Fixpoint Semantics for Extended Logic Programs on Bilattice Based Multivalued Logics and Applications

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IMPERFECT INFORMATION

- Imperfect information
  - Conflicting - mutually contradictory sources
  - Missing - incomplete sources
  - Uncertain - sources of limited reliability

- Multivalued logics – provide one of the most capable approaches to handle all the 3 aspects in imperfect information
BILATTICES AS MULTIVALUED LOGICS

**Definition** A bilattice is a triplet $\langle B, \leq, \leq_i \rangle$ in which the set $B$ forms a complete lattice with each of the orders, called the truth and information orders.

- Induced inf and sup operations: $\wedge, \vee, \otimes, \oplus$
- Adding negation $\neg$ as a unary function:
  - Antimonotone w.r.t. $\leq_t$
  - Monotone w.r.t. $\leq_i$
  - $\neg\neg x = x$
  - $\neg(a \wedge b) = \neg a \vee \neg b$ ...
  - $\neg(a \otimes b) = \neg a \otimes \neg b$ ...

- $\wedge B = \text{false}$ \quad $\vee B = \text{true}$
- $\otimes B = \bot$ \quad $\oplus B = \top$

- infinitely distributive bilattices $\Rightarrow$
  - interlacing laws / monotonicity
BILATTICES – EXAMPLES

Confidence-Doubt logic

\[ L^{CD} = [0,1]^2, (L^{CD}, \leq_i, \leq_t) \]

\( \langle c, d \rangle \) confidence, doubt

\( \langle x, y \rangle \land \langle z, w \rangle = \langle \min(x, z), \max(y, w) \rangle \)
\( \langle x, y \rangle \lor \langle z, w \rangle = \langle \max(x, z), \min(y, w) \rangle \)
\( \langle x, y \rangle \otimes \langle z, w \rangle = \langle \min(x, z), \min(y, w) \rangle \)
\( \langle x, y \rangle \oplus \langle z, w \rangle = \langle \max(x, z), \max(y, w) \rangle \)
\( \neg \langle x, y \rangle = \langle y, x \rangle \)

Belnap’s four valued logic

Experts’ support bilattice \( 2^{\text{ExpSet}} \times 2^{\text{ExpSet}} \)
**DEFINITIONS**

**Formula:** an expression built up of literals and elements of bilattice $B$ using $\land, \lor, \otimes, \oplus, \neg, \exists, \forall$

**Rule:** a construct of the form $H(v_1,\ldots,v_n) \leftarrow F(v_1^{'},\ldots,v_m^{'})$

It is assumed that the free variables from body (right) appear in the head (left)

**Extended Program:** a finite set of rules, assuming that no predicate letter appears in the head of more than one rule (no real restriction – see Clark’s completion)

**Interpretations:** $I: HB_P \rightarrow B$ \hspace{2em} $Int_P = B^{HB_P}$

**Orders on interpretations:**

$I \leq_i J$ if $I(A) \leq_i J(A)$ \hspace{2em} $I \leq_i J$ if $I(A) \leq_i J(A)$

$I \leq_p J$ if $I(A) \neq \bot$ then $I(A) = J(A)$

for any ground atom $A$
FORMULA EVALUATION

Closed formula evaluation:

\[ I(X \land Y) = I(X) \land I(Y) \quad I(X \lor Y) = I(X) \lor I(Y) \]
\[ I(X \otimes Y) = I(X) \otimes I(Y) \quad I(X \oplus Y) = I(X) \oplus I(Y) \]
\[ I(\neg X) = \neg I(X) \]
\[ I(\exists x F(x)) = \lor_{s \in GT} I(F(s)) \quad I(\forall x F(x)) = \land_{s \in GT} I(F(s)) \]

Ultimate evaluation:

The ultimate evaluation \( U(I, C) \) of a closed formula \( C \) w.r.t. \( I \) is a logical value \( \alpha \) defined by:

- if \( J(C) = I(C) \) for any interpretation \( J \geq p I \) then \( \alpha = I(C) \), else \( \alpha = \bot \)

Proposition 1 If \( I(C) = I_\top(C) \) then \( U(I, C) = I(C) \), else \( U(I, C) = \bot \)
REASONING WITH IMPERFECT INFORMATION IN BILATTICES

Two approaches to infer information:

1. Applying the rules
2. Completing missing information with default information
   - Conventional CWA: negative information is advantaged. The value false plays a special role as logical value by default
   - OWA: Any logical value can be assigned by default

- Particular operators are defined for (1), (2)
PROGRAM OPERATORS AND PROPERTIES

Production operator $\Phi_p : Int_p \rightarrow Int_p$
$\Phi_p(I)(A) = U(I, C)$ if there is $A \leftarrow C \in P$, else $\Phi_p(I) = \bot$

Proposition 2: $\Phi_p$ is monotone w.r.t. $\leq_i$ and $\leq_p$.

Revision operator $\text{Rev} : Int_p \times Int_p \rightarrow Int_p$
Revises interpretation $X$ via interpretation $J$:
$\text{Rev}(X, J) = X'$ s.t. $X'(A) = X(A)$ for any ground atom $A$
for which either $J(A) = \bot$ or $X(A) = J(A)$,
and $X'(A) = \bot$ for any other ground atom.
**Program Operators and Properties**

**Refining operator** \( \Psi_p : \text{Int}_p \times \text{Int}_p \rightarrow \text{Int}_p \)

\[ \Psi_p(X, I) = \text{Re}_v(X, \Phi_p(\text{Re}_v(X, I) \oplus I)) \]

**Proposition 3**
Let \( I \) be an interpretation, and \( \mathcal{D} \) a default interpretation. \( (\lambda X)\Psi_p(X, I) \) has a greatest fixpoint below \( \mathcal{D} \) w.r.t. \( \leq_p \) that can be obtained as limit of the decreasing sequence w.r.t. the same order, defined by:

\[ X_\emptyset = \mathcal{D}, \quad X_n = \Psi_p(X_{n-1}, I) \] if \( n \) is a successor ordinal, and

\[ X_n = \inf_{\leq_p, m < n} X_m \] if \( n \) is a limit ordinal.

The limit \( \text{Def}_p^\mathcal{D}(I) \) is the default information to complete missing information.
PROGRAM OPERATORS AND PROPERTIES.

FIXPOINT SEMANTICS

Integrating Operator \( \Gamma_p : \text{Int}_p \rightarrow \text{Int}_p \)

\[ \Gamma_p(I) = \Phi_p(I) \oplus \text{Def}_p(I) \]

**Theorem 1** \( \Gamma_p \) is monotone w.r.t. \( \leq_p \) order and has a least fixpoint given by the limit of the increasing sequence:

\[ I_0 = \text{Const}_\perp; \quad I_n = \Gamma_p(I_{n-1}) \text{ for a successor ordinal } n; \]

\[ I_n = \sup_{\leq_p, m<n} I_m \text{ for a limit ordinal } n \]

The fixpoint semantics of \( P \) is defined as the limit of the sequence from Theorem 1.

**Theorem 2** The fixpoint semantics \( s \) of \( P \) satisfies \( \Phi_p(s) = s \).
RELATION TO OTHER SEMANTICS

**Theorem 3** Let $P$ be an extended program and $\text{mstable}(P)$ be is multivalued stable model as defined by Fitting, which is the smallest in the information/knowledge order. Then the fixpoint semantics of $P$ w.r.t. the default interpretation that assigns the value $false$ to any ground atom coincides with $\text{mstable}(P)$.

**Theorem 4** Given an extended program $P$, for any logical value $\alpha$ from the underlying bilattice, the $\alpha$-fix model of $P$ as previously defined by us, coincides with the fixpoint semantics of $P$ w.r.t. the default interpretation that uniformly assigns the value $\alpha$ to any ground atom.

**Corollary 1** The fixpoint semantics captures the well-founded semantics, Przymusinki’s three-valued stable semantics, and the Kripke-Kleene semantics.
COMPUTATIONAL ASPECTS

Proposition 4 If $Values(P)$ is the set of logical values appearing in $P$ to which one adds the four extreme values of the bilattice, then $\langle Closure(Values(P)), \leq_i, \leq_i \rangle$ is a finite bilattice.

Theorem 5 If $P$ is function free then the computation of its fixpoint semantics finishes in a finite number of steps. Moreover, the complexity class is PTIME.
**Computational aspects**

Algorithm based on a bottom up approach to computing the fixpoint semantics

1. function Rev(Y, Z)
2. W := Y;
3. for every pair (A, v1) ∈ W
4.   if v1 ≠ ⊥ and (A, v2) ∈ Z and v2 ≠ ⊥ and
5.   v2 ≠ v1
6.   then replace (A, v1) with (A, ⊥) in W;
7. return W;
8. function Phi(P, I)
9.   I₀ := I;
10. J := Ø;
11. for any pair (A, v) ∈ I₀
12.   if v = ⊥ then
13.     replace (A, v) with (A, T) in I₀;
14. for any rule A ← B in P
15.   if I(B) = I₀(B) then insert (A, I(B)) in J
16. else insert (A, ⊥) in J;
17. for any atom A not appearing in J insert (A, ⊥) in J;
18. return J;
19. function FixpointSemantics(P, D)
20.   I₀ := Const₁;
21. repeat
22.   I₁ := I₀;
23.   J₀ := D;
24. repeat
25.   J₁ := J₀;
26.   J₀ := Rev(J₁, Phi(P, Rev(J₁, I₁) + I₁))
27. until J₁ = J₀;
28. I₁ := Phi(P, I₁) + J₁
29. until I₁ = I₀;
30. return I₁.
POSSIBLE EXTENSION OF THE APPROACH

- Considering **sets of logical values** assigned to atoms instead of a punctual logical value.

\[ Int_P = (2^B)^{HB_P} \]

- Apart from the 3 orders seen so far on \( Int_P \), there will be a \( 4^{\text{th}} \) order related the idea of imprecision.
APPLICATIONS

- Imperfect information integration
- Uncertain knowledge bases
IMPERFECT INFORMATION INTEGRATION

Example:
integrating imperfect information in medical diagnosis: does patient P have condition C?

\[\text{Diagnosis}(P,C) \leftarrow \text{Tests}(P,C) \land \text{MDsSuspect}(P,C)\]
\[\text{Tests}(P,C) \leftarrow \text{Test1}(P,C) \oplus \text{Test2}(P,C)\]
\[\text{MDsSuspect}(P,C) \leftarrow \text{MD1Suspects}(P,C) \otimes \text{MD2Suspects}(P,C)\]
UNCERTAIN KNOWLEDGE BASES

An Uncertain Knowledge Base is a pair $KB=(F, R)$ in the context of a bilattice as underlying logic

- $F$ set of Facts or stored information. A fact is a pair of an atom and a logical value.
- $R$ set of Rules or the inference mechanism
- The content of KB is expressed by the fixpoint semantics of the associated extended program – facts are transformed in rules that take priority over the rules in $R$

Data Complexity: the time complexity to answer an atomic query w.r.t. the size of $F$

Theorem 6 The data complexity for KB as defined above is PTIME
Uncertain Knowledge Bases

- A query can be defined as being a rule
- A query $Q$ is evaluated by being integrated in the extended program associated to the KB
- Query optimisation – an essential topic in Computer Science
- We are currently studying the problems of query containment and equivalence, and their complexity classes in such a framework
- The framework is restricted to non recursive sets of rules (due to non decidability problem when recursion is allowed)
- Established result: Equivalence of non-recursive queries on $\land, \otimes$ is decidable: NP-complete and $\Pi_p^2$-complete
Thank you!