Algebras Induced by a Unary Term

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3. Generalize the previous situations to algebras induced by a unary term
4. Investigate the subvariety lattice of $p\mathcal{A}$, the variety of positively-pointed Abelian ℓ-groups

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- We observe that Γ is a straight-forward generalization of the Mundici functor.

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- Let **F** be the free Abelian ℓ-group over Y.
- Consider the interval $[e, y \vee e]$ in **F**.
- Let **A** be the *MV*-subalgebra of this interval generated by $\bar{X} = \{(x \vee e) \wedge (y \vee e) \mid x \in X\}$.

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2. Then, defining $g : Y \to G$ by $g(x) = f(x)$, for $x \in X$, and $g(y) = a$, we know that g can be extended to a homomorphism $\bar{g} : \mathbf{F} \to \mathbf{G}$.

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3. Letting $\bar{f}$ be $\bar{g}$ restricted to A, we see that $\bar{f} : \mathbf{A} \to \mathbf{B}$ is a homomorphism that extends f.

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- Since $\mathbb{Z}$ generates the variety of Abelian ℓ-groups, we see that the class of finite MV-chains generates the variety of MV-algebras.
- The MV-algebra $[0, 1] \subseteq \mathbb{R}$ generates the variety of MV-algebras.

Projective MV-algebras

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3. Since $\mathbf{A}$ is projective, there is a homomorphism $k' : \mathbf{A} \to [e, b]$ such that $g' \circ k' = f'$.
4. Extending k' to $k : \mathbf{G} \to \mathbf{K}$, we see that $g \circ k = f$.

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- As in the previous situation, this algebra is the free algebra over X^{-} in the variety $\mathcal{A}^{-}$ of negative cones of Abelian ℓ-groups.

Baker-Beynon-McNaughton-type Theorem

Theorem

For any natural number n, $\mathbf{F}(n)$*, the n-generated free algebra in* $\mathcal{A}^-$*, is a subalgebra of* $(\mathbb{R}^-)^{(\mathbb{R}^-)^n}$*. In particular, the functions that make up* $\mathbf{F}(n)$ *are precisely the continuous, piecewise-linear functions with integer coefficients.*

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2. Then, by the previous slide, we know that $\mathbf{F}(n)$ is generated by the π_i^-, where π_i is the i-th projection of $\mathbb{R}^n$.
3. Lastly, one can show that every continuous, piecewise-linear function with integer coefficients from $(\mathbb{R}^-)^n$ to $\mathbb{R}^-$ is in the subalgebra generated by the π_i^-.

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$$f_\tau(x_1, \ldots, x_n) = \tau^{\mathbf{A}}(f^{\mathbf{A}}(x_1, \ldots, x_n)), \text{ for } x_1, \ldots, x_n \in A_\tau.$$

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- Consider the class $\mathcal{W}$ of subalgebras of algebras of $\mathcal{V}$ induced by τ.
- Consider the functor Λ that sends an algebra $\mathbf{A} \in \mathcal{V}$ to its induced algebra $\mathbf{A}_\tau \in \mathcal{W}$.

Algebras Induced by a Unary Term - 2

Theorem

Let $\Lambda : \mathcal{V} \to \mathcal{W}$ be the functor corresponding to the idempotent term τ. Also, assume that $\mathcal{W}$ contains a non-trivial algebra. Then, for any cardinal κ, $\mathbf{F}_{\mathcal{W}}(\kappa)$ is an L'-subalgebra of $\Lambda(\mathbf{F}_{\mathcal{V}}(\kappa))$.

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- Let $\tau(\mathcal{V})$ be the class of algebras in $p\mathcal{A}$ that Γ maps into $\mathcal{V}$. Let $\sigma(\mathcal{V})$ be the variety generated by the unital correspondents of the algebras in $\mathcal{V}$.
- Then, to each variety $\mathcal{V}$ of MV-algebras, there is an interval $[\sigma(\mathcal{V}), \tau(\mathcal{V})]$ in the subvariety lattice of $p\mathcal{A}$ consisting of all of its subvarieties that Γ maps exactly to $\mathcal{V}$.

Diagram

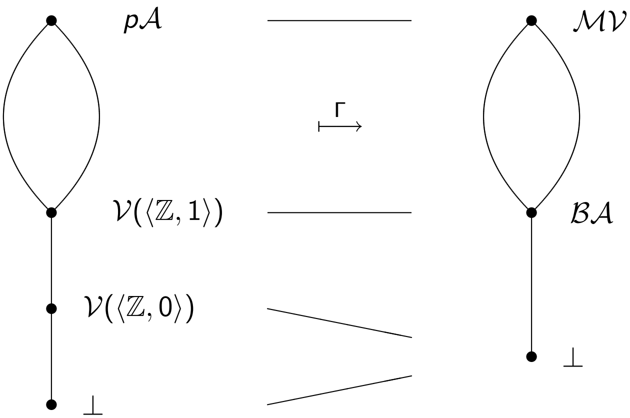

Figure: The subvariety lattices of $p\mathcal{A}$ and $\mathcal{MV}$.

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Theorem

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The functor Γ induces a lattice isomorphism between the subvariety lattice of $p\mathcal{A}$ (excluding the trivial variety) and the subvariety lattice of $\mathcal{MV}$.

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