v) = r = c_k(Z, u)$  and this proves that  $u \equiv_k v$ .

Vice-versa assume that for every atom  $Z$  of  $B_{uv}$  we have  $c_k(Z, u) = c_k(Z, v)$ . Then,

$$\bigvee \{X \in at(B_u) \mid c_k(X, u) = r\} = \bigvee \{Z \in at(B_{uv}) \mid c_k(Z, u) = r\} = \quad (6.33)$$

$$\bigvee \{Z \in at(B_{uv}) \mid c_k(Z, v) = r\} = \bigvee \{Y \in at(B_v) \mid c_k(Y, v) = r\}. \quad (6.34)$$

This proves that  $u \equiv_G v$ .

**Fact 4.** The map  $F : [w] \in B^+ / \equiv_k \rightarrow G(w) \in S_{k+1}[B]$  is an isomorphism of MV-algebras.

Indeed, since  $\equiv_k$  and  $\equiv_G$  coincide, it follows that  $F$  is an isomorphism of semigroups. We have  $F(\mathbf{0}) = G(0^k) = \psi_{0^k}$ , where

$$\psi_{0^k}(r) = \bigvee \{X \in at(B_{0^k}) \mid c_k(X, 0^k) = r\}. \quad (6.35)$$

Since  $B_{0^k} = \{0, 1\}$  then  $\psi_{0^k}$  is the characteristic function of 0. Let us denote  $\neg[w]$  by  $[z]$ . We then have

$$F(\neg[w]) = F([z]) = G(z) = \psi_z. \quad (6.36)$$

By Lemma 6.2.9, for every  $r \in S_{k+1}[B]$  we get

$$\psi_z(r) = \bigvee \{X \in at(B_z) \mid c(X, z) = r\} = \quad (6.37)$$

$$\bigvee \{X \in at(B_w) \mid c(X, w) = \neg r\} = \psi_w(\neg r). \quad (6.38)$$

■

From Definition 6.2.10 it follows that the operation  $\odot$  is the Łukasiewicz conjunction in the MV-algebra  $\mathcal{B}_{(k)}$ . From (6.14) we have

$$[u] \odot [v] = \mathbf{0} \Leftrightarrow c_k(X, u) \odot c_k(X, v) = \mathbf{0} \quad (6.39)$$

for every  $X \in at(B_{uv})$ .

### 6.2.3 Subjective states

De Finetti in [39] used the idea of *fair betting system* as a foundation for the theory of probability (see also [75, 97]). A betting system is a set of events and rates fixed by the bank. A player bet over events and win in accordance with rates. The betting system is said to be unfair if, no matter which event occurs, the player always wins or always loses. If the distribution of rates satisfies the probability rules, then there does not exist any set of bets for which the player or the bank always wins (Dutch book theorem), and the game is fair.

In [96], the author generalizes the classical Dutch Book argument to probability functions for various non-standard propositional logics for example modal, intuitionistic and paraconsistent logics. In [54] we extend this argument to finite-valued Lukasiewicz logic, while the problem for infinite-valued logic is still matter of investigation.

Let us consider the MV-algebra of  $k$ -bets over the finite boolean algebra  $2^\Omega$ ,

$$\mathcal{B}_{(k)} = (B_{(k)} = (2^\Omega)^+ / \equiv_k, \oplus, \neg, [\emptyset^k], [\Omega^k]). \quad (6.40)$$

An element of  $B_{(k)}$  has the form  $[X_1^{m_1} \cdots X_n^{m_n}]$  with  $X_i \subseteq \Omega$  disjoint events and  $(m_i)_{i \in I}$  strictly increasing sequence of positive integers  $\leq k$ .

**Definition 6.2.13** *A subjective quotation in a multiple bets game over the space of events  $\Omega$ , is a function over the MV-algebra  $\mathcal{B}_{(k)}$  of  $k$ -bets*

$$s : B_{(k)} \rightarrow [0, 1] \quad (6.41)$$

**Definition 6.2.14** *A favourable (resp., unfavourable) Dutch book for a subjective MV-quotation  $s$ , is a set of multiple bets  $T$  (that is, a subset of  $B_{(k)}$ ) such that for every  $x \in \Omega$*

$$\sum_{w \in T} s(w) - c_k(x, w) < 0 \quad (\text{resp., } > 0). \quad (6.42)$$

In other words, a favourable (resp., unfavourable) Dutch book  $T$  for a quotation  $s$  is a set of bets such that whatever elementary events  $x$  in  $\Omega$  occurs, Player A wins more (resp., less) than he has paid.

A subjective quotation for which it is not possible to construct a Dutch book, will be called a *coherent quotation*.

**Theorem 6.2.15** *Any coherent quotation satisfies axioms (i), (ii) and (iii) in the definition of states and is therefore a state.*



**Proof.** Let  $s$  be a coherent quotation. First of all note that if  $s([\emptyset^k]) > 0$  then  $\{[\emptyset^k]\}$  would be a favourable Dutch book for  $s$ , since

$$s([\emptyset^k]) - c_k(x, \emptyset^k) = s([\emptyset^k]) - 0 > 0. \quad (6.43)$$

So we have

$$s([\emptyset^k]) = 0. \quad (6.44)$$

Consider a multiple bet  $\alpha = [u]$  and its complement  $\neg\alpha = [w]$ . If  $s([u]) + s([w]) < 1$  then from Lemma 6.2.9 and from (6.9), we get, for every  $x \in S$ ,  $c_k(x, u) = 1 - c_k(x, w)$  and hence

$$s([u]) - c_k(x, u) + s([w]) - c_k(x, w) = s([u]) + s([w]) - 1 < 0. \quad (6.45)$$

Symmetrically, if  $s([u]) + s([w]) > 1$  we are similarly led to an unfavourable Dutch book. So for a coherent quotation we have

$$s(\alpha) + s(\neg\alpha) = 1. \quad (6.46)$$

Further,  $s(1) = s([\Omega^k]) = s(\neg[\emptyset^k]) = 1$ .

Let us consider the case of two disjoint bets  $\alpha = [u]$  and  $\beta = [v]$ , and let  $\alpha \oplus \beta = [w]$  and  $\neg(\alpha \oplus \beta) = [w']$ . By Definition 6.2.10, for every  $x \in \Omega$

$$c_k(x, u) \odot c_k(x, v) = c_k(x, \emptyset^k) = 0 \quad (6.47)$$

whence

$$c_k(x, u) \oplus c_k(x, v) = c_k(x, u) + c_k(x, v). \quad (6.48)$$

Suppose

$$s([u]) + s([v]) + s([w']) < 1. \quad (6.49)$$

From equation (6.48) and Lemmas 6.2.1(i), 6.2.9, for every  $x \in \Omega$  we have

$$c_k(x, w') = 1 - (c_k(x, u) \oplus c_k(x, v)) = 1 - c_k(x, u) - c_k(x, v), \quad (6.50)$$

hence

$$s([u]) + s([v]) + s([w']) - c_k(x, u) - c_k(x, v) - c_k(x, w') < 0. \quad (6.51)$$

So in case  $s([u]) + s([v]) + s([w']) < 1$ ,  $\{[u], [v], [w']\}$  would be a Dutch book.

Symmetrically we get

$$s(\alpha) + s(\beta) + s(\neg(\alpha \oplus \beta)) = 1. \quad (6.52)$$

Thus, using (6.46),

$$s(\alpha) + s(\beta) = s(\alpha \oplus \beta). \quad (6.53)$$

■

### 6.2.4 States on DMV-algebras

The machinery of the previous sections can be used also when considering DMV-algebras instead of MV-algebras [56]. Indeed if  $A$  is a DMV-algebra, then a state of  $A$  is a function  $s : A \rightarrow [0, 1]$  such that  $s$  is a state of the MV-algebra  $A^*$ . One can prove that for every  $x \in A$

$$s(\delta_n x) = \frac{s(x)}{n}. \quad (6.54)$$

Then the set of bets can be constructed in the following way: let us denote by  $\mathcal{B}$  the set

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} B^n \times \mathbb{Q}^n.$$

Let  $k > 0$  be a natural number. For every  $u = ((X_1 \cdots X_n), (r_1, \dots, r_n)) \in \mathcal{B}$  and  $x \in \Omega$  let

$$c_k(x, u) = \frac{\min\{k, \sum_{x \in X_i} r_i\}}{k}.$$

Note that  $c_k(x, u)$  is an element of the DMV-algebra  $\mathbb{Q} \cap [0, 1]$ .

**Definition 6.2.16** *Two elements  $u, v \in \mathcal{B}$  are k-equivalent (and we write  $u \equiv_k v$ ) if for every  $x \in \Omega$ ,  $c_k(x, u) = c_k(x, v)$ .*

Elements of  $\mathcal{B}/\equiv_k$  are said *multiple bets*. If Player A buys from Player B the multiple bet  $\alpha = [u]$  paying  $s(\alpha)$ ,  $c_k(x, u)$  is the *winning* (given by elementary event  $x \in \Omega$ ), of  $x$  in  $u$ . The total *gain* is given by the difference  $c_k(x, u) - s(\alpha)$ . Two multiple bets are equivalent if they led to the same winning.

The following operations in  $\mathcal{B}/\equiv_k$  are well defined. Let

$$\begin{aligned} u &= ((X_1, \dots, X_n), (r_1, \dots, r_n)) \\ v &= ((Y_1, \dots, Y_n), (s_1, \dots, s_n)) \end{aligned}$$

belong to  $\mathcal{B}$ . Then we set

- $[u] \oplus [v] = [((X_1, \dots, X_n, Y_1, \dots, Y_n), (r_1, \dots, r_n, s_1, \dots, s_n))]$
- $\neg[u] = [w]$  where  $w$  is an element of  $\mathcal{B}$  such that for every  $x \in \Omega$ ,  $c_k(x, w) = 1 - c_k(x, u)$
- $\delta_n[u] = [z]$  where  $z$  is an element of  $\mathcal{B}$  such that for every  $x \in \Omega$ ,  $c_k(x, z) = \frac{c_k(x, u)}{n}$

Let  $\mathbf{0} = [((\emptyset, \dots, \emptyset), (r_1, \dots, r_n))] = [((X_1, \dots, X_n), (0, \dots, 0))]$  and

$$\mathbf{1} = [\underbrace{((\Omega, \dots, \Omega))}_{k \text{ times}}, \underbrace{(1, \dots, 1)}_{k \text{ times}}].$$

Clearly, for every  $x \in \Omega$ ,  $c_k(x, \mathbf{0}) = 0$  and  $c_k(x, \mathbf{1}) = 1$ . Further, if  $[w] = [u] \oplus [v]$  then, for every  $x \in \Omega$ ,  $c_k(x, w) = c_k(x, u) \oplus c_k(x, v)$  and if  $[u] \odot [v] = [w]$  then for every  $x \in \Omega$ ,  $c_k(x, w) = c_k(x, u) \odot c_k(x, v)$ .

The structure  $\mathcal{B}_{(k)} = (\mathcal{B}/\equiv_k, \oplus, \neg, \{\delta_n\}_{n \in \mathbb{N}}, \mathbf{0}, \mathbf{1})$  is the algebra of  $k$ -bets.

Following the same guidelines as in [54] one can prove that  $\mathcal{B}_{(k)}$  is isomorphic to the Boolean power  $(\mathbb{Q} \cap [0, 1])[B]$  ([23, 46]) and so it is a DMV-algebra. Indeed, recall that the boolean power  $(\mathbb{Q} \cap [0, 1])[B] = ((\mathbb{Q} \cap [0, 1])[B], \oplus, \neg, \mathbf{0}, \mathbf{1})$  is the set of all function  $f : \mathbb{Q} \cap [0, 1] \rightarrow B$  such that

$$f(a_1) \wedge f(a_2) = 0 \text{ if } a_1 \neq a_2 \text{ and } \bigvee_{a \in \mathbb{Q} \cap [0, 1]} f(a) = 1$$

equipped with operations

$$\begin{aligned} (f \oplus g)(x) &= \bigvee_{h \oplus k = x} f(h) \wedge g(k); \neg f(x) = f(\neg x), \\ \mathbf{0}(x) &= \begin{cases} \Omega & \text{if } x = 0 \\ \emptyset & \text{if } x \neq 0 \end{cases} \\ \mathbf{1}(x) &= \begin{cases} \Omega & \text{if } x = 1 \\ \emptyset & \text{if } x \neq 1. \end{cases} \end{aligned}$$

The boolean power of a DMV-algebra is a DMV-algebra. Then the function  $\Psi : [u] \in \mathcal{B}/\equiv_k \rightarrow \Psi(u) \in (\mathbb{Q} \cap [0, 1])[B]$  such that

$$\Psi(u) : r \in \mathbb{Q} \cap [0, 1] \mapsto \{x \in \Omega \mid c_k(x, u) = r\}$$

is an isomorphism between the algebra of  $k$ -bets and the boolean power  $(\mathbb{Q} \cap [0, 1])[B]$ .

Again it is possible to prove that states on  $\mathcal{B}$  are coherent quotation and further it is also possible to check directly that any coherent quotation has to satisfy (6.54). Indeed if otherwise  $s$  is a coherent quotation such that  $s(\delta_n[u]) > s([u])/n$  and  $[w] = \delta_n[u]$ , then one can prove that the multiset formed by  $[u]$  and by  $n$  repetition of  $[w]$  is a unfavorable Dutch book, and, symmetrically, if  $s(\delta_n[u]) < s([u])/n$  the same multiset is an favorable Dutch book.

### 6.3 Conditioning a state given an MV-event

We extend the notion of conditional states proposed in [43] in light of [101]. For any MV-algebra  $A$  we say that  $B \subseteq A$  is an *MV-bunch* if  $1 \in B$ ,  $0 \notin B$  and  $B$  is closed under  $\oplus$  operation. A typical example of MV-bunch is obtained by considering the set  $B = B_s = \{x \in A \mid s(x) \neq 0\}$  where  $s$  is any state: in this case we will say that  $B$  is the *MV-bunch of  $s$* . For instance the MV-bunch of the state  $p^\sharp$  of Proposition 6.1.2, is given by  $B_p = \{\mu \in L^X \mid \exists x \in \text{Supp}(\mu), p(x) \neq 0\} = \{\mu \in L^X \mid p^\sharp(\mu) \neq 0\}$ .

**Definition 6.3.1** *A conditional state  $s(x \mid y)$  of an MV-algebra  $A$  is a function  $s : A \times B \rightarrow [0, 1]$ , where  $B \subseteq A$  is an MV-bunch, satisfying the following conditions:*

- (i)  $s(- \mid y)$  is a state on  $A$  for every  $y \in B$ , ;
- (ii)  $s(y \mid y) = 1$  for every  $y \in B(A) \cap B$ ;
- (iii)  $s(x \odot y \mid z) = s(y \mid z)s(x \mid y \odot z)$ ; for any  $x \in A$ ,  $y \in B(A)$ ,  $z \in B(A) \cap B$  such that  $y \odot z \in B$  then
- (iv)  $s(x \mid y)s(y \mid 1) = s(y \mid x)s(x \mid 1)$ , for any  $x, y \in B$ .

In this section we shall address the following problem: given a state  $s$  and events  $\mu$  and  $\nu$ , what is the conditional state  $s(\mu \mid \nu)$ ? Throughout this section we set  $L = S_{k+1}$ .

Let  $s$  be a state on  $L^X$  and let  $\mu$  be an element in the MV-bunch of  $s$ , i.e.  $s(\mu) \neq 0$ . We define the state  $s_\mu$  on the MV-algebra  $L^X$  by setting

$$s_\mu(\nu) = \sum_{x \in X} \frac{s(x)\nu(x)\mu(x)}{s(\mu)}, \quad (6.55)$$

for all  $\nu \in L^X$ . In other words, in accordance with Proposition 6.1.2,  $s_\mu$  is the state extending the probability whose distribution is  $\frac{s(x)\mu(x)}{s(\mu)}$ . By Proposition 6.1.3, such extension is unique.

**Proposition 6.3.2** *Let  $B$  be the MV-bunch of a state  $s$  and define  $s : L^X \times B \rightarrow [0, 1]$  by setting  $s(\nu, \mu) = s_\mu(\nu)$ . Then  $s(- \mid -)$  is a conditional state of  $L^X$ .*

**Proof.** First of all, as we have seen above, for every  $\mu \in B$ ,  $s(-|\mu) = s_\mu$  is a state. Further,

$$s(\nu | 1) = \sum_{x \in X} \frac{\nu(x)s(x)}{s(1)} = s(\nu),$$

and so  $s(-, 1)$  is a state. Secondly, if  $X \in B(L^X) \cap B$  then

$$s(X | X) = \sum_{x \in X} \frac{X(x)s(x)}{s(X)} = \sum_{x \in X} \frac{s(x)}{s(X)} = \frac{s(X)}{s(X)} = 1.$$

Thirdly, whenever  $\mu \in L^X$ ,  $X \in B(L^X) = 2^X$ ,  $Z \in 2^X \cap B$ , and  $X \odot Z = X \cap Z \in B$ , then

$$(X \odot \mu)(x) = \min\{X(x) + \mu(x) - 1, 0\} = \begin{cases} \mu(x) & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, we have the identities

$$s(X \odot \mu | Z) = \sum_{x \in X} (X \odot \mu)(x) \frac{Z(x)s(x)}{s(Z)} = \sum_{x \in X \cap Z} \frac{\mu(x)s(x)}{s(Z)}$$

and

$$\begin{aligned} s(X | Z)s(\mu | X \odot Z) &= \frac{s(X \cap Z)}{s(Z)} \cdot \sum_{x \in X \cap Z} \frac{\mu(x)s(x)}{s(X \cap Z)} \\ &= \sum_{x \in X \cap Z} \frac{\mu(x)s(x)}{s(Z)} \end{aligned}$$

Then  $s(X \odot \mu | Z) = s(X | Z)s(\mu | X \odot Z)$ .

To conclude the proof, if  $\mu, \eta \in B$  then we can write

$$\begin{aligned} s(\mu | \eta) \cdot s(\eta | 1) &= \sum_{x \in 1} \frac{\mu(x)\eta(x)s(x)}{s(\eta)} \cdot \sum_{x \in 1} \eta(x)s(x) \\ &= \sum_{x \in 1} \mu(x)\eta(x)s(x) \end{aligned}$$

$$\begin{aligned}
s(\eta | \mu) \cdot s(\mu | 1) &= \sum_{x \in 1} \frac{\eta(x)\mu(x)s(x)}{s(\mu)} \cdot \sum_{x \in 1} \mu(x)s(x) \\
&= \sum_{x \in 1} \eta(x)\mu(x)s(x).
\end{aligned}$$

Thus,  $s(\mu | \eta) \cdot s(\eta | 1) = s(\eta | \mu) \cdot s(\mu | 1)$ . ■

A basic property of the classical conditioning for a probability  $p$  is the iteration rule

$$p(x | y \cap z) = \frac{p(x \cap y | z)}{p(y | z)},$$

for every set  $x$ ,  $y$  and  $z$ . This identity is an immediate consequence of the definition  $p(x | y) = \frac{p(x \cap y)}{p(y)}$ .

By contrast, since condition (iii) of Definition 6.3.1 holds for all  $y$  and  $z$  in the boolean skeleton of the MV-algebra, the same rule does not hold for conditional states, in general. A counterexample is given by the conditional state defined in (6.55). So, given a state  $s$ , in accordance with the classical case, one might try to define the state conditioned by an MV-event  $\mu$  by setting

$$s(\nu | \mu) = \frac{s(\nu \odot \mu)}{s(\mu)}, \quad (6.56)$$

with  $\nu$  in  $L^X$ . Due to the associativity of  $\odot$ , such function satisfies the iteration rule. But (6.56) is not a conditional state, since  $\odot$  is not distributive with respect to  $\oplus$ . We weaken the definition of state and hence of conditional state:

**Definition 6.3.3** *A quasi-state on a MV-algebra  $A$ , is a function  $q : A \rightarrow [0, 1]$  such that:*

- (q-i)  $q$  is monotone;
- (q-ii)  $q(0) = 0$ ;
- (q-iii)  $q(1) = 1$ ;
- (q-iv) whenever  $a, b \in A$  and  $a \wedge b = 0$ , then  $q(a \oplus b) = q(a) + q(b)$ .

Since  $a \wedge b = 0$  implies  $a \odot b = 0$ , every state is a quasi-state. Further the restriction of a quasi-state to the boolean skeleton of  $A$  is a probability.

A quasi-state can be canonically constructed starting from a distribution on the set of  $L$ -singletons, as in classical probability theory. By a  $q$ -s-distribution we mean a function  $q^\diamond$  defined on the  $L$ -singletons such that

- i)  $q^\diamond$  is monotone;
- ii)  $\sum_{x \in X} q^\diamond(\langle x, 1 \rangle) = 1$ .

**Theorem 6.3.4** *A function  $q : L^X \rightarrow [0, 1]$  is a quasi-state if and only if  $q(0) = 0$  and there exists a  $q$ -s-distribution  $q^\diamond$  such that, whenever  $\mu \neq 0$ ,*

$$q(\mu) = \sum_{x \in X} q^\diamond(\langle x, \mu(x) \rangle).$$

**Proof.** Let  $q$  be a quasi-state. Then its restriction  $q^\diamond$  to  $L$ -singletons is a  $q$ -s-distribution. Since, for every  $\mu \in L^X$ ,  $\mu = \bigoplus_{x \in X} \langle x, \mu(x) \rangle$  and  $\langle x, \mu(x) \rangle \wedge \langle y, \mu(y) \rangle = 0$  for every  $x, y \in X$ ,  $x \neq y$ , we have

$$q(\mu) = \sum_{x \in X} q^\diamond(\langle x, \mu(x) \rangle)$$

Conversely, let  $q^\diamond$  be a  $q$ -s-distribution and define  $q$  by setting  $q(0) = 0$  and, for  $\mu \neq 0$ ,  $q(\mu) = \sum_{x \in X} q^\diamond(\langle x, \mu(x) \rangle)$ . We claim that  $q$  is a quasi state. Indeed, (q-ii) and (q-iii) are immediate. Moreover,

(q-i) if  $\alpha, \beta \in L^X$  and  $\alpha \leq \beta$ , then  $\langle x, \alpha(x) \rangle \leq \langle x, \beta(x) \rangle$  for every  $x \in X$ , and so (for (i)):

$$q(\alpha) = \sum_{x \in X} q^\diamond(\langle x, \alpha(x) \rangle) \leq \sum_{x \in X} q^\diamond(\langle x, \beta(x) \rangle) = q(\beta),$$

(q-iv) if  $\alpha, \beta \in L^X$  such that  $\alpha \wedge \beta = 0$  then, denoting by  $X_\alpha$  and  $X_\beta$  the support of  $\alpha$  and  $\beta$  respectively, we have  $X_\alpha \cap X_\beta = \emptyset$ . In conclusion

$$q(\alpha \oplus \beta) = \sum_{x \in X_\alpha} q^\diamond(\langle x, \alpha(x) \rangle) + \sum_{x \in X_\beta} q^\diamond(\langle x, \beta(x) \rangle) = q(\alpha) + q(\beta).$$

■

In particular, if  $p$  is a probability on  $X$ , then by setting

$$q^\diamond(\langle x, \mu(x) \rangle) = p(x) \cdot \mu(x)$$

we obtain the state  $s$  defined in [116].

Notice that the restriction of a  $q$ -s-distribution to the  $L$ -singletons of the form  $\langle x, 1 \rangle$  defines a distribution of probability. Moreover, different  $q$ -s-distributions can define the same probability and this shows that the uniqueness proved for states in Proposition 6.1.3 cannot be extended to the quasi states.

Analogously with Definition 6.3.1 we give the following:

**Definition 6.3.5** *A conditional quasi-state  $q(x | y)$  of an MV-algebra  $A$  is a function  $q : A \times B \rightarrow [0, 1]$ , where  $B \subseteq A$  is an MV-bunch, satisfying the following conditions:*

- (i)  $q(- | y)$  is a quasi-state on  $A$ , for every  $y \in B$ ;
- (ii)  $q(y | y) = 1$  for every  $y \in B(A) \cap B$ ;
- (iii)  $q(x \odot y | z) = q(y | z)q(x | y \odot z)$  for any  $x \in A$  and  $y, z \in B(A)$ , such that  $y \odot z \in B$ ;
- (iv)  $q(x | y)q(y | 1) = q(y | x)q(x | 1)$ , for any  $x, y \in B$ .

Note that in condition (iii) the operation  $\odot$  involves always a boolean element.

An interesting class of conditional quasi-states is given by the following Proposition whose proof is routine.

**Proposition 6.3.6** *Let  $q$  be a quasi-state on  $X$ ,  $\otimes$  a  $t$ -norm (see Definition 1.2.1) and  $\beta$  an element in the MV-bunch of  $q$ . Then the function  $q_{(\beta)}^\diamond$  defined by:*

$$q_{(\beta)}^\diamond(\langle x, \lambda \rangle) = \frac{q(\langle x, \beta(x) \otimes \lambda \rangle)}{q(\beta)}$$

is a  $q$ -s-distribution. Moreover, let  $q_\beta^\otimes$  be the quasi-state associated with  $q_{(\beta)}^\diamond$ , and define the function  $q^\otimes(- | -)$  by setting  $q^\otimes(\mu | \beta) = q_\beta^\otimes(\mu)$ . Then  $q^\otimes$  is a conditional quasi-state on  $L^X$  and

$$q^\otimes(\mu | \beta) = \frac{\sum_{x \in X} q(\langle x, \beta(x) \otimes \mu(x) \rangle)}{q(\beta)}.$$



■  
**Remark.** If  $q$  is a state and the t-norm is the usual product, then  $q^\otimes(-|_-)$  coincides with the conditional state defined in (6.55). Notice that, by defining in an obvious way the  $\otimes$ -intersection of two  $L$ -subsets, we have

$$q_\beta^\otimes(\mu) = \frac{q(\beta \otimes \mu)}{q(\beta)}. \quad (6.57)$$

As a consequence,

$$(q_\beta^\otimes)_\gamma = q_{\beta \otimes \gamma}^\otimes,$$

and therefore the iteration rule always holds for  $q^\otimes$  provided that we refer to  $\otimes$ -intersections.

### 6.3.1 Conditional states and Dempster's rule

Let  $\otimes$  be a t-norm,  $p$  a probability and  $\beta$  an  $L$ -subset. Then by setting  $s$  equal to  $p^\sharp$  as in Proposition 6.1.2, we can consider the quasi-state  $s_\beta^\otimes$ . Also, for any state  $s$  and t-norm  $\otimes$ , the conditional quasi-state  $s_\beta^\otimes$  has the same restriction to the boolean skeleton of  $L^X$  as the conditional state defined in (6.55). Such restriction is the probability given by the distribution

$$s_\beta^\otimes(x) = \frac{s(x)\beta(x)}{s(\beta)}. \quad (6.58)$$

This gives a way to compose a probability  $p$  with a possibility  $\beta$  thus obtaining a probability. In this section we examine the relationship between such probability and the Dempster composition rule in the theory of the belief functions [106].

Let  $\Omega$  be a set. A function  $m : 2^\Omega \rightarrow [0, 1]$  such that

- (1)  $\sum_{X \subseteq \Omega} m(X) = 1$
- (2)  $m(\emptyset) = 0$ ,

is called a *mass distribution* on the *frame*  $\Omega$  and the subsets  $X$  of  $\Omega$  such that  $m(X) > 0$  are called *focal events* of  $m$ . The function  $Bel : 2^\Omega \rightarrow [0, 1]$  defined by setting for any  $E \subseteq \Omega$

$$Bel(E) = \sum_{X \subseteq E} m(X)$$

is called *belief function* (or *lower probability*) associated with  $m$ . The function  $Bel^*$  defined by

$$Bel^*(E) = \sum_{X \cap E \neq \emptyset} m(X)$$

is called *upper probability* of  $m$ . If  $m$  is a mass such that its focal events are singletons of  $\Omega$ , then the functions  $Bel$  and  $Bel^*$  coincide with the probability whose distribution is  $m$ .

Let  $\beta$  be an  $L$ -subset of  $\Omega$  and let  $C(\beta, \lambda) = \{x \in \Omega \mid \beta(x) \geq \lambda\}$ . Then the function  $m_\beta : 2^\Omega \rightarrow [0, 1]$  such that:

$$m_\beta(X) = \begin{cases} \frac{1}{k+1}, & \text{if } X = C(\beta, \lambda) \\ 0, & \text{otherwise} \end{cases}$$

is a mass. In this case the upper probability is given by

$$\begin{aligned} Bel^*(X) &= \sum_{C(\beta, \lambda) \cap X \neq \emptyset} m(C(\beta, \lambda)) = \sum \left\{ \frac{1}{k+1} \mid \sup_{x \in X} \beta(x) \geq \lambda \right\} = \\ &= \frac{\sup_{x \in X} \beta(x)(k+1)}{k+1} = \sup_{x \in X} \beta(x). \end{aligned}$$

Both the  $L$ -subset  $\beta$  and the related upper probability  $Bel^*$  will be called *possibility*. The following Dempster composition rule enable us to combine two masses:

**Definition 6.3.7** Let  $m_1$  and  $m_2$  be two masses on the same frame  $\Omega$ , with focal events  $A_1, \dots, A_k$  and  $B_1, \dots, B_l$  respectively and suppose that (compatibility condition)

$$\sum_{A_i \cap B_j = \emptyset} m_1(A_i)m_2(B_j) < 1 \quad (6.59)$$

Then the function  $m : 2^\Omega \rightarrow [0, 1]$  defined by  $m(\emptyset) = 0$  and

$$m(A) = \frac{\sum_{A_i \cap B_j = A} m_1(A_i)m_2(B_j)}{1 - \sum_{A_i \cap B_j = \emptyset} m_1(A_i)m_2(B_j)}$$

for all non-empty  $A \subset \Omega$  is called *Dempster composition* of  $m_1$  and  $m_2$ .

Note that the Dempster composition of two masses is a mass. The Dempster composition of (the distribution of) a probability  $p$  and another

mass  $m_2$ , for which the compatibility condition holds, is still a probability. Indeed, if the compatibility condition holds:

$$m(A) = \frac{\sum_{\{x_i\} \cap B_j = A} p(x_i) m_2(B_j)}{1 - \sum_{\{x_i\} \cap B_j = \emptyset} p(x_i) m_2(B_j)}$$

so  $m$  is different from zero only on singletons. More precisely, writing  $m(x)$  instead of  $m(\{x\})$  we have

$$m(x) = \frac{\sum_{x \in B_j} p(x) m_2(B_j)}{1 - \sum_{x \notin B_j} p(x) m_2(B_j)}$$

**Proposition 6.3.8** *Let  $p$  be a probability on  $\Omega$ ,  $\otimes$  be a  $t$ -norm,  $q$  a quasi-state extending  $p$ , and  $\beta$  a possibility belonging to the MV-bunch of  $q$ . Then the restriction of  $q^{\otimes}(-, \beta)$  to the skeleton of  $L^\Omega$  is the Dempster composition of the probability  $p$  and the possibility  $\beta$ .*

**Proof.** First of all we have to prove that condition (6.59) is satisfied. Let us denote by  $x_i$  with  $i = 1, \dots, n$  the focal events of  $p$ , and by  $C_j = C(\beta, \frac{j}{k+1})$  with  $j = 1, \dots, k$  the focal events of the mass  $m_\beta$ . Then (6.59) becomes

$$\sum_{\{x_i\} \cap C_j = \emptyset} p(x_i) m(C_j) = \sum_{x_i \notin C_j} \frac{p(x_i)}{k+1} < 1.$$

and hence

$$\sum_{x_i \in C_j} \frac{p(x_i)}{k+1} > 0.$$

On the other hand,

$$\sum_{x_i \in C_j} \frac{p(x_i)}{k+1} = \sum_j \frac{p(C_j)}{k+1}.$$

Noting that in  $\sum_j p(C_j)$  every  $p(x)$  is repeated exactly  $\beta(x) \cdot (k+1)$  times we get

$$\sum_j \frac{p(C_j)}{k+1} = \sum_{x \in \Omega} \frac{p(x) \beta(x) (k+1)}{k+1} = \sum_{x \in \Omega} p(x) \beta(x) > 0.$$

Then the desired composition is given by

$$\frac{\sum_{x \in C_j} p(x) m(C_j)}{1 - \sum_{x_i \notin C_j} p(x_i) m(C_j)} = \frac{\sum_{x \in C_j} \frac{p(x)}{k+1}}{\sum p(x) \beta(x)} =$$

$$= \frac{\sum_j p(C_j)}{(k+1) \sum p(x)\beta(x)} = \frac{p(x)\beta(x)(k+1)}{(k+1) \sum p(x)\beta(x)}.$$

■

Thus, all conditional quasi-states given by Definition 6.3.6 are compatible with the Dempster composition rule.

### 6.3.2 Ulam game

In his book “Adventures of a Mathematician” [113], Ulam describes the following game between two players  $A$  and  $B$ : Player  $B$  chooses a secret number  $x$  in a finite set  $X$ , and Player  $A$  must guess  $x$  by a suitable sequence of questions to which  $B$  can only answer yes or no — being allowed to lie in at most  $k$  of these answers. Here, by a *question*  $Q$ , we understand a subset of  $X$ . The problem is to find strategies for  $A$  that minimize the number of questions in the worst cases, i.e. whatever is the initial choice of the secret number and whatever is the behavior of  $B$  (see [88] and references therein). In case all questions are asked independently of the answers, optimal searching strategies in this game are the same as optimal  $k$ -error-correcting coding strategies (see [18]). Now, in the particular case when  $k = 0$  (corresponding to the familiar game of Twenty Questions) all that Player  $A$  knows about  $x$  is represented by the conjunction in the classical propositional calculus of all the pieces of information obtained from the answers of Player  $B$ . In case  $k > 0$  classical logic no longer yields a natural formalization of the answers. As shown in [88] (and references therein) one may more conveniently use the  $(k+2)$ -valued sentential calculus of Łukasiewicz [79], [32]. In fact, Player  $A$  can record the current knowledge of the secret number by taking the Łukasiewicz conjunction of the pieces of information contained in the answers of  $B$ .

More precisely, let  $L = S_{k+1}$ . For every question  $Q \subseteq X$ , the *positive  $L$ -answer* to  $Q$  is the  $L$ -set  $Q^{yes} : X \rightarrow L$  given by

$$Q^{yes}(y) = \begin{cases} 1, & \text{if } y \in Q; \\ \frac{k}{k+1}, & \text{if } y \notin Q. \end{cases}$$

Elements  $y \in X$  such that  $Q^{yes}(y) = 1$  are said to *satisfy  $L$ -answer  $Q^{yes}$* ; the remaining elements *falsify* the answer.

The *negative  $L$ -answer  $Q^{no}$*  to  $Q$  is the same as the positive answer to the *opposite question*  $\overline{Q} = X - Q$ , in symbols,

$$Q^{no} = \overline{Q^{yes}}.$$

The dependence of  $Q^{yes}$  and  $Q^{no}$  on the actual value of  $k$  is tacitly understood. By definition, the  $L$ -subset  $\mu : X \rightarrow L$  of possible numbers resulting after a sequence of questions  $Q_1, \dots, Q_n$  with their respective answers  $b_1, \dots, b_n$  ( $b_i \in \{yes, no\}$ ), is the Łukasiewicz intersection

$$\mu_n = Q_1^{b_1} \odot \dots \odot Q_n^{b_n}.$$

By definition the  $L$ -subset resulting after the empty sequence of questions is the function constantly equal to 1 over  $X$ . As we will show in the next proposition, we can interpret  $\mu$  as the  $L$ -subset of possible numbers. Initially all numbers are possible and have maximum “truth value” 1 (we have no information), in the final step only one number is possible (we have maximum information). The following Proposition is routine (see [32]).

**Proposition 6.3.9** *Let  $x \in X$  and let  $\mu_n$  be the  $L$ -subset of possible numbers resulting after the questions  $Q_1, \dots, Q_n$  and the answers  $b_1, \dots, b_n$  ( $b_i \in \{yes, no\}$ ). Then:*

$$\mu_n(x) = \begin{cases} 1 - \frac{i}{k+1}, & \text{if } x \text{ falsifies precisely } i \leq k+1 \text{ of the } Q_1^{b_1}, \dots, Q_n^{b_n} \\ 0 & \text{otherwise.} \end{cases}$$

■

Needless to say, the game terminates when the  $L$ -subset  $\mu$  of possible numbers becomes an  $L$ -singleton. More precisely let  $\mu_n$  be the  $L$ -subset of possible numbers resulting after a sequence of questions and related answers, and assume that  $\mu_n$  is an  $L$ -singleton, namely that  $Supp(\mu_n) = \{x\}$ . Then  $x$  is the secret number.

### 6.3.3 Probabilistic Ulam game

In this section we suppose that player  $B$  cannot arbitrarily choose the secret element  $x \in X$  but that  $x$  is chosen in a random way in accordance with a probability distribution  $p$  on  $X$ . Also, we assume that such a distribution is known by Player  $A$ . This variant of Ulam’s game naturally arises when one considers the problem of efficient transmission in a noisy channel with feedback [18].

At first we will examine the case with no lies, where, as is well known, optimal strategies use balanced questions that minimize the expected value of entropy.

In an attempt to extend this result to the case  $k > 0$ , we shall develop a notion of entropy of a conditioned quasi-state.

**Entropy and strategies: the case with no lies**

Assume that the secret number  $x$  is defined in a random way in accordance with a probability measure  $p_0 : 2^X \rightarrow [0, 1]$  and that no lie is admitted. In this case no choice is possible for Player B, and the game can be considered as a one-person game. Then, we can consider the *entropy of  $p_0$*

$$H(p_0) = - \sum_{x \in X} p_0(x) \cdot \log p_0(x),$$

where we admit the usual convention that  $0 \log 0 = 0$ . If such an entropy is equal to zero, then, as is well known,  $p_0$  is nonzero for precisely one  $x \in X$  and we can conclude that  $x$  is the secret number. Otherwise, a question  $Q$  exists such that both  $p_0(Q)$  and  $p_0(\overline{Q})$  are nonzero. Set  $Q_1 = Q$  and assume that  $b_1 \in \{yes, no\}$  is the answer to the question  $Q_1$ . In this case we have to consider *the conditional probability*  $p_1 = p_0(-, Q^{b_1})$ , where  $Q^{yes} = Q$  and  $Q^{no} = \overline{Q}$ , since no lie is admitted. If the entropy of  $p_1$  is zero, then we are done, since there is only an element  $x$  such that  $p_1(x) \neq 0$  and this is the secret number. Otherwise, we consider a question  $Q_2$  such that both  $p_1(Q_2)$  and  $p_1(\overline{Q_2})$  are nonzero. More generally, assume that at the  $i^{th}$  step of this process questions  $Q_1, \dots, Q_i$  have been asked and  $b_1, \dots, b_i$  are their respective answers. Then the knowledge about  $x$  available to Player A is represented by the set  $M_i = Q_1^{b_1} \cap \dots \cap Q_i^{b_i}$  of possible numbers and by the conditional probability  $p_i = p_{i-1}(-, Q^{b_i})$ . An application of the conditioning iteration rule, yields that  $p_i = p_0(-, M_i)$ . Now, entropy minimization suggests to us how the next question  $Q_{i+1}$  should be chosen. Indeed, given a question  $Q$  and a probability distribution  $p$ , let  $E(H)$  be the expected value of the entropy of  $p$  after  $Q$ , in symbols,

$$E(H) = p(Q^{yes}) \cdot H(p_{Q^{yes}}) + p(Q^{no}) \cdot H(p_{Q^{no}}).$$

As is well known, we must choose a question  $Q$  that minimizes the value of  $E(H)$ .

Let  $Q$  be a question. Then *the entropy of the scheme*

$$\begin{pmatrix} Q^{yes} & Q^{no} \\ p(Q^{yes}) & p(Q^{no}) \end{pmatrix}$$

is given by

$$H_Q = -p(Q^{yes}) \log p(Q^{yes}) - p(Q^{no}) \log p(Q^{no}).$$

The proof of the following result is routine.

**Proposition 6.3.10** *Given a probability  $p$  on  $X$  and a question  $Q$ , we have:*

$$E(H) = H(p) - H_Q.$$

From this proposition it follows that, in order to minimize  $E(H)$  we have to maximize  $H_Q$  and therefore to choose a question  $Q$  that is *balanced*, i.e.,  $p(Q)$  is as close as possible to  $p(\overline{Q})$ .

**Entropy and strategies: the case with  $k$  lies**

Let us consider Ulam game with  $k > 0$  lies on the search space  $X$ , on which a probability distribution  $p_0$  is defined. Suppose that after  $i$  questions the game is described by the  $L$ -set  $\mu$  and by a probability  $p_i$ . We can canonically extend probabilities  $p_0$  and  $p_i$  respectively to states  $s_0 = p_0^\flat$  and  $s_i = p_i^\flat$  as in Proposition 6.1.2. If a new question  $Q$  is asked, and answer  $b$  is given, there are at least three different ways to define the updated state  $s_{i+1}$ :

(1) Letting

$$s'_{i+1}(\alpha) = s_i(\alpha \mid Q^b) = \frac{\sum_{x \in X} s_i(x) \alpha(x) Q^b(x)}{s_i(Q^b)}.$$

In other words,  $s'_{i+1}$  is the state result of conditioning  $s_i$  by the  $L$ -subset  $Q^b$ . This conditional state is always different from zero since for every question  $Q$  and answer  $b$ ,  $Q^b$  is different from zero. If we adopt this definition, it may happen that discarded elements have a nonzero probability.

(2) Letting

$$s''_{i+1}(\alpha) = s_0(\alpha \mid \mu \odot Q^b) = \frac{\sum_{x \in X} s_0(x) \alpha(x) (\mu \odot Q^b)(x)}{s_0(\mu \odot Q^b)},$$

i.e.,  $s''_{i+1}$  is the state result of conditioning  $s_0$  by  $\mu \odot Q^b$ .

(3) Letting

$$s'''_{i+1}(\alpha) = s_i^\circ(\alpha \mid Q^b) = \frac{s_i(\alpha \odot Q^b)}{s_i(Q^b)} = \frac{s_0(\alpha \odot \mu \odot Q^b)}{s_0(\mu \odot Q^b)} = s_0^\circ(\alpha \mid \mu \odot Q^b),$$

i.e.,  $s'''_{i+1}$  is the quasi-state obtained conditioning  $s_i$  by  $Q^b$  that is equivalent to consider the quasi-state obtained conditioning  $s_0$  by  $\mu \odot Q^b$ .

In the following we shall adopt this last approach. Note that the restriction of  $s'''$  to the boolean skeleton of  $L^X$  is a probability equal to the probability obtained by the restriction of  $s''$ .

In the case of a game with no-lies equipped with an initial distribution of probability, Player  $B$  must answer in the right way. So it makes no sense to consider a game in which Player  $B$  gives the answers in a malicious way. We can suppose that the expected value of the entropy is calculated considering that the probability to have a positive (negative) answer to the question  $Q$  is  $p(Q^{yes}) = p(Q)$  (respectively  $p(Q^{no}) = p(\bar{Q})$ ). So the rate between positive and negative answers is  $\frac{p(Q^{yes})}{p(Q^{no})}$ .

By contrast, when the number of lies is different from zero then Player  $B$  can decide whether or not to give a false answer, in order to minimize the amount of information given to  $A$ . We have a typical two persons game and, in accordance,  $A$  can adopt a minimax strategy. More precisely:

*in searching strategies with a malicious Player  $B$ , Player  $A$  must choose at the  $i^{th}$  step a question  $Q$  minimizing the quantity*

$$\max \{H(s_i(-, Q^{yes})), H(s_i(-, Q^{no}))\}.$$

### Random lies

A different case is when Player  $B$  gives the answer in a random way, equivalently when lies are randomly generated. In this case it makes sense to apply a minimization of the expected value of entropy with respect to the probability distribution on answers *yes* and *no*. In order to have a complete analogy with Section 6.3.3, we suppose that, given the quasi-state  $s = s_i$ , at the  $i^{th}$  step of the game, the ratio between positive and negative answers is  $\frac{s(Q^{yes})}{s(Q^{no})}$ , and hence

$$\text{probability (positive answer to } Q) = \frac{s(Q^{yes})}{s(Q^{yes}) + s(Q^{no})} \quad (6.60)$$

and

$$\text{probability (negative answer to } Q) = \frac{s(Q^{no})}{s(Q^{yes}) + s(Q^{no})}. \quad (6.61)$$

Accordingly, we stipulate that the *expected value of the entropy* of the quasi-state  $s$  after the question  $Q$ , is given by

$$E(H, Q) = \frac{s(Q^{yes})}{s(Q^{yes}) + s(Q^{no})} \cdot H(s(-|Q^{yes})) + \frac{s(Q^{no})}{s(Q^{yes}) + s(Q^{no})} \cdot H(s(-|Q^{no})).$$



The entropy of the scheme

$$\begin{pmatrix} Q^{yes} & Q^{no} \\ s(Q^{yes}) & s(Q^{no}) \end{pmatrix}$$

is given by

$$\begin{aligned} H_Q &= -\frac{s(Q^{yes})}{s(Q^{yes}) + s(Q^{no})} \log \frac{s(Q^{yes})}{s(Q^{yes}) + s(Q^{no})} + \\ &\quad -\frac{s(Q^{no})}{s(Q^{yes}) + s(Q^{no})} \log \frac{s(Q^{no})}{s(Q^{yes}) + s(Q^{no})}. \end{aligned}$$

We denote by  $\nu$  the normalization factor  $s(Q^{yes}) + s(Q^{no})$ . Denoting by  $\frac{\mathbf{k}}{\mathbf{k}+1}$  the  $L$ -subset constantly equal to  $\frac{k}{k+1}$ , it is immediate to prove that

$$\nu = 1 + s\left(\frac{\mathbf{k}}{\mathbf{k}+1}\right).$$

So  $\nu$  does not depend on  $Q$ . Also, we denote by  $E\left(\frac{\mathbf{k}}{\mathbf{k}+1}\right)$  the entropy of  $\frac{\mathbf{k}}{\mathbf{k}+1}$ , i.e.,

$$E\left(\frac{\mathbf{k}}{\mathbf{k}+1}\right) = -\sum_{\mathbf{x} \in \mathbf{X}} s\left(\langle \mathbf{x}, \frac{\mathbf{k}}{\mathbf{k}+1} \rangle\right) \log \left( s\left(\langle \mathbf{x}, \frac{\mathbf{k}}{\mathbf{k}+1} \rangle\right) \right).$$

Now, we can prove the following extension of Proposition 6.3.10.

**Proposition 6.3.11** *Adopt the above notation. For any quasi-state  $s$  and question  $Q$  we have the identity*

$$\nu \cdot E(H, Q) = H(s) - H_Q - E\left(\frac{\mathbf{k}}{\mathbf{k}+1}\right).$$

**Proof.** By definition

$$\begin{aligned} \nu \cdot E(H, Q) &= s(Q^{yes})H(s(-, Q^{yes})) + s(Q^{no})H(s(-, Q^{no})) \\ &= s(Q^{yes}) \cdot \left( -\sum_{x \in X} s_{Q^{yes}}(x) \log s_{Q^{yes}}(x) \right) + \\ &\quad s(Q^{no}) \cdot \left( -\sum_{x \in X} s_{Q^{no}}(x) \log s_{Q^{no}}(x) \right). \end{aligned}$$

From

$$s_{Q^{yes}}(x) = \frac{s(\langle x, Q^{yes}(x) \rangle)}{s(Q^{yes})} \quad \text{and} \quad s_{Q^{no}}(x) = \frac{s(\langle x, Q^{no}(x) \rangle)}{s(Q^{no})},$$

it follows that

$$\begin{aligned} \nu \cdot E(H, Q) &= - \sum_{x \in X} s(\langle x, Q^{yes}(x) \rangle) \log \left( \frac{s(\langle x, Q^{yes}(x) \rangle)}{s(Q^{yes})} \right) \\ &\quad - \sum_{x \in X} s(\langle x, Q^{no}(x) \rangle) \log \left( \frac{s(\langle x, Q^{no}(x) \rangle)}{s(Q^{no})} \right) \end{aligned}$$

By direct verification we got

$$\begin{aligned} \nu \cdot E(H, Q) &= - \sum_{x \in X} s(\langle x, Q^{yes}(x) \rangle) \log s(\langle x, Q^{yes}(x) \rangle) \\ &\quad + \sum_{x \in X} s(\langle x, Q^{yes} \rangle) \log s(Q^{yes}) \\ &\quad - \sum_{x \in X} s(\langle x, Q^{no}(x) \rangle) \log s(\langle x, Q^{no}(x) \rangle) \\ &\quad + \sum_{x \in X} s(\langle x, Q^{no} \rangle) \log s(Q^{no}). \end{aligned}$$

By definitions of  $Q^{yes}$  and  $Q^{no}$ ,

$$\begin{aligned} \nu \cdot E(H, Q) &= s(Q^{yes}) \log s(Q^{yes}) + s(Q^{no}) \log s(Q^{no}) \\ &\quad - \sum_{x \in Q} s(x) \log s(x) - \sum_{x \notin Q} s(x) \log s(x) + \\ &\quad - \sum_{x \notin Q} s(\langle x, \frac{k}{k+1} \rangle) \log \left( s(\langle x, \frac{k}{k+1} \rangle) \right) + \\ &\quad - \sum_{x \in Q} s(\langle x, \frac{k}{k+1} \rangle) \log \left( s(\langle x, \frac{k}{k+1} \rangle) \right) \\ &= s(Q^{yes}) \log s(Q^{yes}) + s(Q^{no}) \log s(Q^{no}) + H(s) \\ &\quad - \sum_{x \in X} s(\langle x, \frac{k}{k+1} \rangle) \log \left( s(\langle x, \frac{k}{k+1} \rangle) \right). \end{aligned}$$

■

As in the case no lies, the questions that (in the average) give more information are balanced in the appropriate sense:

**Proposition 6.3.12** *For any quasi-state  $s$  and question  $Q$ ,*

$$s(Q^{yes}) - s(Q^{no}) = s(Q) - s(\overline{Q}) + s\left(\frac{\mathbf{k}}{\mathbf{k} + \mathbf{1}}\right).$$

*Consequently, the minimum of the expected value of the entropy  $E(H, Q)$  is achieved asking balanced questions, in the sense that  $s(Q)$  has to be as close as possible to  $s(\overline{Q})$ .*

**Proof.** Since

$$\begin{aligned} s(Q^{yes}) - s(Q^{no}) &= \\ &= \sum_{x \in X} s(\langle x, Q^{yes}(x) \rangle) - \sum_{x \in X} s(\langle x, Q^{no} \rangle) = \\ &= \sum_{x \in Q} s(x) - \sum_{x \notin Q} s(x) + \sum_{x \notin Q} s\left(\langle x, \frac{k}{k+1} \rangle\right) - \sum_{x \in Q} s\left(\langle x, \frac{k}{k+1} \rangle\right) \\ &= s(Q) - s(\overline{Q}) + \sum_{x \in X} s\left(\langle x, \frac{k}{k+1} \rangle\right), \end{aligned}$$

the first part of the proposition is proved. Moreover, by Proposition 6.3.11, the minimum value of  $E(H, Q)$  is achieved in correspondence of the maximum value of  $H_Q$ , i.e.,  $s(Q^{yes})$  has to be as close as possible to  $s(Q^{no})$ . ■

Thus if Player  $A$  knows that for every question  $Q$  the probability of a positive or a negative answer is given by (6.60),(6.61), then the strategy of balanced question is such that the higher the probability of  $x$ , the smaller the number of questions to find it.

## Chapter 7

# An application: fuzzy collaborative filtering

Recommendation systems [104, 61, 107] are nowadays attracting growing attention as successful web applications. They are of special interest for web sites devoted to electronic commerce, and they are widely used to support user choices for virtually any kind of goods or services. In general, a recommendation system deals with a finite set of users  $U$  and a finite set of items  $I$ . The task is to give an estimate, for each  $u \in U$  and for each  $i \in I$ , of the *degree of preference*  $d(u, i)$  of user  $u$  for item  $i$ .

Recommendation systems can be classified in two categories according to the kind of information they use in order to accomplish their task: *content-based* methods and *collaborative filtering* methods.

The more conventional *content-based* methods require a description of each item  $i \in I$  and, possibly, of each user  $u \in U$ . Usually, these descriptions are given in terms of a *feature space*: each item  $i \in I$  is associated with an  $n$ -tuple  $(f_{i1}, \dots, f_{in})$  of *features* (or, *attributes*), where each  $f_{ij}$  belongs to the set  $F_j$  of all possible values of the  $j$ th feature. Analogously, a user may be represented by an  $m$ -tuple of features in some set  $G_1 \times G_2 \times \dots \times G_m$ . Relying upon these descriptions, a content-based method gives an estimate  $d(u, i)$  for each pair  $(u, i) \in U \times I$  for which  $u$  still has not given any explicit value for her/his preference for the item  $i$ . Often, similarity measures are computed between any pair  $(u, u') \in U^2$  and any pair  $(i, i') \in I^2$ , and the estimated value for  $d(u, i)$  is computed in terms of these similarities.

On the other hand, *collaborative filtering* techniques [61, 107] are helpful when a feature space representation of items (and users) is not available, or gives unreliable or insufficient information. In these cases, to formulate

an estimate for  $d(u, i)$ , collaborative systems only rely on the collection  $W \subseteq U \times I$  of pairs  $(u', i')$  for which user  $u'$  has already given an explicit value for her/his degree of preference for the item  $i'$ . No other information is used. Usually, the information available in the function  $d : W \rightarrow [0, 1]$  is used to compute similarities between the users in terms of the preferences they have shown so far. Those similarities then play the role of weighting factors in some suitably defined aggregate measure, which in turns gives the estimate for  $d(u, i)$  for each  $i \in I$ .

Collaborative filtering techniques are particularly useful in web applications where in many real cases users of a service want to remain anonymous: then any feature-based representation of users is unreliable or even impossible to get. At the same time, feature-based representation of items is often difficult to design, or inaccurate for the task to be accomplished.

The typical example of a recommendation system based on collaborative filtering techniques is concerned with movies [19, 81, 22]: in the standard interaction between the user and the recommendation system user  $u$  asks suggestions for the next movie to see. The system will suggest a film among those still not seen by  $u$  which has been graded well by the users which in the past have shown preferences similar to those expressed by  $u$ . The movie scenario can be dealt with content-based techniques as well, as it is not difficult to design a feature space for movies. Collaborative methods have been proved to be competitive here [22].

In some domains — a notable example being music [107, 69, 6, 7] — the identification of a feature space related to the subjective tastes of users is intrinsically very difficult. Moreover, there are domains where the feature-based approach is questionable, as for instance in judging the relevance of an article [103] for a user on a certain topic just by looking at the number of occurrences of certain associated keywords: it makes more sense to base the relevance estimation upon the judgment of previous readers of the same article. In all those cases, collaborative filtering techniques are to be considered as an alternative to content-based methods. Hybrid approaches mixing collaborative and content-based techniques are also interesting.

## 7.1 The standard collaborative filtering algorithm

In spite of the growing demand for recommendation systems based on collaborative filtering techniques, only a few different algorithms have been proposed in the literature.

Most of the algorithms for collaborative filtering are variants of the fol-

lowing schema.

Without loss of generality, throughout this section the degree  $d(u, i)$  is defined on  $W \subseteq U \times I$ . The value  $d(u, i)$  is an element of  $[0, 1]$  expressing the preference of user  $u$  for item  $i$ . A value in  $[0, 1/2)$  means that  $u$  dislikes  $i$ , while a value in  $(1/2, 1]$  represents a positive judgment. The value  $1/2$  is taken as a neutral judgment.

**Definition 7.1.1 (Standard collaborative filtering)** For each  $u \in U$  let  $I_u = \{i \in I \mid (u, i) \in W\} \neq \emptyset$  be the set of items  $i$  such that user  $u$  has explicitly expressed degree of preference  $d(u, i)$  for item  $i$  and let  $\bar{d}(u) = \sum_{i \in I_u} d(u, i) / |I_u|$  be the average degree of preference of user  $u$ . For each  $(u, i) \in U \times I$ , the predicted value  $p(u, i)$  for  $d(u, i)$  is given by

$$p(u, i) = \bar{d}(u) + \frac{\sum_{v \in U_i} \rho_{u,v} (d(v, i) - \bar{d}(v))}{\sum_{v \in U_i} |\rho_{u,v}|}$$

where  $U_i$  is the subset of  $U$  containing all users  $v$  that have given an explicit value  $d(v, i)$  for item  $i$ , while  $\rho(u, v)$  is the Pearson correlation coefficient between  $u$  and  $v$ :

$$\rho(u, v) = \frac{\sum_{j \in J} (d(u, j) - \bar{d}(u))(d(v, j) - \bar{d}(v))}{\sqrt{\sum_{j \in J} (d(u, j) - \bar{d}(u))^2 \sum_{j \in J} (d(v, j) - \bar{d}(v))^2}}$$

The Pearson coefficient takes values in  $[-1, 1]$  and measures linear correlation of the functions  $d(u, \cdot), d(v, \cdot) : J \rightarrow [0, 1]$ , where  $(I_u \cap I_v) \subseteq J$  is a subset of  $I$  computed according to some fixed criteria. The Pearson coefficient is used as a weighting factor in the weighted mean giving the predicted value  $p(u, i)$ . That is, the more the preference profiles between user  $u$  and user  $v$  are linearly correlated, the better is the advice that  $v$  gives to  $u$  about  $i$ .

Variants on this schema compute the weighting coefficients  $\rho(u, v)$  using other similarity measures, as the Spearman coefficient, or the vector similarity coefficient (see [22]). An alternative algorithm, in which collaborative filtering is regarded as a classification problem and is tackled by applying singular value decomposition techniques to the matrix  $\{d(u, i)\}_{u \in U, i \in I}$ , is presented in [19]. Attempts to conjugate collaborative filtering with other learning paradigms are found in, for instance, [6, 7, 69].

## 7.2 Fuzzy collaborative filtering

We introduce collaborative filtering in the context of fuzziness and many-valued logic. We describe an infinite-valued logical framework in which the

collaborative approach is formalized. This framework yields an alternative algorithm to the ones introduced above. We further consider some extensions to the basic schema. We introduce in our algorithm the use of a measure of confidence on the predicted values for  $d(u, i)$ . Many-valued logic is shown to be flexible enough to allow the realization of hybrid systems combining the collaborative and the content-based perspectives.

**Definition 7.2.1** A fuzzy similarity [117] on a set  $A$  with respect to a  $t$ -norm  $*$  is a function  $s : A \times A \rightarrow [0, 1]$  satisfying, for every  $x, y, z \in A$ ,

- reflexive property:  $s(x, x) = 1$ ;
- symmetric property:  $s(x, y) = s(y, x)$ ;
- transitivity property:  $s(x, y) * s(y, z) \leq s(x, z)$ .

**Example 7.2.2** If  $x, y \in [0, 1]$ , the relation  $x \leftrightarrow y$  is a similarity relation on  $[0, 1]$  with respect to the Lukasiewicz conjunction  $\odot$ .

### Description of the system

A collaborative decision problem is characterized by a finite set of users  $U$  and a finite set of items  $I$ . Besides the preference function  $d$  defined on a subset  $W$  of  $U \times I$ , in our approach the system contains two functions  $p$  (the predicted value for the degree of preference) and  $c$  (the confidence degree about the predicted value) assigning to each pair  $(u, i) \in U \times I$  a value in  $[0, 1]$ . As in the standard approach, values of  $p^t(u, i)$  in  $[0, 1/2)$  are considered negative (that is,  $u$  dislikes  $i$ ). Values in  $(1/2, 1]$  are positive ( $u$  likes  $i$ ).  $1/2$  is considered a neutral judgment.

The meaning of  $c$  and  $p$  is that the greater is  $c(u, i)$ , the better the system trust  $p(u, i)$  as a good estimate of  $d(u, i)$ . If for a pair  $(u, i) \in U \times I$ ,  $c(u, i) = 1$  then  $p(u, i) = d(u, i)$ .

Functions  $d$ ,  $p$  and  $c$  evolve in time. We shall denote by  $d^t$ ,  $c^t$  and  $p^t$  confidence and predicted value functions at time  $t$ , for  $0 \leq t \in \mathbb{Z}$ .

We assume that at time  $t = 0$  the function  $c^0$  takes values in  $\{0, 1\}$ . We start from a situation where the degrees  $p^0(u, i)$  such that  $c^0(u, i) = 1$  have been explicitly given by users. On the other hand, we assume no confidence on values  $p^0(u, i)$  such that  $c^0(u, i) = 0$ . Then we set  $p^0(u, i) = 1/2$  when  $c^0(u, i) = 0$ .

The system acquires new knowledge every time a user  $u$  rates a new item  $i$  (we assume that users never retract any of her/his previous judgments, even though the system still can work if this is allowed to happen).

When a new value  $d^t(u, i)$  is given to the system, the internal state of knowledge evolves from time  $t$  to  $t + 1$ , and new values of  $p^{t+1}$  and  $c^{t+1}$  are computed. The function  $b^t : U \times I \rightarrow \{0, 1\}$  gives value 1 to pairs  $(u, i)$  such that at time  $t$  user  $u$  gave an explicit preference degree for  $i$ .

There is no loss of generality in assuming a fixed size for  $U$  and  $I$ : we are considering evolution of the system from time  $t = 0$  to time  $T$  where the system exactly contained  $|U|$  users and  $|I|$  items. Then  $t \leq T := |U||I| - |\{(u, i) \mid b^0(u, i) = 1\}|$ .

### 7.2.1 The logical formulation

In this section, the functions  $c^t$ ,  $p^t$  and  $b^t$  and the user inputs  $d^t$  will be described as propositional variables or propositional formulas of Rational Lukasiewicz calculus as in Chapter 5.

**Definition 7.2.3** *An influence function is a function  $\triangleleft : [0, 1]^2 \rightarrow [0, 1]$  such that, for every  $x, y, z \in [0, 1]$ ,*

- $0 \triangleleft y = 1/2, \quad 1 \triangleleft y = y, \quad x \triangleleft 1/2 = 1/2;$
- *If  $y_1 \leq y_2$  then  $x \triangleleft y_1 \leq x \triangleleft y_2$  ;*
- *If  $y \in [0, 1/2)$  and  $x_1 \leq x_2$  then  $x_1 \triangleleft y \geq x_2 \triangleleft y$ , if  $y \in (1/2, 1]$  and  $x_1 \leq x_2$  then  $x_1 \triangleleft y \leq x_2 \triangleleft y$ .*

The second variable  $y$  of an influence function  $\triangleleft$  is meant to be a degree of preference, thus having  $1/2$  as neutral judgment. The intended meaning of  $\triangleleft$  is weighting  $y$  by means of  $x$ . When  $x = 0$  then  $x \triangleleft y$  is constantly  $1/2$  for all  $y \in [0, 1]$ . The greater the value of  $x$ , the closer to  $y$  the value of  $x \triangleleft y$ .

**Example 7.2.4** *The following are formulas whose truth-table is an influence function (see Figure 7.1 for their graphs):*

$$\begin{aligned} x \triangleleft_1 y &:= \delta_2(\neg x \vee 2y) \oplus \delta_2(x \wedge y^2) \\ x \triangleleft_2 y &:= \delta_2((x \rightarrow y) \vee 2y) \oplus \delta_2((x \odot y) \wedge y^2) \\ x \triangleleft_3 y &:= \delta_2(x \rightarrow 2y) \oplus \delta_2(x \odot y^2). \end{aligned}$$

Note that  $\triangleleft_3$  is the most “restrictive” influence function among those in Example 7.2.4: indeed  $x \triangleleft_3 y = y$  only if  $x = 1$  or  $y = 1/2$ .

For any pair  $(u, i) \in U \times I$  and  $0 \leq t \leq T$ ,  $d^t(u, i)$  and  $b^t(u, i)$  are propositional variables. We shall consider only *legal* assignments. A *legal* assignment  $v : Form \rightarrow [0, 1]$  (cf. Definition 1.1.5) must be such that all the following formulas

$$(A1) \quad b^t(u, i) \leftrightarrow (b^t(u, i) \odot b^t(u, i))$$



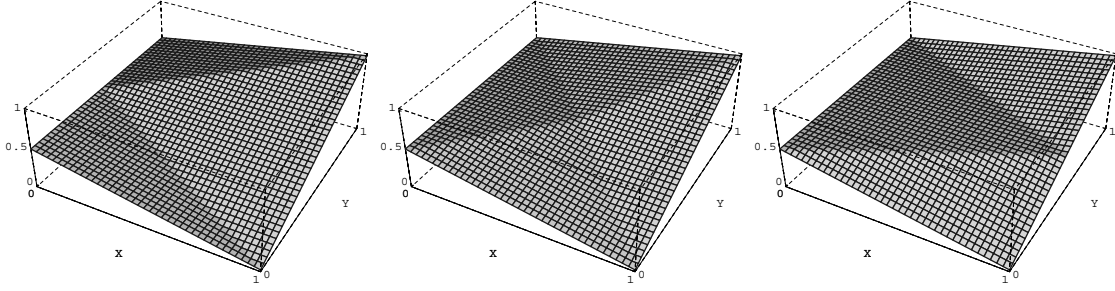


Figure 7.1: From left to right, the graphs of truth-table functions resp. associated with:  $\triangleleft_1, \triangleleft_2, \triangleleft_3$ .

$$(A2) \quad \bigvee_{(u,i) \in U \times I} b^t(u,i) \wedge \bigwedge_{(u,i) \neq (v,j)} (-b^t(u,i) \vee -b^t(v,j)) \quad \text{for any } 0 < t \leq T$$

$$(A3) \quad b^t(u,i) \rightarrow \bigwedge_{t' \neq t} -b^{t'}(u,i)$$

evaluate to 1 under  $v$ . (A1) asserts that under any evaluation  $b^t(u,i)$  takes values in  $\{0, 1\}$ . (A2) guarantees that at each  $t > 0$  only one user  $u$  inputs a new judgment for an item  $i$ . (A3) asserts that a judgment is never retracted. We define the following sets of formulas:

$$\begin{aligned} c^0(u,i) &:= b^0(u,i) \\ p^0(u,i) &:= (d^0(u,i) \odot b^0(u,i)) \oplus (\frac{1}{2} \odot -b^0(u,i)). \end{aligned}$$

At time  $t$  a *weighted similarity* among users is computed by the formula

$$s^t(u,v) := \bigodot_{i \in I} [(k \rightarrow c^t(u,i)) \odot (k \rightarrow c^t(v,i))] \rightarrow (p^t(u,i) \leftrightarrow p^t(v,i)) \quad (7.1)$$

The meaning of (7.1) is that users  $u$  and  $v$  are similar if, for every item  $i$ , whenever the confidence of  $u$  and  $v$  about  $i$  is high then their degree of preference for item  $i$  is about the same.

**Proposition 7.2.5** *The function  $s^t$  is symmetric and reflexive. Further, if  $v \in U$  is such that for any  $i \in I$ ,  $c^t(v,i) \geq k$ , then  $s^t(u,v) \odot s^t(v,w) \leq s^t(u,w)$ .*

**Proof.** Symmetry follows from definition. For every  $u \in U$  and  $i \in I$ ,  $p^t(u,i) \leftrightarrow p^t(u,i) = 1$  hence  $[(k \rightarrow c^t(u,i)) \odot (k \rightarrow c^t(u,i))] \rightarrow (p^t(u,i) \leftrightarrow p^t(u,i)) = 1$  and  $s(u,u) = 1$ . Whence  $s$  is reflexive. By definition, for every  $i \in I$ ,

$$(((k \rightarrow c^t(u,i)) \odot (k \rightarrow c^t(v,i))) \rightarrow (p^t(u,i) \leftrightarrow p^t(v,i))) \odot$$

$$\begin{aligned}
& \odot \left( ((k \rightarrow c^t(v, i)) \odot (k \rightarrow c^t(w, i))) \rightarrow (p^t(v, i) \leftrightarrow p^t(w, i)) \right) \leq \\
& (k \rightarrow c^t(u, i)) \odot (k \rightarrow c^t(v, i)) \odot (k \rightarrow c^t(v, i)) \odot (k \rightarrow c^t(w, i)) \rightarrow \quad (7.2) \\
& \rightarrow ((p^t(u, i) \leftrightarrow p^t(v, i)) \odot (p^t(v, i) \leftrightarrow p^t(w, i)))
\end{aligned}$$

Supposing than for every  $i \in I$  the confidence degree  $c^t(v, i)$  is greater than  $k$ , recalling Example 7.2.2, we have

$$(7.2) \leq ((k \rightarrow c^t(u, i)) \odot (k \rightarrow c^t(w, i))) \rightarrow ((p^t(u, i) \leftrightarrow p^t(w, i))).$$

Then  $s^t(u, v) \odot s^t(v, w) \leq s^t(u, w)$ . ■

Observe that the higher is the confidence of the system about user  $v$ , the higher is the truth value of  $s^t(u, v) \odot s^t(v, w) \rightarrow s^t(u, w)$ .

Let  $n = |U|$  and  $\triangleleft$  be an influence connective. The (*aggregate*) *advice* of user  $u$  for item  $i$  at time  $t$  is the formula

$$a^t(u, i) = \bigoplus_{v \in U} \delta_n \left( (k \rightarrow c^t(v, i)) \triangleleft (s^t(u, v) \triangleleft p^t(v, i)) \right). \quad (7.3)$$

The use of the derived connective  $\triangleleft$  is justified by the fact that, in order to join  $c^t$ ,  $s^t$  and  $p^t$  one must take into account that values of  $p^t$  less than  $1/2$  are considered as negative judgments. Then equation (7.3) says that the predicted value of the preference of user  $u$  for item  $i$  is given by considering the average value, for every user  $v$  having a high confidence about  $i$ , of the preference of  $v$  for  $i$  suitably weighted by the similarity between  $u$  and  $v$ .

**Proposition 7.2.6** *The formula  $\varphi = (3\delta_2 c \odot (c \triangleleft x)) \oplus (3\delta_2 \neg c \odot (\neg c \triangleleft y))$  is such that for any assignment  $v$ :*

$$\begin{aligned}
v(c) = 0 & \text{ implies } v(\varphi) = v(y); \\
v(c) = 1 & \text{ implies } v(\varphi) = v(x); \\
v(c) = 1/2 & \text{ implies } v(\varphi) = (v(x) + v(y))/2.
\end{aligned}$$

Suppose that at time  $t+1$  user  $u^*$  expresses an explicit preference degree  $d^{t+1}(u^*, i^*)$  for item  $i^*$ . Then  $b^{t+1}(u^*, i^*)$  is 1 and predicted preference and confidence degrees become:

$$\begin{aligned}
p^{t+1}(u, i) & := \left( (b^{t+1}(u, i) \wedge d^{t+1}(u, i)) \vee (\neg b^{t+1}(u, i) \wedge \right. \\
& \left. ((3\delta_2 c^t(u, i) \odot (c^t(u, i) \triangleleft p^t(u, i))) \oplus (3\delta_2 \neg c^t(u, i) \odot (\neg c^t(u, i) \triangleleft a^t(u, i)))) \right) \quad (7.4)
\end{aligned}$$

$$c^{t+1}(u, i) := b^{t+1}(u, i) \vee (p^t(u, i) \leftrightarrow p^{t+1}(u, i)) \quad (7.5)$$

**Proposition 7.2.7** *If  $t < t'$  and  $b^t(u, i) = 1$  then  $c^{t'}(u, i) = 1$  and  $p^{t'}(u, i) = p^t(u, i) = d^t(u, i)$ .*

**Proof.** If  $b^t(u, i) = 1$ , by (7.4)  $p^t(u, i) = d^t(u, i)$  and by (7.5),  $c^t(u, i) = 1$ . Since by (A3), for every  $t' > t$ ,  $b^{t'}(u, i) = 0$ , then by (7.4) and Proposition 7.2.6,  $p^{t+1}(u, i) = p^t(u, i)$  and by (7.5)  $c^{t+1}(u, i) = 1$ . Hence for every  $t' > t$ ,  $c^{t'}(u, i) = 1$  and  $p^{t'}(u, i) = p^t(u, i) = d^t(u, i)$ . ■

The set of formulas described in this section provides an algorithm for collaborative filtering:

**Definition 7.2.8** *For any  $0 \leq t \leq T$ , let  $v$  be an assignment to the propositional variables  $d^t(i, u)$  and  $b^t(i, u)$  satisfying (A1), (A2), (A3), and let  $p^t(u, i)$  and  $c^t(u, i)$  be defined by (7.4) and (7.5). Then, the predicted degree of preference of user  $u$  for item  $i$  at time  $t$  is the value  $v(p^t(u, i))$ .*

Since truth-table functions of Rational Lukasiewicz logic are piecewise linear, Definition 7.2.8 immediately yields an effective procedure to compute the predicted values  $p^t(u, i)$ .

### 7.3 Combining content-based and collaborative filtering

Suppose we have associated with each item  $i \in I$  an  $n$ -tuple of features  $(f_{i1}, \dots, f_{in})$  belonging to a feature space  $F_1 \times F_2 \times \dots \times F_n$ . Assume also that we have defined a fuzzy similarity measure  $\mu_j : F_j^2 \rightarrow [0, 1]$  on each component  $F_j$  of the feature space. For any items  $i_1, i_2 \in I$ , we define the *content-based similarity* between  $i_1$  and  $i_2$  as

$$cs(i_1, i_2) = \bigodot_{j=1}^n \mu_j(f_{i_1j}, f_{i_2j}).$$

A purely content-based method can be described by defining the *content-based prediction* on  $(u, i)$  at time  $t + 1$  by the formula

$$cp^t(u, i) = \bigoplus_{j \in I \setminus \{i\}} \delta_{|I|-1} (cs(i, j) \triangleleft cp^t(u, j)),$$

where an average on the set  $I \setminus \{i\}$  weighted by similarity of items is taken as the predicted degree of preference of user  $u$  for item  $i$ .

We propose two ways of designing hybrid methods combining content-based and collaborative filtering techniques.

The *external combination* simply runs the content-based and the collaborative methods independently, then takes a combination of the two predictions, giving, for the pair  $(u, i) \in U \times I$  the prediction (where  $\nu \in [0, 1]$  is a parameter of the system):

$$hp^t(u, i) = \nu p^t(u, i) \oplus \neg \nu cp^t(u, i).$$

The *internal combination* mixes content-based and collaborative aspects at a deeper level. Here we show how the content-based similarity  $cs$  can reinforce the measure of the aggregate advice  $a^t$  in the collaborative filtering algorithm of Definition 7.2.8: this is obtained by setting  $n = 2|U|$ ,  $m = 2(|I| - 1)$  and by replacing the definition (7.3) of formula  $a^t(u, i)$  by:

$$a^t(u, i) := \bigoplus_{v \in U} \delta_n((k \rightarrow c^t(v, i)) \triangleleft (s^t(u, v) \triangleleft p^t(v, i))) \oplus \bigoplus_{j \in I \setminus \{i\}} \delta_m((k \rightarrow c^t(u, j)) \triangleleft (cs(i, j) \triangleleft p^t(u, j))).$$

At present we are developing tests to measure the performance of our algorithms compared to the standard (see Definition 7.1.1) algorithm and some of its variants. We base our tests upon the dataset EachMovie [81] publicly available at DEC Systems Research Center web site.

Future directions will consider how to use more actively feedback from users (a formula for errors can be defined as  $e^t(u, i) = \neg(p^t(u, i) \leftrightarrow d^t(u, i))$ , assuming that at time  $t$  user  $u$  expresses his degree  $d^t(u, i)$  for item  $i$ ). Also, an interesting direction is in the research of selection criteria of the underlying infinite-valued logic and of the influence functions.

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