Fixpoint Semantics for Extended Logic Programs on Bilattice based Multivalued Logics and Applications

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IMPERFECT INFORMATION

- Imperfect information
  - Conflicting - mutually contradictory sources
  - Missing - incomplete sources
  - Uncertain - sources of limited reliability

- Multivalued logics – provide one of the most capable approaches to handle all the 3 aspects in imperfect information

BILATTICES AS MULTIVALUED LOGICS

Definition A bilattice is a triplet $\langle B, \leq_t, \leq_i \rangle$ in which the set B forms a complete lattice with each of the orders, called the truth and information orders.

- Induced inf and sup operations: $\wedge, \vee, \otimes, \oplus$
- Adding negation $\neg$ as a unary function:
  - Antimonotone w.r.t. $\leq_t$
  - Monotone w.r.t. $\leq_i$
  - $\neg\neg x = x$
  - $\neg(a \wedge b) = \neg a \vee \neg b$ ... $\neg(a \otimes b) = \neg a \otimes \neg b$ ...
- $\wedge B = \mathit{false}$ $\quad \vee B = \mathit{true}$

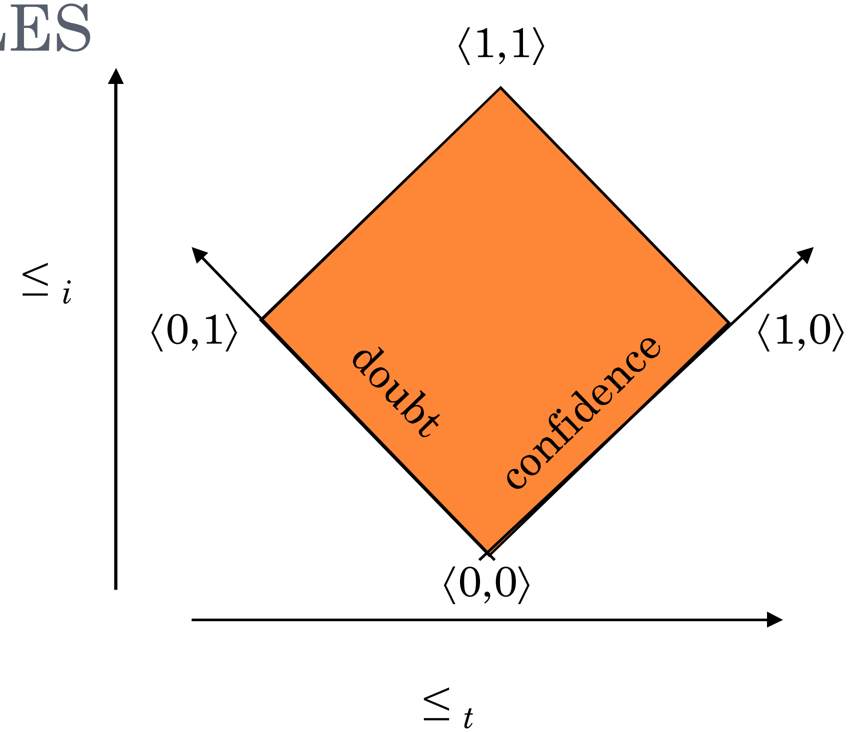
  $\otimes B = \bot$ $\quad \oplus B = \top$
- infinitely distributive bilattices =>

  interlacing laws / monotonicity

Bilattices – examples

Confidence-Doubt logic

$$L^{CD} = [0,1]^2, (L^{CD}, \le_t, \le_i)$$



$\langle c, d \rangle$ confidence, doubt

$$\langle x, y\rangle \wedge \langle z, w\rangle = \langle \min(x,z), \max(y,w)\rangle$$

$$\langle x, y\rangle \vee \langle z, w\rangle = \langle \max(x,z), \min(y,w)\rangle$$

$$\langle x, y\rangle \otimes \langle z, w\rangle = \langle \min(x,z), \min(y,w)\rangle$$

$$\langle x, y\rangle \oplus \langle z, w\rangle = \langle \max(x,z), \max(y,w)\rangle$$

$$\neg\langle x, y\rangle = \langle y, x\rangle$$

Belnap's four valued logic

Experts' support bilattice $2^{ExpSet} \times 2^{ExpSet}$

DEFINITIONS

Formula: an expression built up of literals and elements of bilattice B using $\wedge, \vee, \otimes, \oplus, \neg, \exists, \forall$

Rule: a construct of the form $H(v_1,\ldots,v_n) \leftarrow F(v_1',\ldots,v_m')$

It is assumed that the free variables from body (right) appear in the head (left)

Extended Program: a finite set of rules, assuming that no predicate letter appears in the head of more than one rule (no real restriction – see Clark's completion)

Interpretations: $I: HB_P \rightarrow B$ $\qquad Int_P = B^{HB_P}$

Orders on interpretations:

$I \leq_t J$ if $I(A) \leq_t J(A)$ $\qquad I \leq_i J$ if $I(A) \leq_i J(A)$

$I \leq_p J$ if $I(A) \neq \bot$ then $I(A) = J(A)$

for any ground atom A

Formula evaluation

Closed formula evaluation:

$I(X \wedge Y) = I(X) \wedge I(Y)$ $\quad$ $I(X \vee Y) = I(X) \vee I(Y)$

$I(X \otimes Y) = I(X) \otimes I(Y)$ $\quad$ $I(X \oplus Y) = I(X) \oplus I(Y)$

$I(\neg X) = \neg I(X)$

$I(\exists x F(x)) = \vee_{s \in GT} I(F(s)$ $\quad$ $I(\forall x F(x)) = \wedge_{s \in GT} I(F(s)$

Ultimate evaluation:

The ultimate evaluation $U(I,C)$ of a closed formula C w.r.t. I is a logical value α defined by:
if $J(C)=I(C)$ for any interpretation $J \geq_p I$ then $\alpha=I(C)$, else $\alpha=\perp$

Proposition 1 If $I(C)=I_{\top}(C)$ then $U(I,C)=I(C)$, else $U(I,C)=\perp$

Reasoning with imperfect information in bilattices

Two approaches to infer information:

1. Applying the rules
2. Completing missing information with default information
   - Conventional CWA: negative information is advantaged. The value false plays a special role as logical value by default
   - OWA: Any logical value can be assigned by default

     Default Interpretation $\mathcal{D}$

- Particular operators are defined for (1), (2)

Program operators and properties

Production operator $\Phi_P : Int_P \rightarrow Int_P$

$\Phi_P(I)(A) = U(I, C)$ if there is $A \leftarrow C \in P$, else $\Phi_P(I) = \perp$

Proposition 2: Φ_P is monotone w.r.t. $\leq_i$ and $\leq_p$.

Revision operator $\mathrm{Re}v : Int_P \times Int_P \rightarrow Int_P$

Revises interpretation X via interpretation J:

$\mathrm{Re}v(X, J) = X'$ s.t. $X'(A) = X(A)$ for any ground atom A

for which either $J(A) = \perp$ or $X(A) = J(A)$,

and $X'(A) = \perp$ for any other ground atom.

Program operators and properties

Refining operator $\Psi_P : Int_P \times Int_P \to Int_P$

$$\Psi_P(X, I) = \mathrm{Rev}(X, \Phi_P(\mathrm{Rev}(X, I) \oplus I))$$

Proposition 3

Let I be an interpretation, and $\mathcal{D}$ a default interpretation. $(\lambda X)\Psi_P(X, I)$ has a greatest fixpoint below $\mathcal{D}$ w.r.t. $\leq_p$ that can be obtained as limit of the decreasing sequence w.r.t. the same order, defined by: $X_0 = \mathcal{D}$; $X_n = \Psi_P(X_{n-1}, I)$ if n is a successor ordinal, and $X_n = \inf_{\leq_p, m<n} X_m$ if n is a limit ordinal.

The limit $Def_P^{\mathcal{D}}(I)$ is the default information to complete missing information

Program operators and properties. Fixpoint semantics

Integrating Operator $\Gamma_P : Int_P \rightarrow Int_P$

$$\Gamma_P(I) = \Phi_P(I) \oplus Def_P^{\mathcal{D}}(I)$$

Theorem 1 Γ_P is monotone w.r.t. $\leq_p$ order and has a least fixpoint given by the limit of the increasing sequence:

$$I_0 = Const_{\perp}; \quad I_n = \Gamma_P(I_{n-1}) \text{ for a successor ordinal n;}$$

$$I_n = \sup_{\leq_p, m<n} I_m \text{ for a limit ordinal n}$$

The fixpoint semantics of P is defined as the limit of the sequence from Theorem 1.

Theorem 2 The fixpoint semantics s of P satisfies $\Phi_P(s) = s$

RELATION TO OTHER SEMANTICS

Theorem 3 Let *P* be an extended program and *mstable(P)* be is multivalued stable model as defined by Fitting, which is the smallest in the information/ knowledge order. Then the fixpoint semantics of *P* w.r.t. the default interpretation that assigns the value *false* to any ground atom coincides with *mstable(P)*.

Theorem 4 Given an extended program *P*, for any logical value α from the underlying bilattice, the α-fix model of *P* as previously defined by us, coincides with the fixpoint semantics of *P* w.r.t. the default interpretation that uniformly assigns the value α to any ground atom.

Corollary 1 The fixpoint semantics captures the well-founded semantics, Przymusinki's three-valued stable semantics, and the Kripke-Kleene semantics.

Computational aspects

Proposition 4 If *Values(P)* is the set of logical values appearing in *P* to which one adds the four extreme values of the bilattice, then $\langle Closure(Values(P)), \leq_t, \leq_i \rangle$ is a finite bilattice.

Theorem 5 If *P* is function free then the computation of its fixpoint semantics finishes in a finite number of steps. Moreover, the complexity class is PTIME.

COMPUTATIONAL ASPECTS

Algorithm based on a bottom up approach to computing the fixpoint semantics

```
function Rev(Y, Z)
W := Y;
for every pair (A, v1) ∈ W
    if v1 ≠ ⊥ and (A, v2) ∈ Z and v2 ≠ ⊥ and
    v2 ≠ v1
        then replace (A, v1) with (A, ⊥) in W;
return W;

function Phi(P, I)
I_⊤ := I;
J = ∅;
for any pair (A, v) ∈ I_⊤
    if v = ⊥ then
        replace (A, v) with (A, ⊤) in I_⊤;
for any rule A ← B in P
    if I(B) = I_⊤(B) then insert (A, I(B)) in J
        else insert (A, ⊥) in J;
for any atom A not appearing in J insert (A, ⊥) in J;
return J;

function FixpointSemantics(P, D)
I2 := Const_⊥;
repeat
    I1 := I2;
    J_2 := D;
    repeat
        J1 := J2;
        J2 := Rev(J1, Phi(P, Rev(J1, I1) ⊕ I1))
    until J1 = J2;
    I2 := Phi(P, I1) ⊕ J1
until I1 = I2;
return I1.
```

Possible extension of the approach

- Considering sets of logical values assigned to atoms instead of a punctual logical value.

$$Int_P = (2^B)^{HB_P}$$

- Apart from the 3 orders seen so far on Int_P, there will be a 4th order related the idea of imprecision.

Applications

- Imperfect information integration
- Uncertain knowledge bases

Imperfect Information Integration

Example:

integrating imperfect information in medical diagnosing: does patient P have condition C?

Diagnosis(P,C) ← Tests(P,C) ∧ MDsSuspect(P,C)
Tests (P,C)← Test1(P,C) ⊕ Test2(P,C)
MDsSuspect(P,C) ← MD1Suspects(P,C) ⊗ MD2Suspects(P,C)

UNCERTAIN KNOWLEDGE BASES

An Uncertain Knowledge Base is a pair *KB=(F, R)* in the context of a bilattice as underlying logic

- *F* set of Facts or stored information. A fact is a pair of an atom and a logical value.
- *R* set of Rules or the inference mechanism
- The content of KB is expressed by the fixpoint semantics of the associated extended program – facts are transformed in rules that take priority over the rules in R

Data Complexity: the time complexity to answer an atomic query w.r.t. the size of *F*

Theorem 6 The data complexity for KB as defined above is PTIME

Uncertain Knowledge Bases

- A query can be defined as being a rule
- A query Q is evaluated by being integrated in the extended program associated to the KB
- Query optimisation – an essential topic in Computer Science
- We are currently studying the problems of query containment and equivalence, and their complexity classes in such a framework
- The framework is restricted to non recursive sets of rules (due to non decidability problem when recursion is allowed)
- Established result: Equivalence of non-recursive queries on $\wedge, \otimes$ is decidable: NP-complete and Π_P^2 -complete

Thank you!