Linear Algebra Theorems for Fuzzy Relation Equations

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Outline

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Introduction

Dialogues with Antonio Di Nola

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Algebraic analysis of fuzzy systems☆

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Outline

Semiring

A semiring $\mathcal{R} = (R, +, \cdot, 0, 1)$ is an algebra where

- $(R, +, 0)$ is a commutative monoid,
- $(R, \cdot, 1)$ is a monoid,
- for all $\alpha, \beta, \gamma \in R, \quad \alpha\cdot(\beta+\gamma) = \alpha\cdot\beta+\alpha\cdot\gamma, \quad (\beta+\gamma)\cdot\alpha = \beta\cdot\alpha+\gamma\cdot\alpha,$
- for all $\alpha \in R, \quad 0 \cdot \alpha = \alpha \cdot 0 = 0.$

A semiring is **commutative** if $(R, \cdot, 1)$ is a commutative monoid.
A semiring is **zerosumfree** if $\alpha + \beta = 0$ implies $\alpha = \beta = 0$.

Example

$(\mathbb{R} \cup -\infty, \max, +, -\infty, 0)$ - max-plus (schedule) algebra.

Incline

Definition

Incline is a commutative semiring $\mathcal{R}$ where for all $\alpha \in R$, $\alpha + 1 = 1$.

In details,

$\mathcal{R} = (R, +, \cdot, 0, 1)$ is an **incline** if

- $(R, +, 0)$ is a semilattice,
- $(R, \cdot, 1)$ is a commutative monoid,
- for all $\alpha, \beta, \gamma \in R$, $\quad \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$,
- for all $\alpha, \beta \in R$, $\quad \alpha + \alpha \cdot \beta = \alpha$.

Example

$(L, \vee, *, 0, 1)$ - semiring reduct of a residuated lattice.

Outline

Algebra of Matrices

Let

- $\mathcal{R} = (R, +, \cdot, 0, 1)$ – semiring,
- $R^{n\times m}$,$n, m \geq 1$, – set of $n \times m$-matrices over R,
- $A, B \in R^{n\times m}$, $C \in R^{m\times k}$

Operations

- $\mathbf{0}_{n\times m}$ – zero matrix, E_n – unit square matrix,
- $(A + B)_{n\times m} = (a_{ij} + b_{ij})$,
- $(\lambda A)_{n\times m} = (\lambda \cdot a_{ij})$,
- $(A \cdot C)_{n\times k} = \sum_{j=1}^{m}(a_{ij} \cdot c_{jl})$.

Semiring of Matrices

Let

- $M_n(R) = R^{n \times n}$, $n \geq 1$, – set of square matrices over R.

Then

- $(M_n(R), +, \cdot, \mathbf{0}_{n \times n}, E_n)$ – semiring.

Elementary Transformations of Matrices

Let $A, B \in R^{n \times m}$.

Elementary transformations of rows (columns)

Addition of a row (column) multiplied by a non-zero element from R to another row (column).

Elementary transform $A \Rightarrow^* B$

$A \Rightarrow^* B$ – matrix B is an **elementary transform** of A, if B can be obtained from A by a finite sequence of elementary transformations of rows (columns).

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Invertible Matrices

Definition

A square $n \times n$ matrix $A \in M_n(R)$ over semiring R is called **right (left) invertible** if there exists matrix $B \in M_n(R)$ such that

$$A \cdot B = E_n \quad (B \cdot A = E_n).$$

Proposition (Reutanauer, Straubing, Tan)

For matrix $A \in M_n(R)$ where R is a commutative semiring, the following statements are equivalent:

- A is right invertible,
- A is left invertible,
- A is invertible,
- $A \cdot A^T$ is an invertible diagonal matrix,
- $A^T \cdot A$ is an invertible diagonal matrix.

Similarity Matrices

Let $(L, \vee, *, 0, 1)$ be a semiring reduct of a residuated lattice.

Definition

A square $n \times n$ matrix S over L is called a **similarity matrix** if for all $i, j, k = 1, \ldots, n$,

- $s_{ii} = 1$, reflexivity
- $s_{ij} = s_{ji}$, symmetry
- $s_{ij} * s_{jk} \leq s_{ik}$, transitivity.

Permanent

Definition

Let R be a semiring. A **permanent** per A of a $n \times m$, $m \leq n$, matrix $A \in M_n(R)$ is

$$\operatorname{per} A = \sum_{\sigma \in S_{m,n}} a_{1\sigma(1)} \cdot \ldots \cdot a_{m\sigma(m)},$$

where $S_{m,n}$ is a set of all injective mappings from the set $\overline{1,m}$ to the set $\overline{1,n}$.

Bideterminant

Definition (J. Kuntzman, M. Minoux)

Let R be a semiring,

- A – square $n \times n$ matrix over R,
- P (Q) – set of even (odd) permutations of $\overline{1,n}$.

A **bideterminant** $|A|$ of A is an ordered pair

$$|A| = (|A|_1, |A|_2)$$

where

$$|A|_1 = \sum_{\sigma \in P} a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdot \ldots \cdot a_{n\sigma(n)},$$

and

$$|A|_2 = \sum_{\sigma \in Q} a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdot \ldots \cdot a_{n\sigma(n)}.$$

Properties of Bideterminant

Let R be a commutative semiring.

Bideterminant of the Unit Matrix

If $E_n \in R^{n\times n}$ is the **unit matrix**, then $|E_n| = (1, 0)$.

Zero Row

Let $A \in R^{n\times n}$ and for at least one $k \in \{1, \ldots, n\}$ and every $j = 1, 2, \ldots, n$, $a_{k,j} = 0$. Then $|A| = (0, 0)$.

Properties of Bideterminant

Let R be a commutative semiring.

Zero Bideterminant

$|A| \equiv 0$, if $|A|_1 = |A|_2$.

Equivalent bideterminants

If $A \Rightarrow^* B$ then there exists $C \in R^{n\times n}$ such that $|C| \equiv 0$ and $|B| = |A| + |C|$. We say that bideterminants $|A|$ and $|B|$ are **equivalent** and denote: $|A| \equiv |B|$.

Properties of Bideterminant

Let R be a commutative semiring.

Two equal rows

Let $A \in R^{n\times n}$ be a matrix where for some k and for some l such that $k \neq l$, $a_{k,j} = a_{l,j}$, $j = 1, \ldots, n$.

Then $|A| \equiv 0$.

Exchange of Two Rows

Let $A \in R^{n\times n}$, and $|A| = (|A|_1, |A|_2)$. If matrix $\tilde{A}$ arises from A after exchange of two rows then

$$|\tilde{A}| = (|A|_2, |A|_1).$$

Properties of Bideterminant

Let R be a commutative semiring.

Linearity

Let $A \in R^{n\times n}$ be a matrix such that $a_{k,j} = \lambda * b_{k,j} + \mu * c_{k,j}$, $j = 1, \ldots, n$. Then

$$|A| = \lambda * \begin{vmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ b_{k,1} & \ldots & b_{k,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{vmatrix} + \mu * \begin{vmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ c_{k,1} & \ldots & c_{k,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{vmatrix}$$

Properties of Bideterminant

Let R be a commutative semiring.

Row expansion formula

Let $A \in R^{n\times n}$. The following analog of the known row expansion formula is valid for bideterminants:

$$|A| = \sum_{\{j\leq n \,|\, i+j \text{ is even}\}} a_{i,j} * (|A'_{i,j}|_1, |A'_{i,j}|_2) + \sum_{\{j\leq n \,|\, i+j \text{ is odd}\}} a_{i,j} * (|A'_{i,j}|_2, |A'_{i,j}|_1).$$

Three Notions of Rank

Let R be a commutative semiring.

Discriminant Rank

A **rank** $r(A)$ of A is a maximal number k of rows $\bar{a}_{i_1}, \dots, \bar{a}_{i_k}$ (columns $\bar{a}^{j_1}, \dots, \bar{a}^{j_k}$) such that there exists a nonzero k-order minor of the $k \times m$ matrix $A(\bar{a}_{i_1}, \dots, \bar{a}_{i_k})$.

Three Notions of Rank

Let R be a commutative semiring.

Column Rank

A **column rank** $r_c(A)$ of A is the least number of linearly independent column vectors of A that are generators of the set $\{\bar{a}^1, \bar{a}^2, \ldots, \bar{a}^m\}$.

Three Notions of Rank

Let R be a commutative semiring.

Factor Rank

A **factor rank** $r_f(A)$ is the least positive integer k, $k \leq \min(m, n)$, such that there exist matrices $B \in R^{n \times k}$, $C \in R^{k \times m}$ satisfying $A = BC$.

Ranks and Linear Independence

- $r_f(A) \leq r_c(A)$,
- $r(A) \leq r_c(A)$.

- If $A \in R^{n \times m}$, $m \leq n$ and $r_f(A) = m$, then column vectors of A are linearly independent.
- If row-vectors $\bar{a}_1, \ldots, \bar{a}_k \in R^m$, $k \leq \min(n, m)$ of A are linearly dependent then

$$r(A(\bar{a}_1, \ldots, \bar{a}_k)) < k.$$

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- If row-vectors $\bar{a}_1, \ldots, \bar{a}_k \in R^m$, $k \leq \min(n, m)$ of A are linearly dependent then

$$r(A(\bar{a}_1, \ldots, \bar{a}_k)) < k.$$

Outline

System of Linear-like Equations

Let R be a commutative semiring, $A \in R^{n\times m}$.

$$\begin{aligned} a_{11}\cdot x_1 + \cdots + a_{1m}\cdot x_m = b_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\cdot x_1 + \cdots + a_{nm}\cdot x_m = b_n, \end{aligned} \qquad (1)$$

is a **system of equations** with respect to the unknown vector $\bar{x} = (x_1 \ldots, x_m)^T \in R^m$. The short denotation of (1):

$$A \cdot \bar{x} = \bar{b},$$

Kronecker-Capelli Theorem

Theorem (necessity)

If the system $A \cdot \bar{x} = \bar{b}$ is solvable then

- $r(A) = r(A\bar{b})$,
- $r_f(A) = r_f(A\bar{b})$.

What about sufficiency?

Sufficiency

- The Kronecker-Capelli Theorem (the form "if and only if") does not hold for the column rank,
- The Kronecker-Capelli Theorem (the sufficiency form) does not hold for discriminant and factor ranks.

Kronecker-Capelli Theorem

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Kronecker-Capelli Theorem. Particular Case

Theorem (Shu, Wang, 2012)

Let

- R – commutative zerosumfree semiring,
- every non-zero element from R is invertible.

Then the system $A \cdot \bar{x} = \bar{b}$ is solvable if and only if

- columns $\bar{a}_1, \ldots, \bar{a}_m$ of A are orthogonal,
- for every $i = 1, \ldots, m$, $(\bar{a}_i, \bar{a}_i)$ is invertible.

Moreover, the solution is unique.

Outline

Cramer Rule in Zerosumfree Semiring

Theorem (Tan, 2007)

Let

- R – commutative zerosumfree semiring,
- $A \in M_n(R)$ – invertible matrix.

Then the system $A \cdot \bar{x} = \bar{b}$ has a unique solution $\bar{x} = (d^{-1} \cdot d_1, \ldots, d^{-1} \cdot d_n)^T$ where $d = \text{per}\, A$ and for all $j = 1, \ldots, n$,

$$d_j = \text{per} \begin{pmatrix} a_{11} & \ldots & a_{i,j-1} & b_1 & a_{i,j+1} & \ldots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \ldots & a_{n,j-1} & b_n & a_{n,j+1} & \ldots & a_{nn} \end{pmatrix}.$$

Cramer Rule in Incline

Theorem (Han, Li 2004)

Let

- R – incline,
- $A \in M_n(R)$ – invertible matrix.

Then the system $A \cdot \bar{x} = \bar{b}$ has a unique solution $A^T \bar{b} = (d_1, \ldots, d_n)^T$ where for all $j = 1, \ldots, n$,

$$d_j = \text{per} \begin{pmatrix} a_{11} & \ldots & a_{i,j-1} & b_1 & a_{i,j+1} & \ldots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \ldots & a_{n,j-1} & b_n & a_{n,j+1} & \ldots & a_{nn} \end{pmatrix}.$$

That's nice. But invertible matrices have rather simple structure ...

Equation with Similarity Matrix. Preliminaries

Let

- $(L, \vee, *, 0, 1)$ – semiring reduct of a residuated lattice,
- $S \in M_n(L)$ – similarity matrix over L.

Propositions

- Any similarity matrix can be obtained from E_n by a finite sequence of elementary transformations of rows.
- There exists a sequence of matrices $\{E_n, \dots, S_i, S_{i+1}, \dots, S\}$ such that a bideterminant of each second matrix in this sequence is equivalent to a bideterminant of the previous one.
- $|S| \equiv (1, 0)$.

Cramer Rule for Equation with Similarity Matrix

The **greatest solution** of the solvable system

$$S \cdot \bar{x} = \bar{b}$$

is equal to $(\hat{x}_1, \ldots, \hat{x}_n)^T$ where

$$\hat{x}_i = \Delta_1 \to \Delta_{i1}, \quad i = 1, \ldots, n,$$

and

- $|S| \equiv (\Delta_1, \Delta_2)$ so that $\Delta_1 = 1, \Delta_2 = 0$,
- $|S_i| \equiv (\Delta_{i1}, \Delta_{i2})$ so that $\Delta_{i1} = b_i, \Delta_{i2} = 0$.

Cramer Rule for Equation with Similarity Matrix. Particular Case

The **greatest solution** of the solvable system

$$S \cdot \bar{x} = \bar{b}$$

where $S = (s_{i,j})$ and for all i, j, k, l, $s_{i,j} \geq s_{k,l}$ if $|i - j| \leq |k - l|$, is equal to $(\hat{x}_1, \ldots, \hat{x}_n)^T$, such that

$$\hat{x}_i = \Delta_1 \to \Delta_{i1}, \quad i = 1, \ldots, n,$$

and

- $|S| = (\Delta_1, \Delta_2) = (1, \Delta_2)$,
- $|S_i| = (\Delta_{i1}, \Delta_{i2}) = (b_i, \Delta_{i2})$.

Conclusion

- An overview of solvability of matrix equations in various algebras were given
- Generalized notions of determinant and rank have been discussed,
- Solvability of matrix equations in terms of ranks and generalized determinants has been discussed.

Happy Birthday, Antonio !