IT4Innovations
Centre of Excellence

Models and Submodels in Higher-Order Fuzzy Logic

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In honour of Antonio Di Nola's 65th birthday

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IRAFM

Outline

History

- Founder: *B. Russel (1903,1908), R. Carnap, K. Gödel, A. Tarski, A. Turing*
- Type theory as a (higher-order) **logic**

  A. Church (1940), L. Henkin (1950, 1963), P. Andrews, W. Farmer
- Type theory as an effective theoretical tool in **computer science**

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Seven virtues of simple type theory

W. Farmer

1. STT has a simple and highly uniform syntax
2. The semantics of STT is based on a small collection of well-established ideas
3. STT is a highly expressive logic
4. STT admits categorical theories of infinite structures
5. There is a simple, elegant, and powerful proof system for STT
6. Techniques of first-order model theory can be applied to STT; distinction between standard and nonstandard models is illuminated
7. There are practical extensions of STT that can be effectively implemented

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Why *FUZZY* type theory

(i) Using FTT, a model of some deep manifestations of the vagueness phenomenon is formed

(ii) Using FTT, the type-theoretical model of concepts and linguistic semantics can be extended to include vagueness

(iii) Using FTT, the FLb-logic (*Fuzzy Logic in Broader Sense*) can be further developed; *formal theory of commonsense reasoning can be brought closer to the human way of thinking*.

Expressive power of FTT makes the task easier

Need for a well developed model theory of FTT

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Structure of truth values

EQ-algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$$

(E1) $\langle E, \wedge \rangle$ is a $\wedge$-semilattice with the top element $\mathbf{1}$

(E2) $\langle L, \otimes, \mathbf{1} \rangle$ is a monoid
$\otimes$ is isotone w.r.t. $\leq$ ($a \leq b$ iff $a \wedge b = a$)

(E3) $a \sim a = \mathbf{1}$ (reflexivity)

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$ (substitution)
(*Leibnitz rule of indiscernibility of identicals*)

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ (congruence)

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ (monotonicity)

(E7) $a \otimes b \leq a \sim b$ (boundedness)

Special definitions in EQ-algebras

- good if $a \sim \mathbf{1} = a$
- $a \to b = (a \wedge b) \sim a$ (implication)
- $\mathcal{E}$ is separated if

$$a \sim b = \mathbf{1} \quad \text{iff} \quad a = b$$

- If $\mathcal{E}$ contains $\mathbf{0}$ then $\neg a = a \sim \mathbf{0}$ (negation)

In linearly ordered structure of truth values:

$$\Delta(a) = \begin{cases} \mathbf{1} & \text{if } a = \mathbf{1}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

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Types and formulas

Elementary types: *o* (truth values), ϵ (objects)

Types

(i) $\epsilon, o \in$ *Types*,

(ii) If $\alpha, \beta \in$ *Types* then $(\alpha\beta) \in$ *Types*.

Formulas

(i) Variables and constants of type α are formulas.

(ii) If $B_{\beta\alpha}$ and A_α are formulas then $(B_{\beta\alpha}A_\alpha)$ is a formula of type β.

(iii) If A_β is a formula and $x_\alpha \in J$ a variable then $\lambda x_\alpha\, A_\beta$ is a formula of type $\beta\alpha$.

Formulas A_o are propositions

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Formulas A_o are propositions

Fuzzy equality

Special formula

$(A_\alpha \equiv B_\alpha)$ — *formula of type o*

$m, m' \in M_\alpha$

(i) $\doteq(m, m) = 1$ (reflexivity)

(ii) $\doteq(m, m') = \doteq(m', m)$ (symmetry)

(iii) $\doteq(m, m') \otimes \doteq(m', m'') \leq \doteq(m, m'')$ ($\otimes$-transitivity)

$h, h' \in M_\beta^{M_\alpha}$

$$[h \doteq h'] = \bigwedge_{m \in M_\alpha} [h(m) \doteq_\beta h'(m)]$$

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General frame

$$\mathcal{M} = \langle \{M_\alpha, \doteq_\alpha \mid \alpha \in Types\}, \mathcal{E}_\Delta \rangle$$

Each set M_α is associated with the corresponding type

(i) M_o — set of truth values

(ii) M_ϵ — some (non-empty) set

(iii) $\doteq_\alpha$ is a fuzzy equality on M_α

(iv) (a) **Standard model:** $M_{\beta\alpha} = M_\beta^{M_\alpha}$

(b) **General model:** $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$

(v) $M_{oo} \cup M_{(oo)o}$ is closed w.r.t. operations on truth values

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Scheme of general frame

$$(M_o = \{a \mid a \in L\}, \leftrightarrow) \qquad (M_\epsilon = \{u \mid \varphi(u)\}, =_\epsilon)$$

$$(M_{oo} \subseteq \{g_{oo} \mid g_{oo} : M_o \longrightarrow M_o\}, =_{oo})$$

$$(M_{o\epsilon} \subseteq \{f_{o\epsilon} \mid f_{o\epsilon} : M_\epsilon \longrightarrow M_o\}, =_{o\epsilon})$$

$$(M_{\epsilon\epsilon} \subseteq \{f_{\epsilon\epsilon} \mid f_{\epsilon\epsilon} : M_\epsilon \longrightarrow M_\epsilon\}, =_{\epsilon\epsilon}), \ldots$$

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Interpretation of formulas

Example (interpretation)

$\mathcal{M}(A_o) \in L$ is a truth value
$\mathcal{M}(A_{o\epsilon})$ is a fuzzy set in M_ϵ
$\mathcal{M}(A_{(o\epsilon)\epsilon})$ is a fuzzy relation on M_ϵ
$\mathcal{M}(A_{\epsilon\epsilon})$ is a function on objects

In a general model $\mathcal{M}$, each formula A_α must have an interpretation

$$\mathcal{M}_p(A_\alpha) \in M_\alpha$$

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Many useful properties

Theorem (Completeness)

(a) *A theory T of FTT is consistent iff it has a general model $\mathcal{M}$.*

(b) *For every theory T of FTT and a formula A_o*

$$T \vdash A_o \quad \text{iff} \quad T \models A_o.$$

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Finite models

A model $\mathcal{M}$ is finite if all sets M_α are finite

Theorem

If a model $\mathcal{M}$ is finite then it is standard.

Frugal model

A (general) model $\mathcal{M}$ is frugal if $\mathrm{Card}(M_\alpha) \leq \mathrm{Card}(J) + \aleph_0$, $\alpha \in$ *Types*

Theorem

If T a consistent theory and $\mathcal{E}_\Delta$ is finite and then the model $\mathcal{M} = \langle \{M_\alpha, \doteq_\alpha |\ \alpha \in \mathit{Types}\}, \mathcal{E}_\Delta \rangle$ frugal.

Relations between models

$$\mathcal{M}^1 = \langle \left(M^1_\alpha, \doteq^1_\alpha \right)_{\alpha \in Types}, \mathcal{E}^1 \rangle$$
$$\mathcal{M}^2 = \langle \left(M^2_\alpha, \doteq^2_\alpha \right)_{\alpha \in Types}, \mathcal{E}^2 \rangle$$

be two models. Let us consider a set of functions

$$\mathfrak{f} = \{ f^\alpha : M^1_\alpha \longrightarrow M^2_\alpha \mid \alpha \in Types \} \qquad (1)$$

Relations between models

(i) For all $\alpha, \beta \in$ *Types*, $(f^\alpha, f^\beta, f^{\beta\alpha})$ forms a commuting triple.

$$\begin{array}{ccc} M^1_\alpha & \xrightarrow{f^\alpha} & M^2_\alpha \\ {\scriptstyle m^1_{\beta\alpha}}\downarrow & & \downarrow{\scriptstyle m^2_{\beta\alpha} = f^{\beta\alpha}(m^1_{\beta\alpha})} \\ M^1_\beta & \xrightarrow{f^\beta} & M^2_\beta \end{array}$$

(ii) $f^o : E^1 \longrightarrow E^2$ preserves all existing infima.

(iii) For each constant c_α

$$f^\alpha(\mathcal{M}^1(c_\alpha)) = \mathcal{M}^2(c_\alpha).$$

Homomorphism of $\mathcal{M}_1$ and $\mathcal{M}_2$

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Relations between models

Embedding

$$\mathfrak{f} : \mathcal{M}^1 \longrightarrow \mathcal{M}^2$$

all functions f^α are injections

Lemma

Each homomorphism $\mathfrak{f}$ is necessarily an embedding of the model $\mathcal{M}^1$ in the model $\mathcal{M}^2$

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Lemma

Each homomorphism $\mathfrak{f}$ is necessarily an embedding of the model $\mathcal{M}^1$ in the model $\mathcal{M}^2$

Special definitions

Definition

Let $\mathfrak{f} : \mathcal{M}^1 \longrightarrow \mathcal{M}^2$ be an embedding.

(i) $\mathcal{M}^1$ is a submodel of $\mathcal{M}^2$, $\mathcal{M}^1 \subset \mathcal{M}^2$, if f^o and f^ϵ are identities and $\mathcal{E}^1$ is a subalgebra of $\mathcal{E}^2$.

(ii) Isomorphism between $\mathcal{M}^1$ and $\mathcal{M}^2$

$$\mathcal{M}^1 \cong \mathcal{M}^2.$$

All functions in $\mathfrak{f}$ are bijections

Special definitions

Definition

(i) Elementary equivalent models $\mathcal{M}^1$ and $\mathcal{M}^2$, $\mathcal{M}^1 \equiv \mathcal{M}^2$:

$$\mathcal{M}^1(A_o) = \mathbf{1}^1 \quad \text{iff} \quad \mathcal{M}^2(A_o) = \mathbf{1}^2$$

for arbitrary sentence $A_o \in Form_o$.

(ii) Strongly elementary equivalent models $\mathcal{M}^1$ and $\mathcal{M}^2$, $\mathcal{M}^1 \triangleq \mathcal{M}^2$, if $\mathcal{E}^1 = \mathcal{E}^2$ and

$$\mathcal{M}^1(A_o) = \mathcal{M}^2(A_o)$$

for arbitrary sentence $A_o \in Form_o$

Special definitions

Definition

(i) Elementary embedding $\mathfrak{f} : \mathcal{M}^1 \longrightarrow \mathcal{M}^2$

$$f^o(\mathcal{M}^1_p(A_o)) = \mathcal{M}^2_{p \circ \mathfrak{f}}(A_o)$$

for arbitrary formula $A_o \in Form_o$ and assignment p

(ii) A model $\mathcal{M}^1$ is an elementary submodel of $\mathcal{M}^2$, $\mathcal{M}^1 \prec \mathcal{M}^2$, if $\mathcal{M}^1 \subset \mathcal{M}^2$ and

$$\mathcal{M}^1_p(A_o) = \mathcal{M}^2_{p \circ \mathfrak{f}}(A_o)$$

for arbitrary formula $A_o \in Form_o$ and assignment p.

Theorem

(a) *If* $\mathcal{M}^1 \triangleq \mathcal{M}^2$ *then* $\mathcal{M}^1 \equiv \mathcal{M}^2$. *If* $\mathcal{M}^1 \cong \mathcal{M}^2$ *then* $\mathcal{M}^1 \triangleq \mathcal{M}^2$.

(b) *If* $\mathcal{M}^1 \prec \mathcal{M}^2$ *then* $\mathcal{M}^1 \equiv \mathcal{M}^2$.

(c) *If* $\mathfrak{f} : \mathcal{M}^1 \longrightarrow \mathcal{M}^2$ *and* $\mathfrak{g} : \mathcal{M}^2 \longrightarrow \mathcal{M}^3$ *are elementary embeddings then* $\mathfrak{f} \circ \mathfrak{g} : \mathcal{M}^1 \longrightarrow \mathcal{M}^3$ *is an elementary embedding.*

(d) *If* $\mathcal{M}^1 \prec \mathcal{M}^2$ *and* $\mathcal{M}^2 \prec \mathcal{M}^3$ *then* $\mathcal{M}^1 \prec \mathcal{M}^3$.

(e) *The relations* $\cong, \equiv, \triangleq$ *are equivalences.*

Standard model $\mathcal{M}$

(i) If M_o, M_ϵ are finite then all M_α for $\alpha \in$ *Types* are finite.

(ii) Let $\mathrm{Card}(M_\epsilon) < \aleph_0$ and $\mathrm{Card}(M_o) \in \{\aleph_0, \aleph_1\}$.
If α does not contain the type *o* then $\mathrm{Card}(M_\alpha) < \aleph_0$.

(iii) Thus, if $\mathrm{Card}(M_\alpha) = \aleph_\eta$ and $\mathrm{Card}(M_\beta) < \aleph_0$ then $\mathrm{Card}(M_{\beta\alpha}) = \aleph_{\eta+1}$ and $\mathrm{Card}(M_{\alpha\beta}) = \aleph_\eta$.

(iv) Analogously for $\mathrm{Card}(M_o) < \aleph_0$ and $\mathrm{Card}(M_\epsilon) \in \{\aleph_0, \aleph_1\}$.

(v) If $\mathrm{Card}(M_o), \mathrm{Card}(M_\epsilon) \in \{\aleph_0, \aleph_1\}$, or one (or both) of the former are finite then $\mathrm{Card}(M_\alpha) < \aleph_\omega$ for all $\alpha \in$ *Types*.

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(iv) Analogously for $\mathrm{Card}(M_o) < \aleph_0$ and $\mathrm{Card}(M_\epsilon) \in \{\aleph_0, \aleph_1\}$.

(v) If $\mathrm{Card}(M_o), \mathrm{Card}(M_\epsilon) \in \{\aleph_0, \aleph_1\}$, or one (or both) of the former are finite then $\mathrm{Card}(M_\alpha) < \aleph_\omega$ for all $\alpha \in$ *Types*.

Standard model $\mathcal{M}$

(i) If M_o, M_ϵ are finite then all M_α for $\alpha \in$ *Types* are finite.

(ii) Let $\mathrm{Card}(M_\epsilon) < \aleph_0$ and $\mathrm{Card}(M_o) \in \{\aleph_0, \aleph_1\}$.
If α does not contain the type *o* then $\mathrm{Card}(M_\alpha) < \aleph_0$.

(iii) Thus, if $\mathrm{Card}(M_\alpha) = \aleph_\eta$ and $\mathrm{Card}(M_\beta) < \aleph_0$ then $\mathrm{Card}(M_{\beta\alpha}) = \aleph_{\eta+1}$ and $\mathrm{Card}(M_{\alpha\beta}) = \aleph_\eta$.

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Analogue of the downward Löwenheim-Skolem theorem

Theorem

let $\mathcal{M}$ be a model with infinite M_o, M_ϵ. Let κ be a cardinal such that for $\gamma \in \{o, \epsilon\}$

$$\max(p(M_o), \mathrm{Card}(Form), \aleph_0) \leq \kappa \leq \mathrm{Card}(M_\gamma)$$

Then there is an elementary submodel $\mathcal{Y} \prec \mathcal{M}$ such that $\mathrm{Card}(Y_\alpha) \leq \kappa$, $\alpha \in Types$.

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