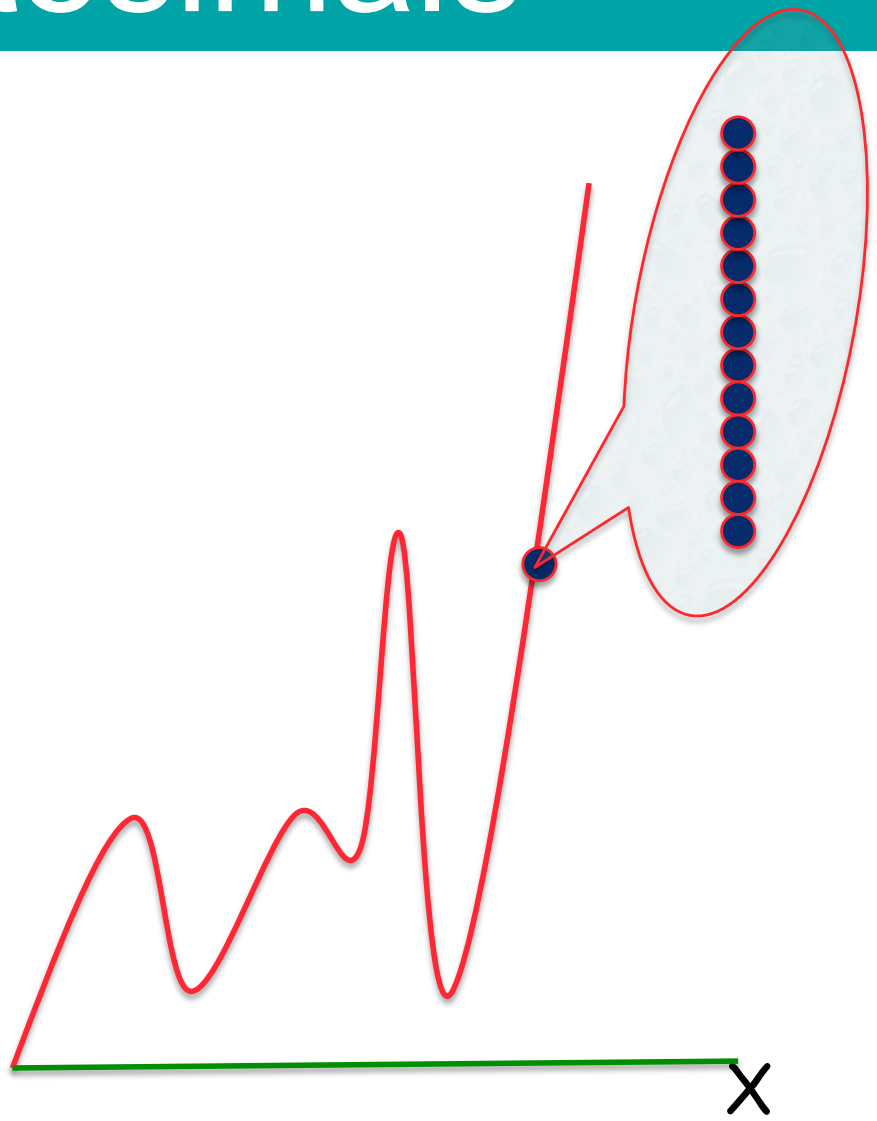
# Strongly complete real semantics for infinite-valued Lukasiewicz logic

Daniele Mundici
Department of Mathematics "Ulisse Dini"
University of Florence,
Florence, Italy
mundici@math.unifi.it

# Infinitesimals

we all know Di Nola's theorem: whenever an MV-algebra A is not semisimple, there is a **nonprincipal** ultrapower J of [0,1] such that A is isomorphic to an MV-algebra of J-valued functions over the set X of prime ideals of A



# Aim of this talk

to devise a controlled generation of MV-algebraic infinitesimals, in an attempt to make nonstandard reals as real as possible

# Strong semisimplicity

- **maximal** congruence of $A$ is one which is maximal among those $\neq A^2$
- the **principal** congruence generated by two elements a and b of $A$ is the smallest congruence $\approx$ of $A$ such that $a \approx b$
- $A$ is **strongly semisimple** if every principal congruence of $A$ is an intersection of maximal congruences of $A$
- DUBUC AND POVEDA introduce this notion in 2010 (Ann. Pure. Appl. Logic, vol. 161): ***an MV-algebra A is strongly semisimple if every principal ideal of A is an intersection of maximal ideals of A***

# the case of boolean algebras

- in every boolean algebra $A$, every prime ideal is maximal
- every ideal of $A$ is an intersection of prime ideals
- so in particular every principal ideal is an intersection of maximals
- and $A$ is strongly semisimple
- a very general problem: which groups, lattices, Heyting algebras, semigroups, lattice-ordered groups, rings, vector lattices, Banach algebras, etc., are strongly semisimple?

# Severi-Bouligand tangents

*Semiretta tangente* ad $I$ in $O$ è una semiretta $a$, di origine $O$, la quale sia di accumulazione per *ogni* insieme di semirette proiettanti da $O$ i punti di $I$, distinti da $O$, situati in un *qualunque* intorno di $O$; cioè una semiretta tale che, assegnati comunque un numero $\varepsilon > 0$ ed un angolo $\delta$, esiste in $I$ qualche punto avente da $O$ una distanza non nulla minor di $\varepsilon$, il quale è proiettato da $O$ secondo una semiretta formante con $a$ un angolo non nullo, $\varphi$, minor di $\delta$.

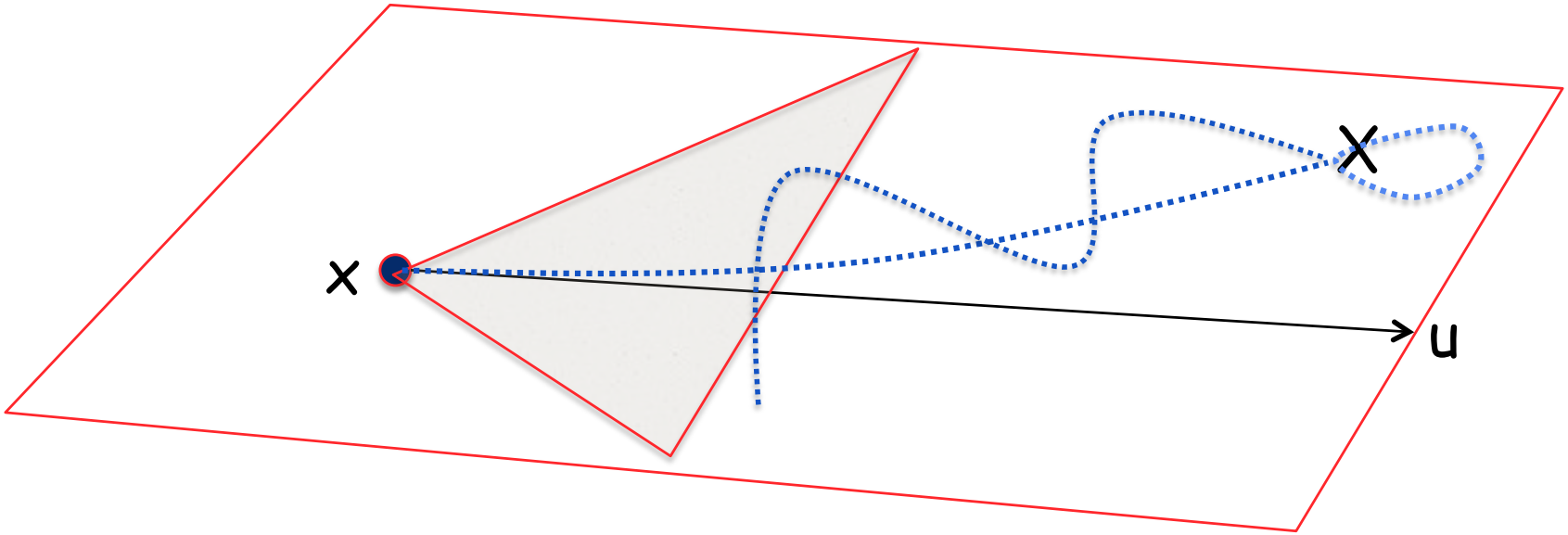
F.Severi, Ann.Soc.Polon. Math. 9 (**1931**) 97-108. Firstly defined, Rome, Stab.Tipogr. Genio Civile and Zanichelli **1927**

1. Par ***contingent*** en un point d'accumulation $A$ d'un ensemble $E$ composé de points de l'espace euclidien, j'entends *le système des demi-tangentes* $A\Delta$ en $A$, ou encore des demi-droites auxquelles on peut attacher une suite infinie de points $M_K$, tous distincts, appartenant à $E$, tendant vers $A$ et tels que les angles $M_K A\Delta$ tendant vers zèro. On reconnait que $A\Delta$ est demi-tangente en $A$ à $E$ si *tout* cône circulaire droit de sommet $A$, d'axe $A\Delta$, enferme toujours des points de $E$ distincts de $A$. Il s'ensuit que, pour tout ensemble $E$, *le contingent est fermé.*

H.Bouligand, Ann.Soc.Polonaise Math. 9 (**1930**) 32-41

# u is a tangent unit vector of X in $R^2$ at x

Any triangle with vertex x containing [x,x+u] in its interior, contains ∞ many points of X



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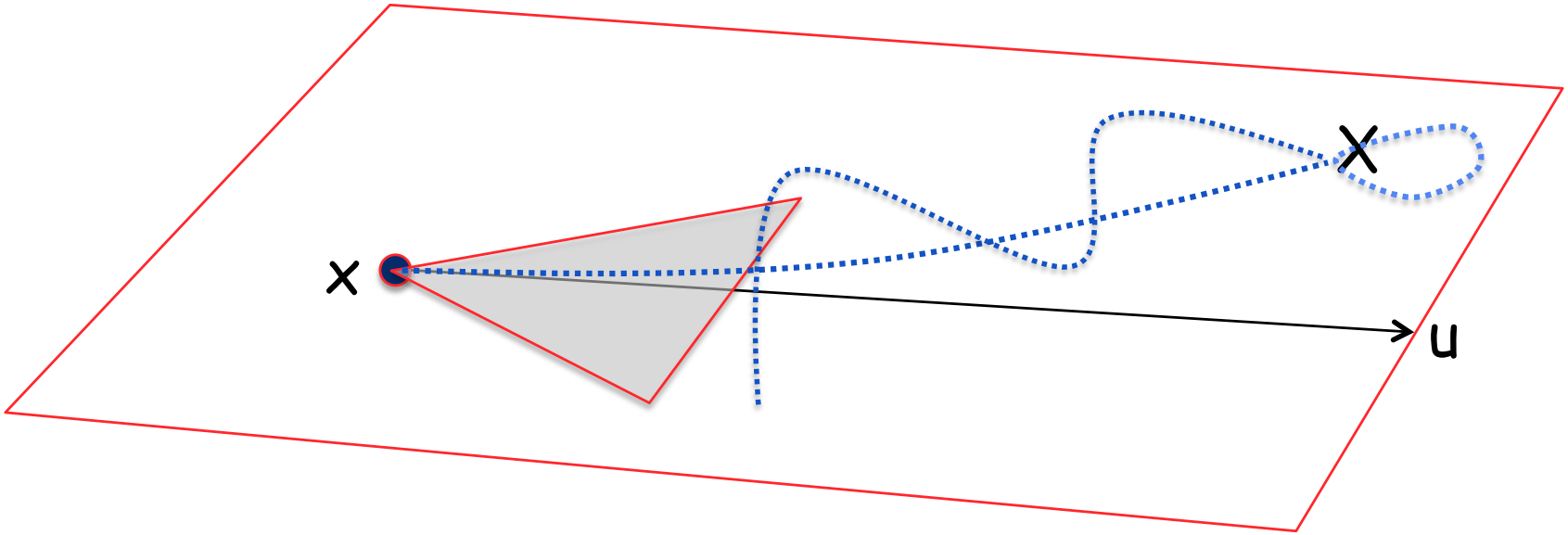

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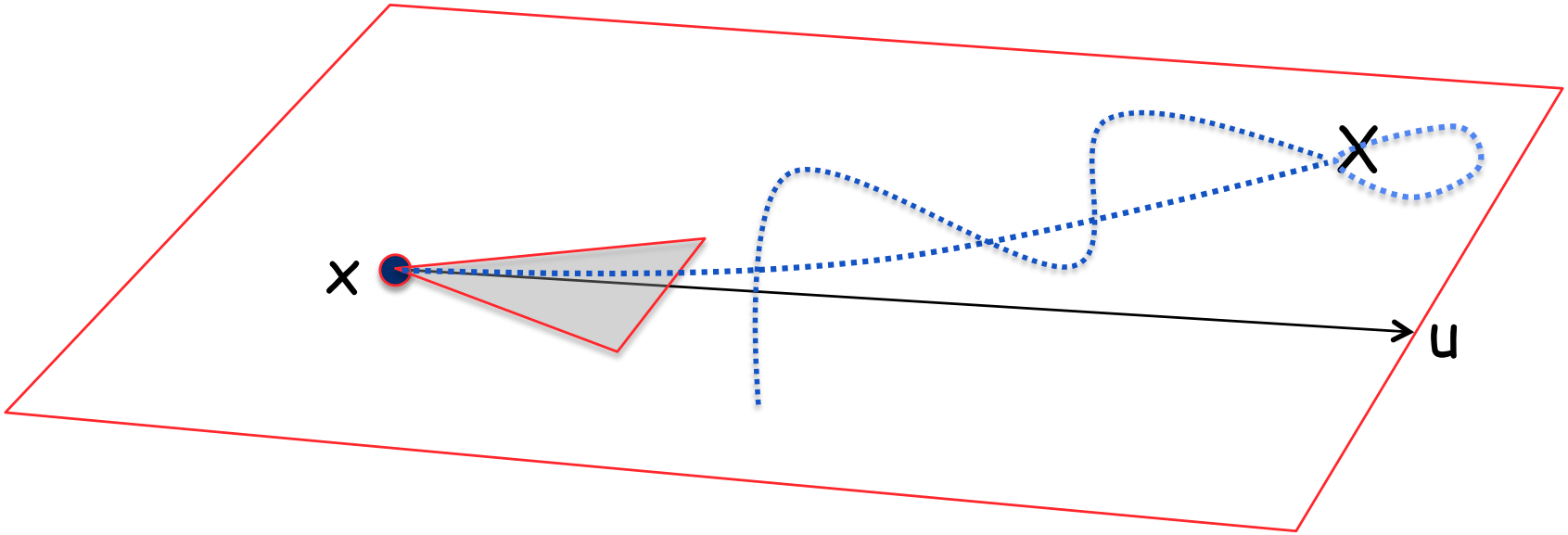

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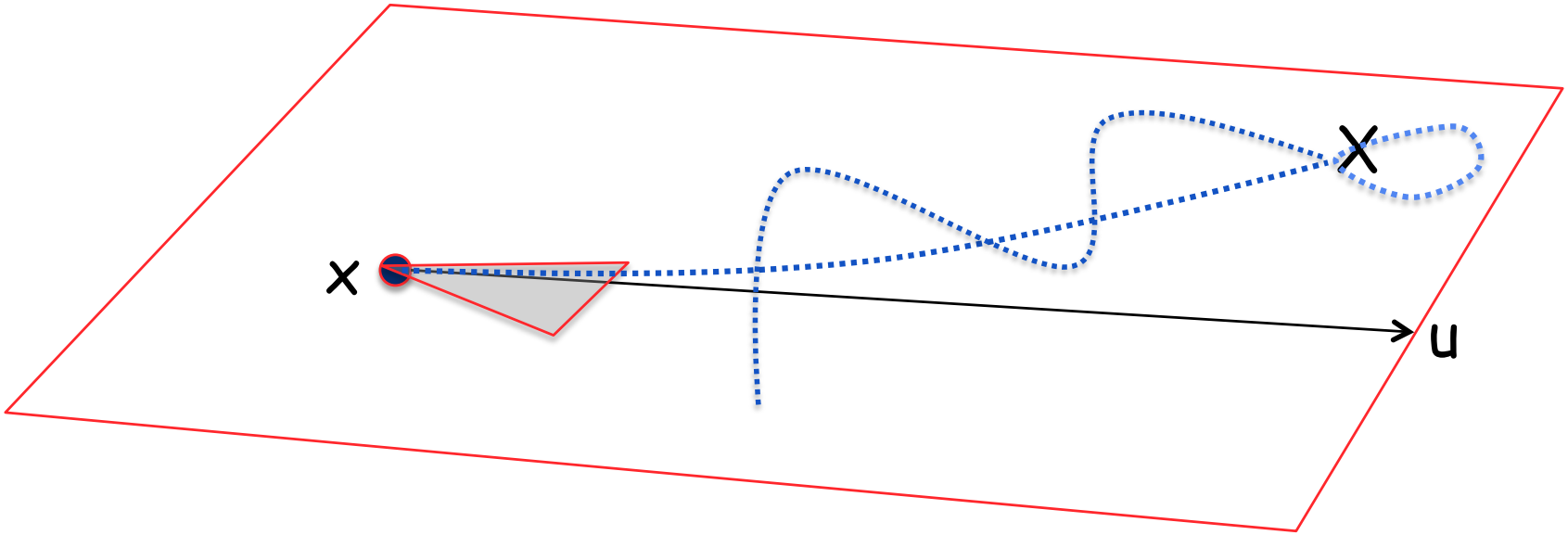
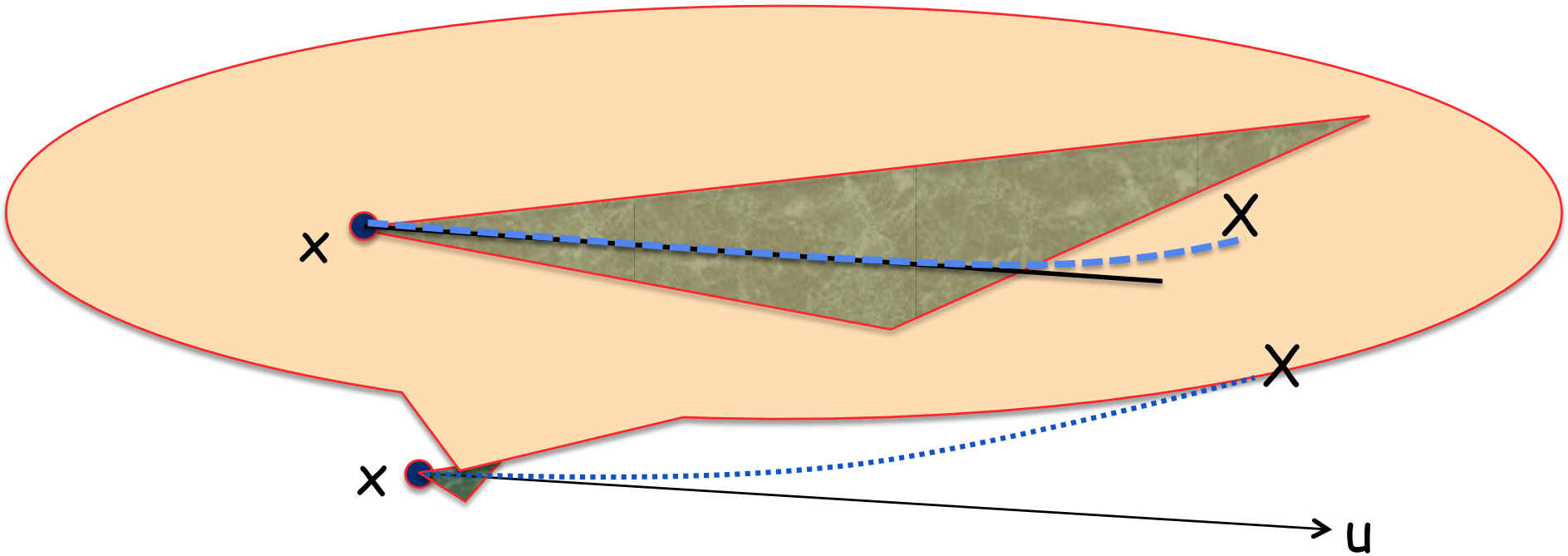


# in all small cones, [x,x+u] intersects X only at x



# definition of tangent vector *u* of a closed **set** *X* in euclidean space $\mathbf{R}^n$ at a point *x*

For all small $\partial$, h > 0, the cone C with vertex x, axis parallel to u, angle $\partial$, and height h,
**contains infinitely many points of X, but none of them except x lies in the segment [x,x+u]**

NOTE: X is an **arbitrary** closed set in euclidean space

# classical consequence (B. Bolzano) deals with models=valuations=interpretations

we write **f(m)=1** instead of "**m is a model of f**"

*A formula f is a* ***consequence (in the sense of Bolzano)*** *of a set P of formulas if every model m of every p in P is also a model of f:*

$$\textit{If} \quad p(m) = 1 \quad \textit{for all} \quad p \quad \textit{in} \quad P \qquad \textit{then} \quad f(m) = 1$$

A **completeness theorem** *often gratifies this definition by showing that all the consequences of P can be computed by a logical calculus of tautologies and Modus Ponens*

# the set of all {0,1}-models has a very poor topological-differential structure

for instance, let P be a set of boolean formulas in the variables $X_1,...X_n$ . Every valuation m is uniquely determined by the tuple $X_1(m),...X_n(m)$.

Thus a valuation is a point in the space $\{0,1\}^{\{X1,...Xn\}}=\{0,1\}^n$

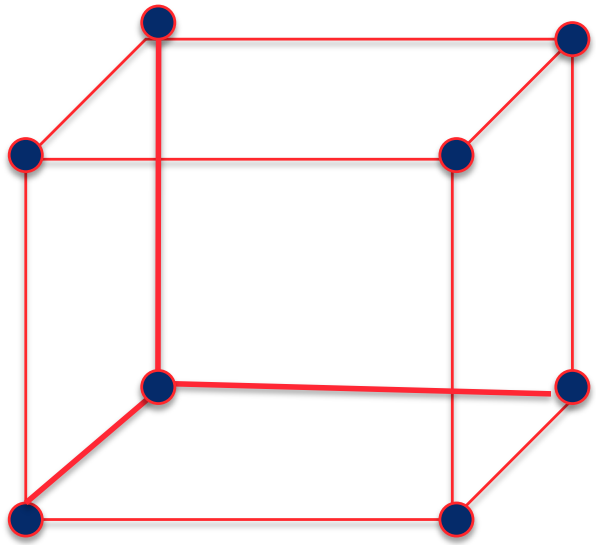
This space of models has the most rudimentary structure:

**it is *the* finite discrete topological space with $2^n$ elements**

all finite-valued logics have this zerodimensional structure

# particular case: boolean consequence

given a set $S = \{X_1,\ldots,X_n\}$ of propositional variables,
the set of all possible interpretations of these symbols
= the set of boolean functions on the vertex of the n-cube
= the set of **models** of formulas in the variables of S
= the set of valuations of these formulas



n=3

# the set of all [0,1]-models usually has a very rich differential structure

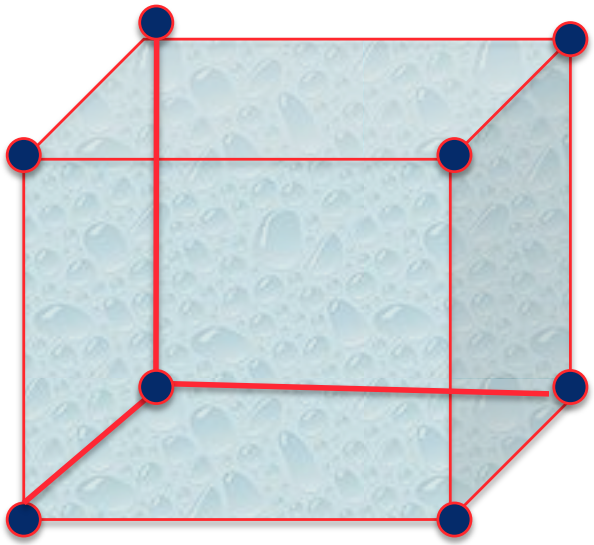
Valuations are functions from $\{X_1,\ldots X_n\}$ into [0,1]

Writing $[0,1]^{\{X1,\ldots Xn\}} = [0,1]^n$ the space of models inherits the rich topological, algebraic, linear, **differential** structure of the $n$-cube $[0,1]^n$

Any [0,1]-valued logic enjoys this structure

# [0,1]-valuations

given a set $S = \{X_1,\ldots,X_n\}$ of propositional variables,
the set of all possible interpretations of these symbols
= [0,1]-valued functions on the vertex of the n-cube
= [0,1]-**models** of formulas in the variables of S
= [0,1]-valuations of these formulas



n=3

# an example

the difference between
{0,1}-valuations
and [0,1]-valuations

# Italian **co-fund** system: the recent call for Projects of National Relevance, PRIN

**Call for projects date:** February 15, 2012, hrs 15.18

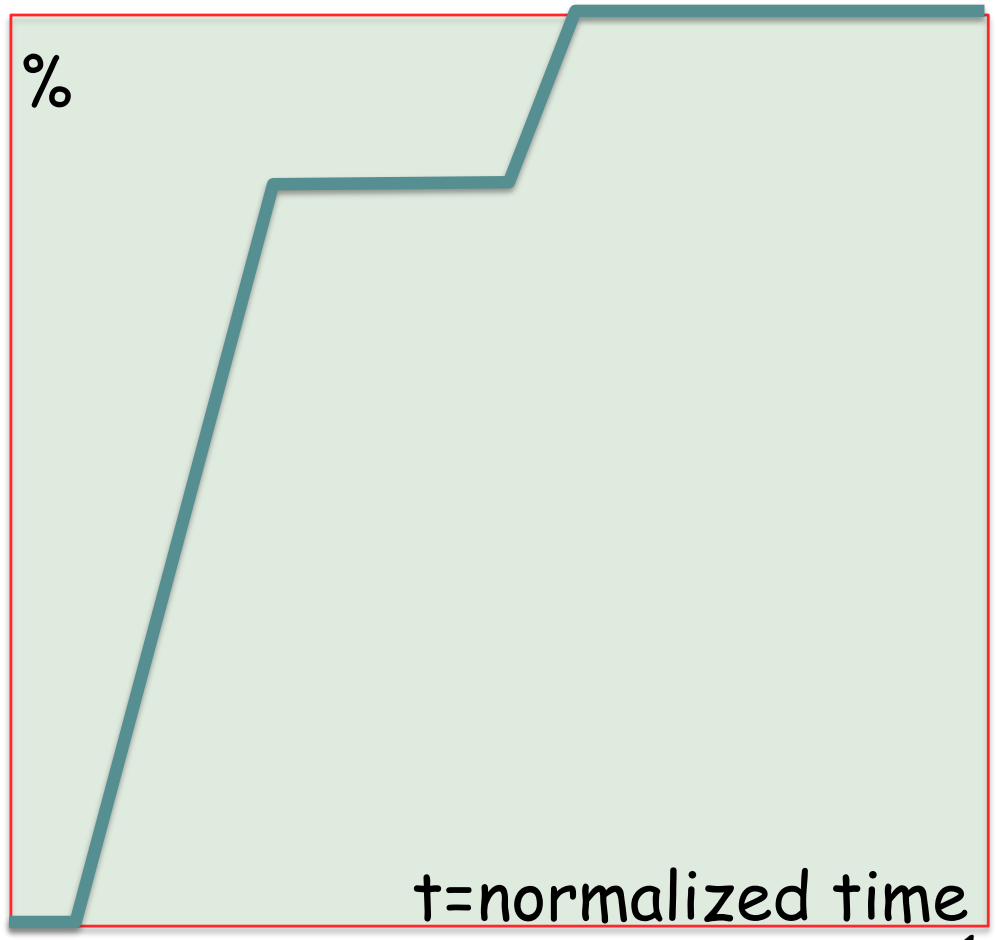
**Application submission deadline**: March 9, 2012, 17.00 hrs

**Co-fund rule:** Applicants must provide 30% of the required funding from sources ≠ the Ministry of Scientific Research.

**Co-funding deadline**: BEFORE March 9, 2012, 17.00 hrs

# Applicant A's co-funding record

%=certified cofunding percentage

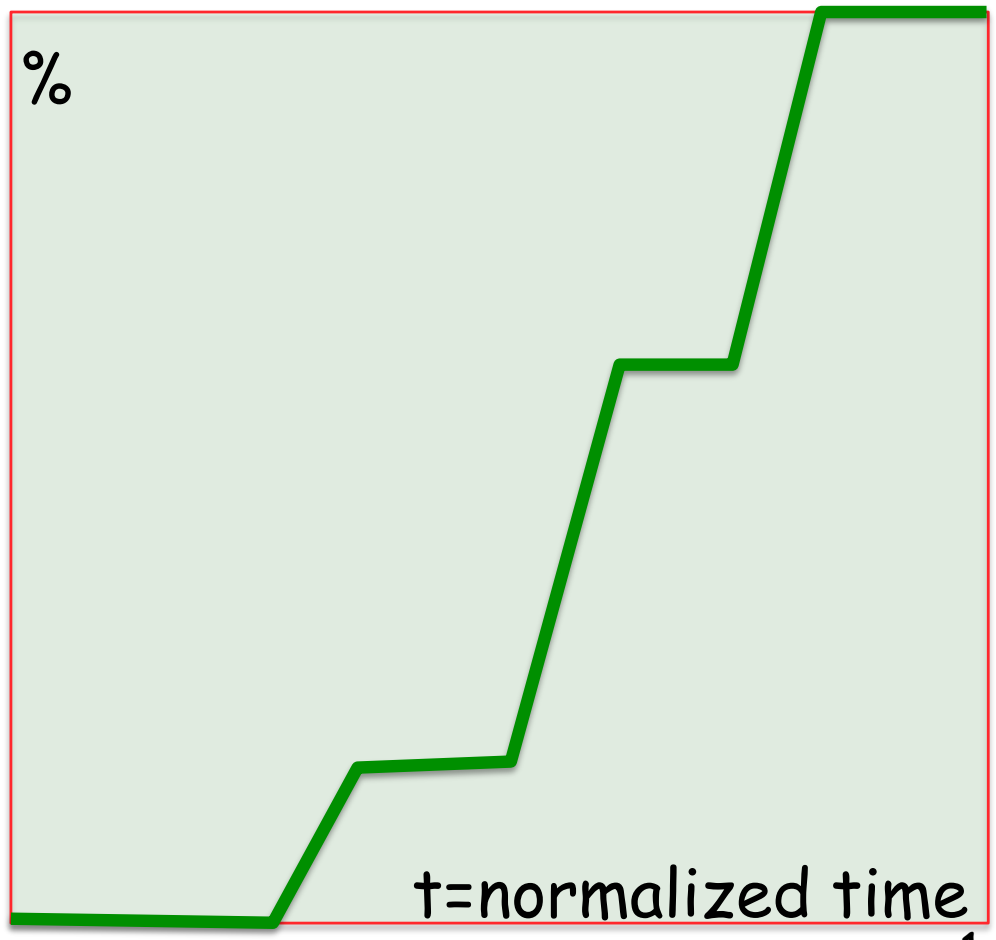



0=February 15, 15.18

1=March 9, 17.00

# Applicant B's cofunding record

%=certified cofunding percentage

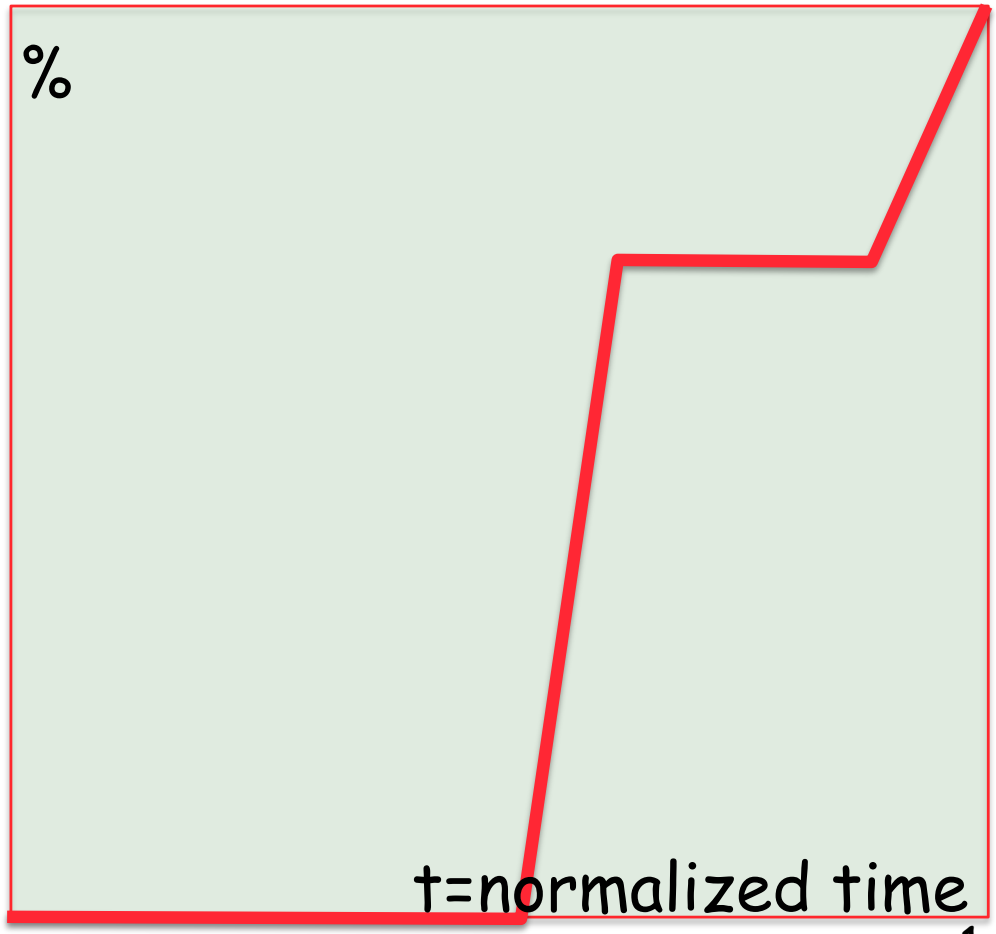



0=February 15, 15.18

1=March 9, 17.00

# Applicant C's cofunding record

%=certified cofunding percentage




0=February 15, 15.18

1=March 9, 17.00

**too late!**

**the system didn't accept this application**

# classical {0,1}-consequence (B. Bolzano)

*A formula $f$ is a consequence of a set $P$ of premises if every model $m$ of every $p$ in $P$ is also a model of $P$*

$$p(m)=1 \text{ for all } p \text{ in } P$$

$$\Downarrow$$

$$f(m)=1$$

# [0,1]-consequence looks around

*A formula $f$ is a consequence of a set $P$ of premises if every model $m$ of every $p$ in $P$ is also a model of $P$*

***but now any "model" looks around all directions d***

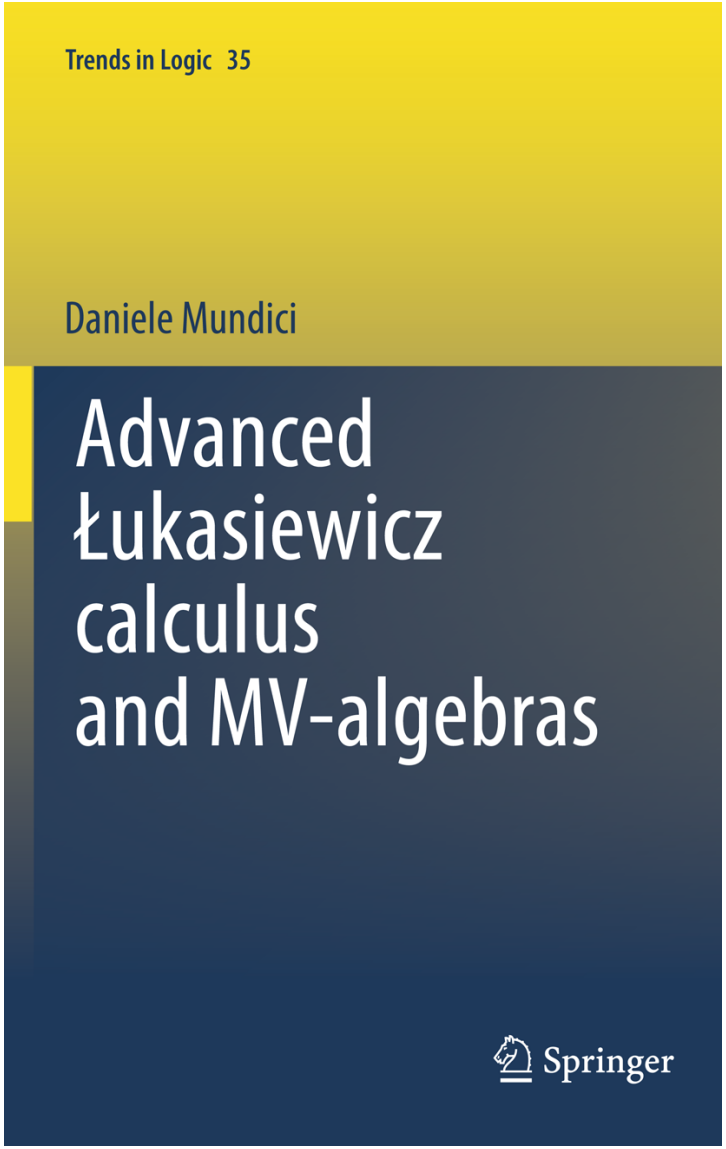
$p(m)=1$ *for all* $p$ *in* $P$ ***and*** $\partial p(m)/\partial d=0$ *for all* $p$ *in* $P$

$\Downarrow$

$f(m)=1$ ***and*** $\partial f(m)/\partial d=0$

# telling the difference between < and ≤

- Applicant C got the full percentage of cofunding **at the very instant** when the deadline expired.
- in Lukasiewicz logic $L_\infty$ we can write a set P of formulas such that a formula f is a consequence of P iff f describes a cofunding record where the required cofunding is obtained **before** (<) the deadline.
- $L_\infty$ *distinguishes between < and ≤*



# Lukasiewicz axioms (logic)
# Chang axioms (algebra)

$A \to (B \to A)$

$(A \to B) \to ((B \to C) \to (A \to C))$

$(\neg A \to \neg B) \to (B \to A)$

$((A \to B) \to B) \to ((B \to A) \to A)$

let us write $a+b$
instead of $\neg a \to b$

**MV-algebras** are involutive abelian monoids with 1, satisfying
$x+1=1$ and $\neg(\neg x+y)+y=\neg(\neg y+x)+x$

# the prototypical MV-algebra

the unit real interval [0,1]

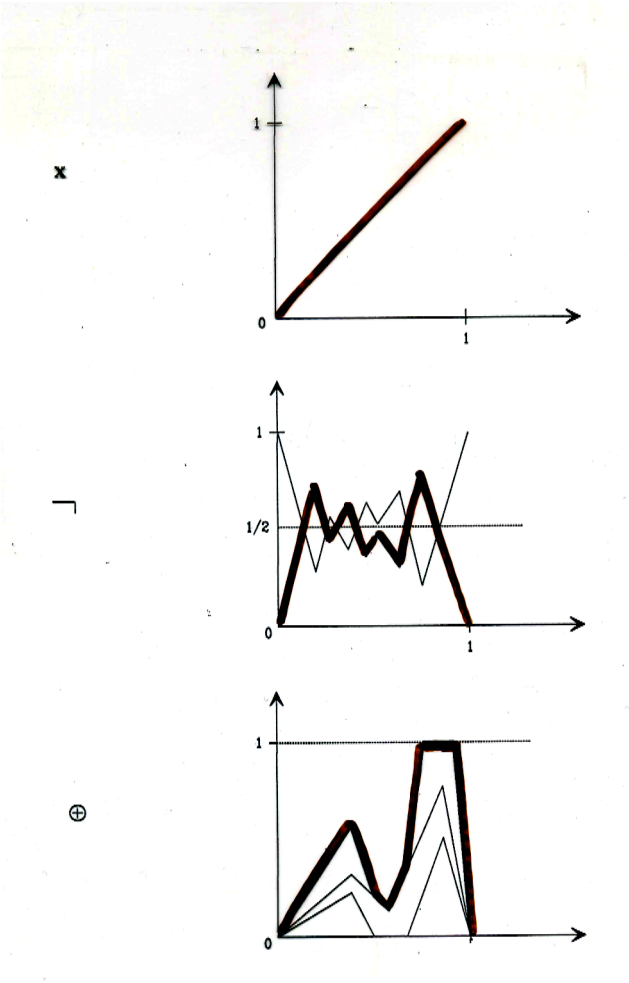
equipped with the distinguished constant 0

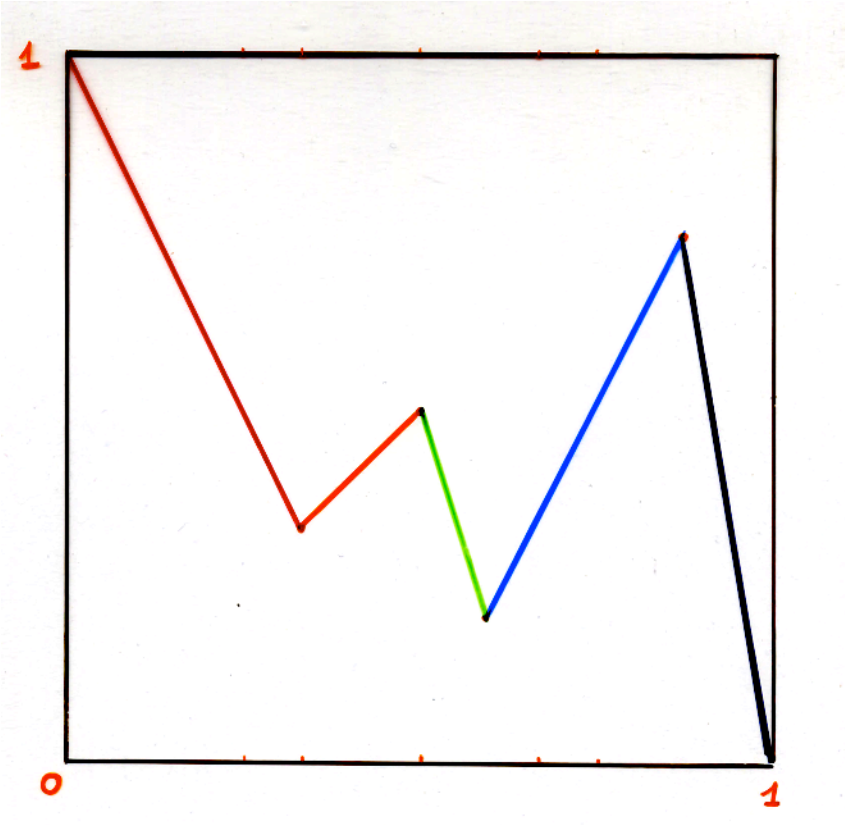
with the unary operation $\neg x = 1-x$

with the binary operation $x \oplus y = \min(1, x+y)$

THEOREM (Chang) **MV=HSP**[0,1]

the free MV-algebra on 1 generator $FREEMV_1$ is the set of functions $f:[0,1]\rightarrow[0,1]$ obtained from the identity function $x$ by pointwise application of the operations of the prototypical MV-algebra




a typical element of $FREEMV_1$

# the free MV-algebra over one generator has enough expressive power to describe the Italian co-fund system

let P be a set of all functions in $FREEMV_1$ which have value 1 arbitrarily close to the deadline

**then f is accepted by the Italian co-fund system iff it complies with the conditions in P. These conditions ask that f must get value 1 at the deadline 1, and must also keep value 1 over some left neighbourhood of 1.**

$$f(1) = 1 \quad \text{and} \quad \partial f(1)/\partial(x^-) = 0$$

# the expressive power of Lukasiewicz logic goes beyond the Bolzano paradigm

f is accepted by the Italian co-fund system iff it complies with the conditions in P.

These conditions ask that f must get value 1 at the deadline 1, **and must also keep value 1 over some left neighbourhood of 1.**

$$\partial f(1)/\partial\text{-}x = 0$$

yes

no

# Lukasiewicz **stable** consequence

*For a formula $f$ to be a* ***stable consequence*** *of a set $P$ of premises the following conditions are necessary:*

(***Bolzano condition for* {0,1}*-logics***) *every model $m$ of every $p$ in $P$ is also a model of $f$,*

(***Stability condition for* [0,1]*-logics***) *if every $p$ in $P$ is stably true along some direction $d$, then so must be $f$*

$$p(m)=1 \quad \textit{and} \quad \partial p(m)/\partial d=0 \quad \textit{for all } p \textit{ in } P$$

$$\Downarrow$$

$$f(m)=1 \quad \textit{and} \quad \partial f(m)/\partial d=0$$

stable consequence is gratified by a

**strong completeness theorem**

which is not the case of so called “semantic” consequence

THEOREM. *A formula $F$ is a stable consequence of $P$ iff Modus Ponens derives $F$ from $P$ and the tautologies*

THEOREM. *The consequence relation $G \vdash F$ is coNP-complete.*

# algebraic content of stable consequence

upon defining two formulas F and G to be *T-equivalent* iff T proves both F→G and G→F,

we get the **Lindenbaum algebra** Lind(T) of T

we then have a correspondence between deductively closed sets T of sentences and ideals $I_T$ in the free MV-algebra F of all formulas

# geometric content of stable consequence

a set T of formulas in $L_\infty$ contains a wealth of information not only on the set Mod(T) of models of T

but also on the **tangent space** of Mod(T) as a subset of the set $[0,1]^n$ of all possible models

when T is finitely axiomatizable, Mod(T) is a rational **polyhedron**

**unification and admissibility** involve a lot of geometry (Cabrer, Ciabattoni, Jerabek, Marra, Metcalfe, Spada,...)

**we will say "consequence" instead of "stable consequence"**

# problem 1

**Notation**

$T^{\vdash}$ = the set of consequences of T

$T^{\models}$ = the set of Bolzano consequences

PROBLEM 1:

When does $T^{\vdash}$ coincide with $T^{\models}$?

# problem 2

**Definition**

Given a set T of formulas we say that $T^{|=}$ **strongly** coincides with $T^{|-}$ if for any formula b we have $(T + b)^{|=} = (T+ b)^{|-}$

PROBLEM 2:

When does $T^{|-}$ **strongly** coincide with $T^{|=}$?

# C.C.Chang's (1959) answer to Problem 1

*Let* $\boldsymbol{T}$ *be a set of formulas, with its set* $\boldsymbol{T}^{\models}$ *of Bolzano consequences, and its set* $\boldsymbol{T}^{\vdash}$ *of consequences. The following conditions are equivalent:*

$\boldsymbol{T}^{\models}$ *coincides with* $\boldsymbol{T}^{\vdash}$ **.**

*The Lindenbaum algebra of* $\boldsymbol{T}$ *is* semisimple

(i.e, the 0 ideal is intersection of maximal ideals)

particular case: when T is finitely axiomatizable

# Answer to Problem 2 (D.M., 2012)

**Theorem**

*For any set* $T$ *of formulas the following conditions are equivalent:*

$T^{\models}$ ***strongly*** *coincides with* $T^{\vdash}$ , i.e., $(T + b)^{\models} = (T + b)^{\vdash}$

*whenever a new axiom b is added to T*

*The Lindenbaum algebra of T is* ***strongly*** *semisimple*

(not only 0, but any principal ideal is an intersection of maximals)

# strongly semisimple→semisimple

**A is semisimple:**
the zero ideal is intersection of maximal ideals:
A is semisimple
= A is archimedean
= A is algebra of real-valued functions
= A does not have infinitesimals

**A is strong semisimple:**
every principal ideal is an
intersection of maximals

# semisimple→?strongly semisimple

**semisimple:**
the zero ideal is an intersection of maximal ideals

**strongly semisimple:**
every principal ideal is an intersection of maximals

# semisimple→?strongly semisimple

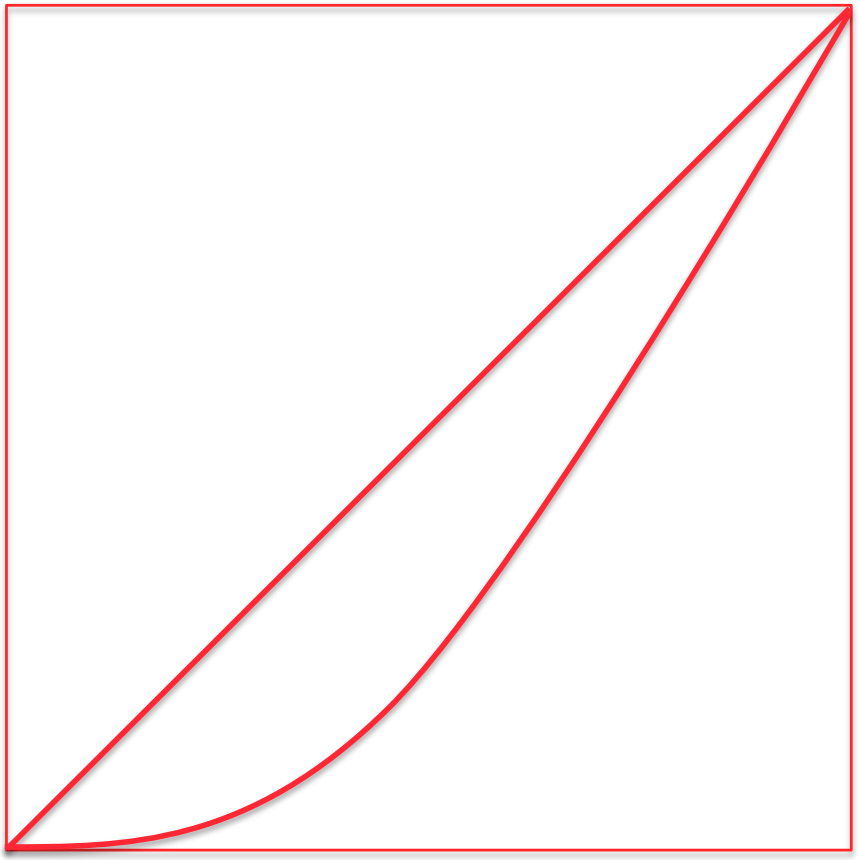
**semisimple:**
the zero ideal is an intersection of maximal ideals

example

the subalgebra A of C([0,1]) generated by x and $x^2$ is semisimple; the ideal P generated by $x^2$ is principal, but differs from the only maximal ideal M above P: x belongs to M but not to P; no multiple of $x^2$ dominates x near 0

**strongly semisimple:**
every principal ideal is an intersection of maximals

# no multiple of $x^2$ dominates x



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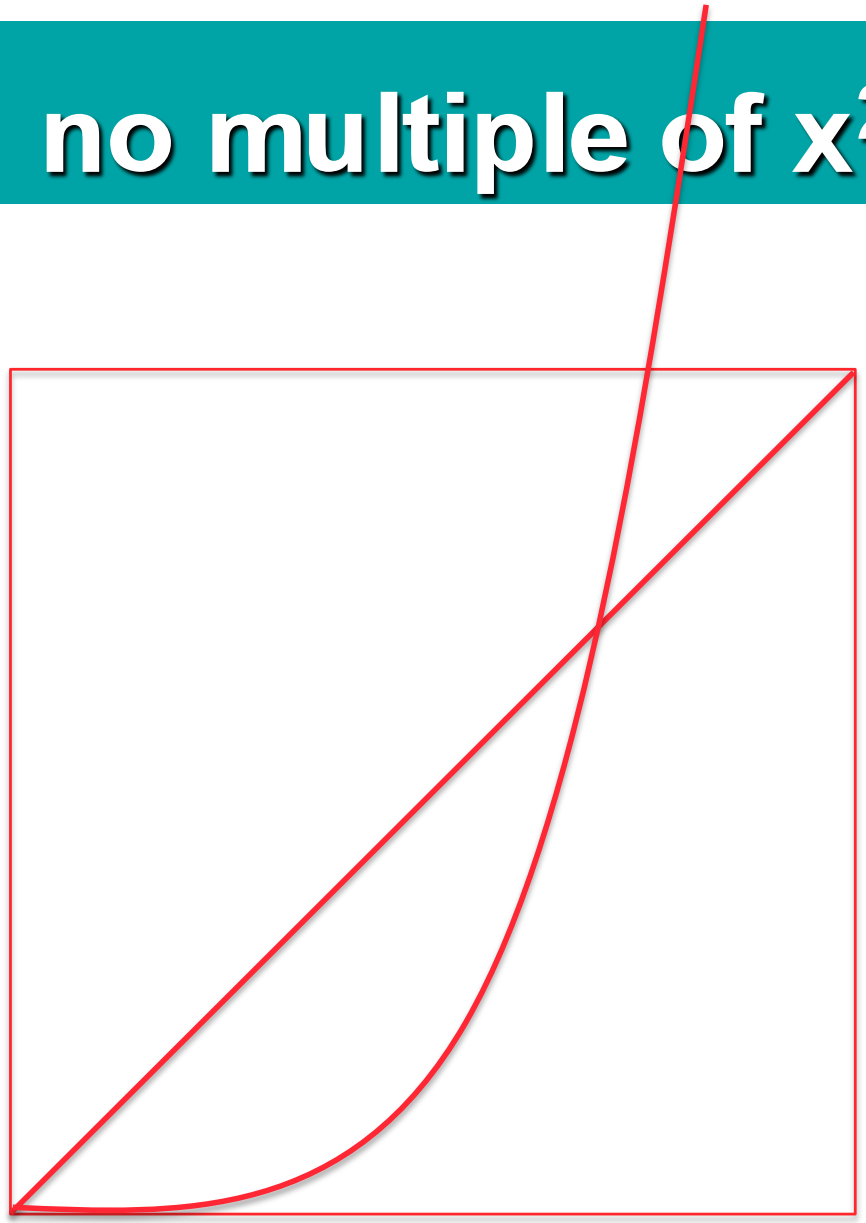
# no multiple of $x^2$ dominates x



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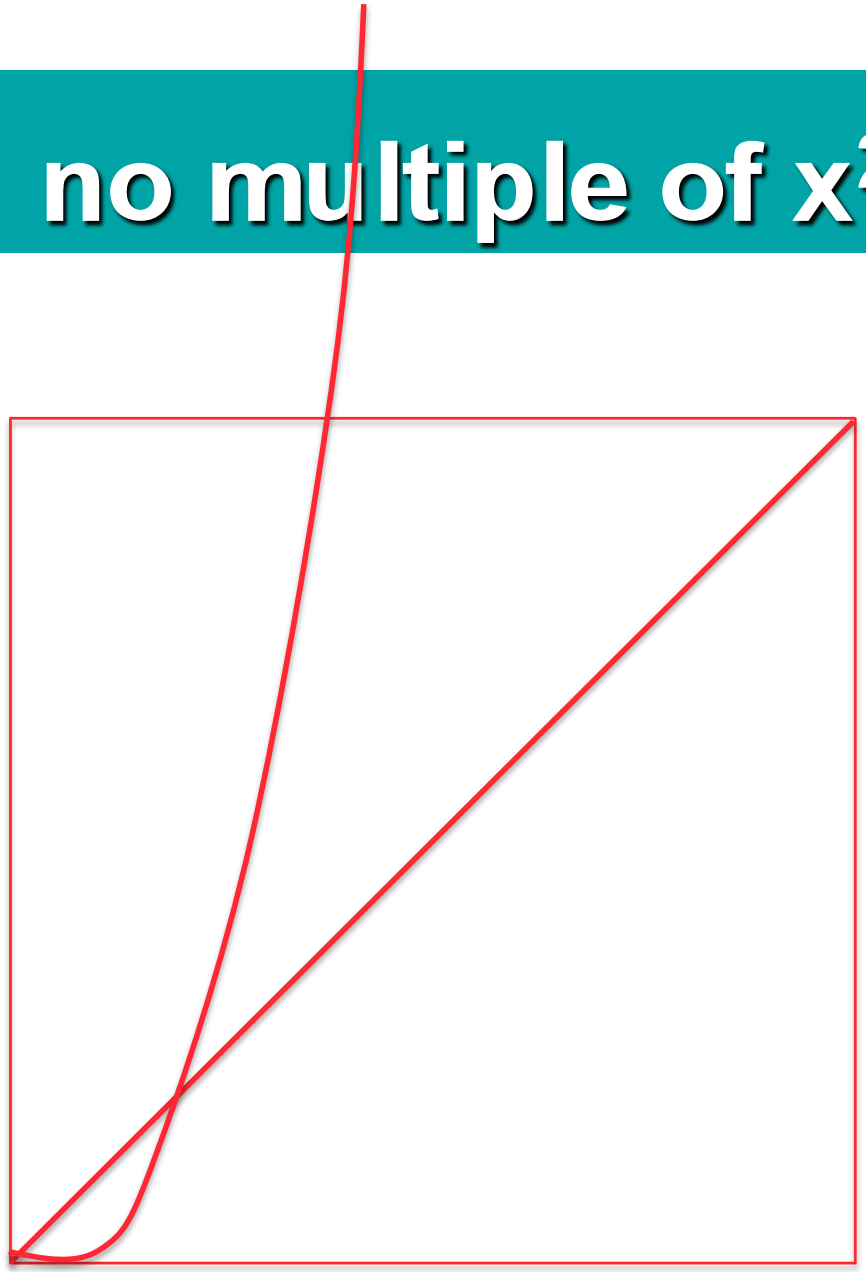

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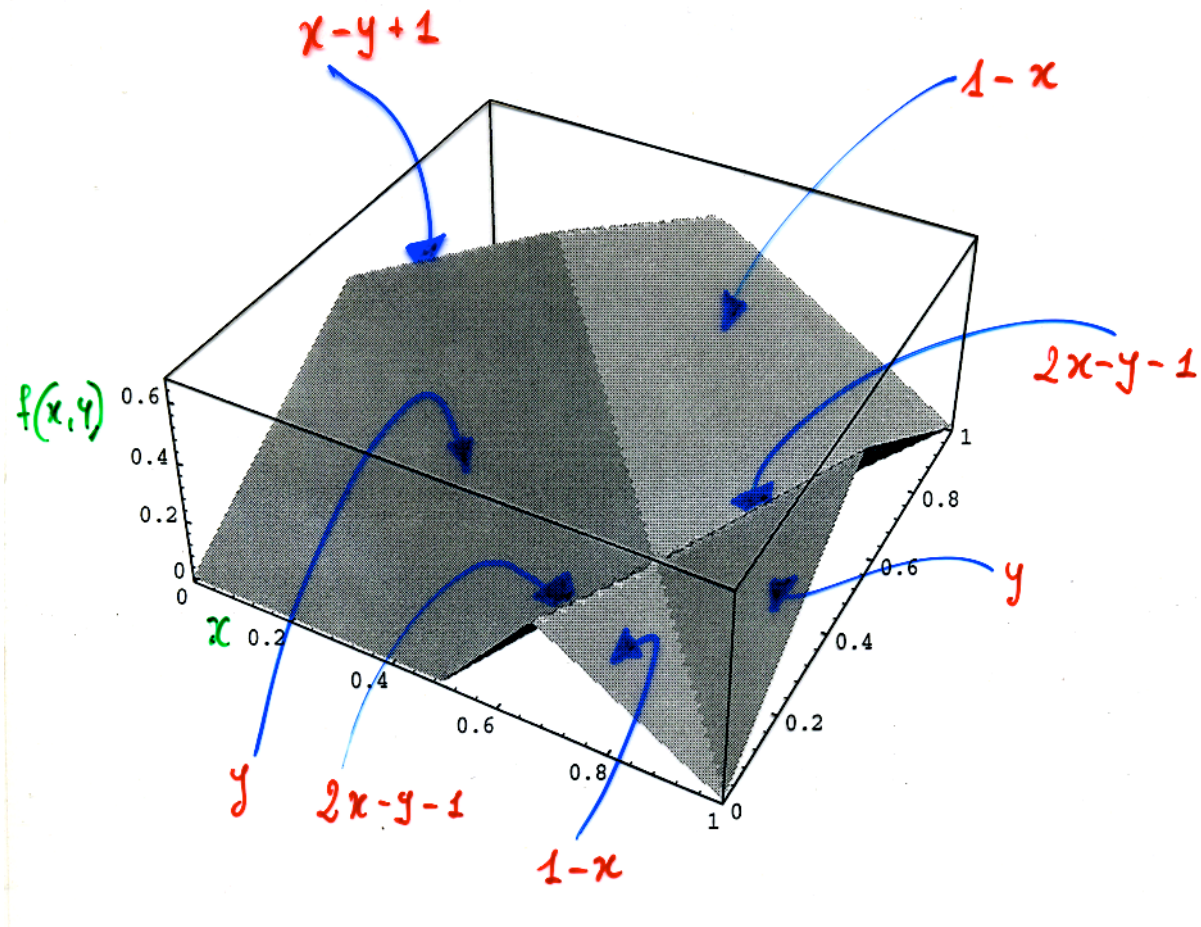
# understanding failure of strong semisimplicity in semisimple algebras

joint work with Manuela Busaniche

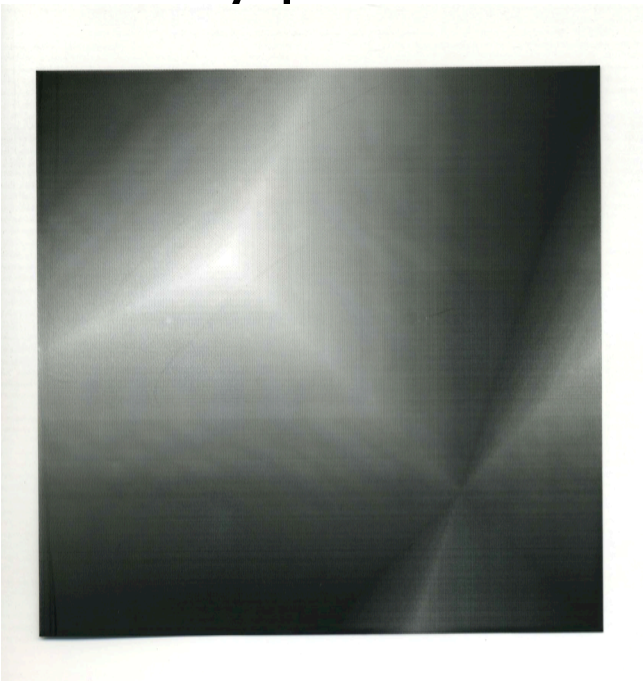
**we need at least two dimensions, because**

for one-generator MV-algebras, semisimplicity coincides with strong semisimplicity

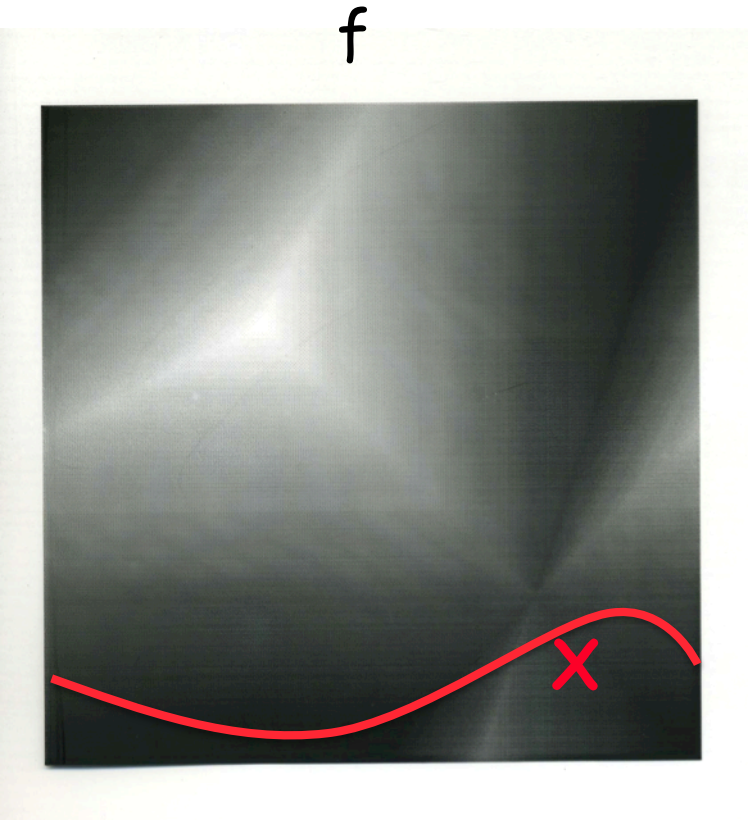
the free MV-algebra on 2 generators is the set of functions $f:[0,1]^2 \rightarrow [0,1]$ obtained from the identity functions x and y by pointwise application of the operations of the prototypical MV-algebra

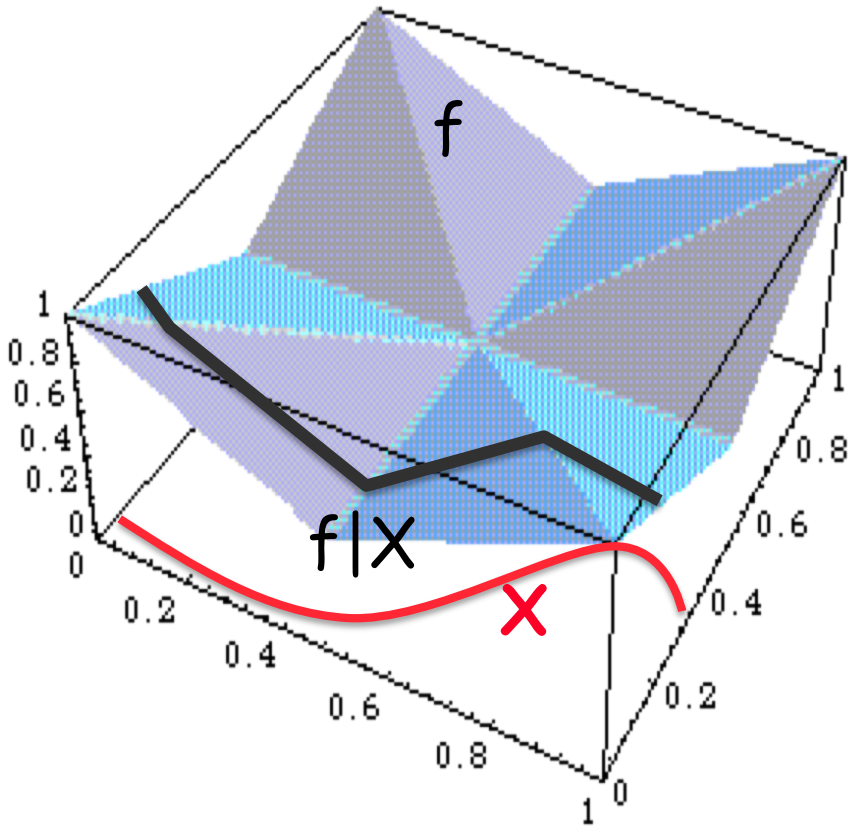



a typical element of $FREEMV_2$ and its density plot



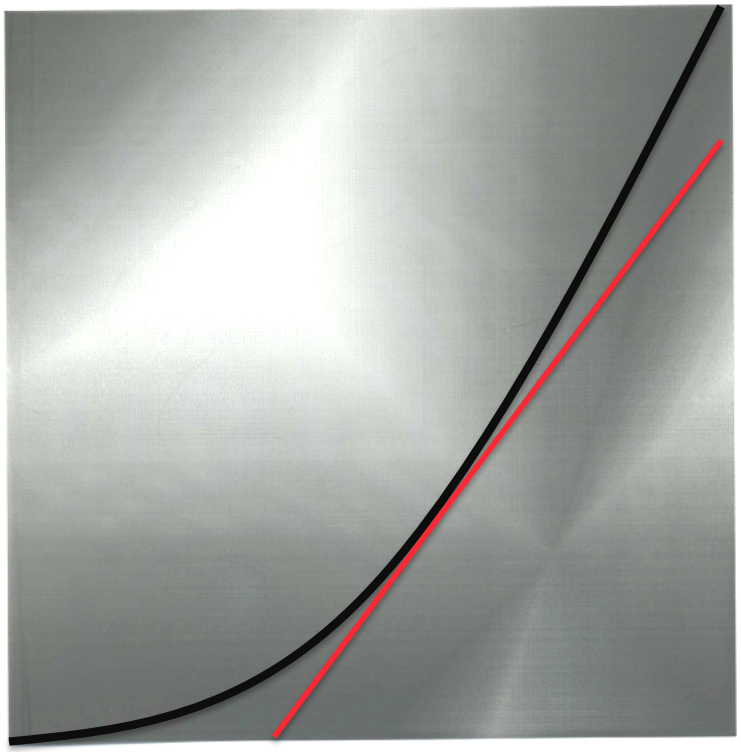
# the semisimple quotient operation on $FREEMV_2$




let X be a closed set in $[0,1]^2$. Restrict every function f of $FREEMV_2$ to X. Then the MV-algebra of restrictions to X is the most general semisimple two-generator MV-algebra

# the MV-algebra of $x$ and $x^2$

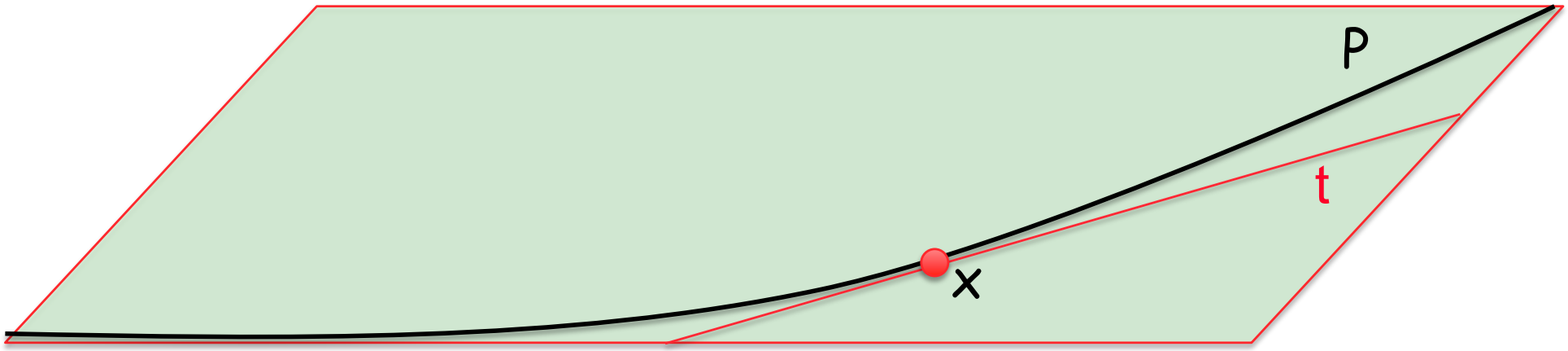


By mapping $X \to x$, $Y \to x^2$ we get a parabola $P$. We let $t$ be the tangent of $P$ at the point $(1/2, 1/4)$.

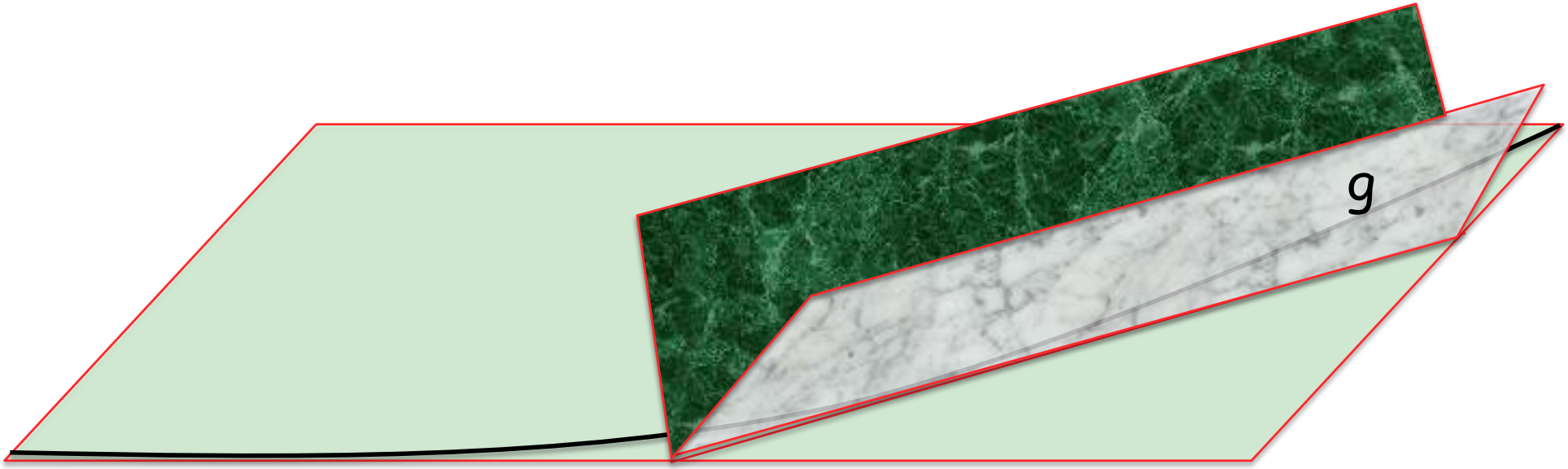
Further, we let

**M|P**

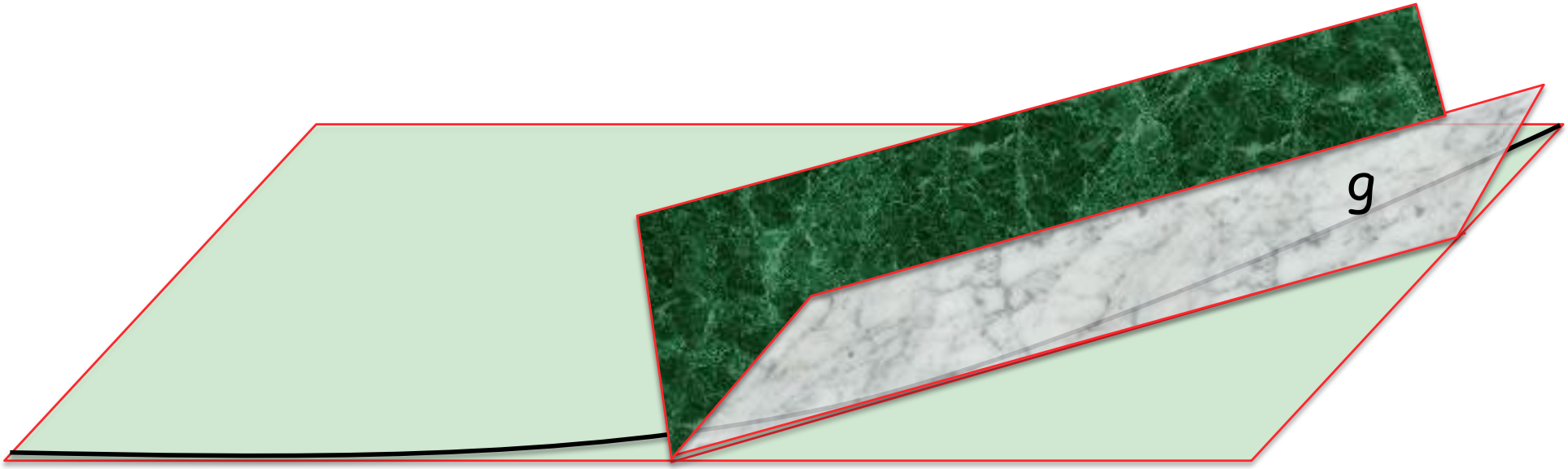
denote the algebra of all restrictions to $P$ of the functions of the free algebra $FREEMV_2$


P
t
x

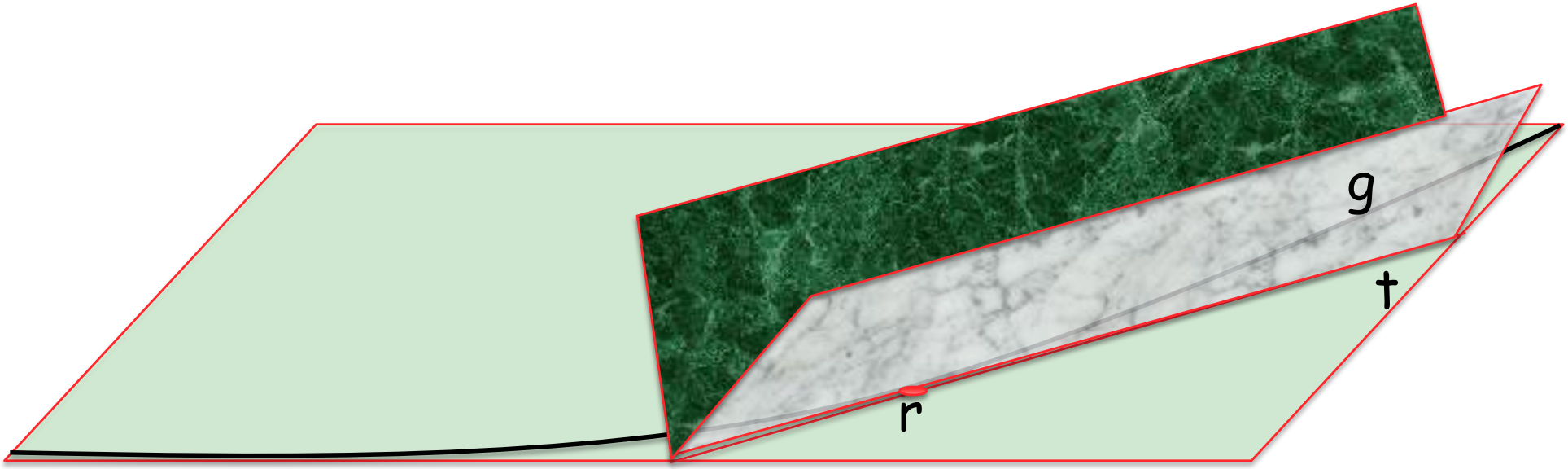
# FREEMV$_2$ contains a function $g : [0,1]^2 \rightarrow [0,1]$ only vanishing along $t$



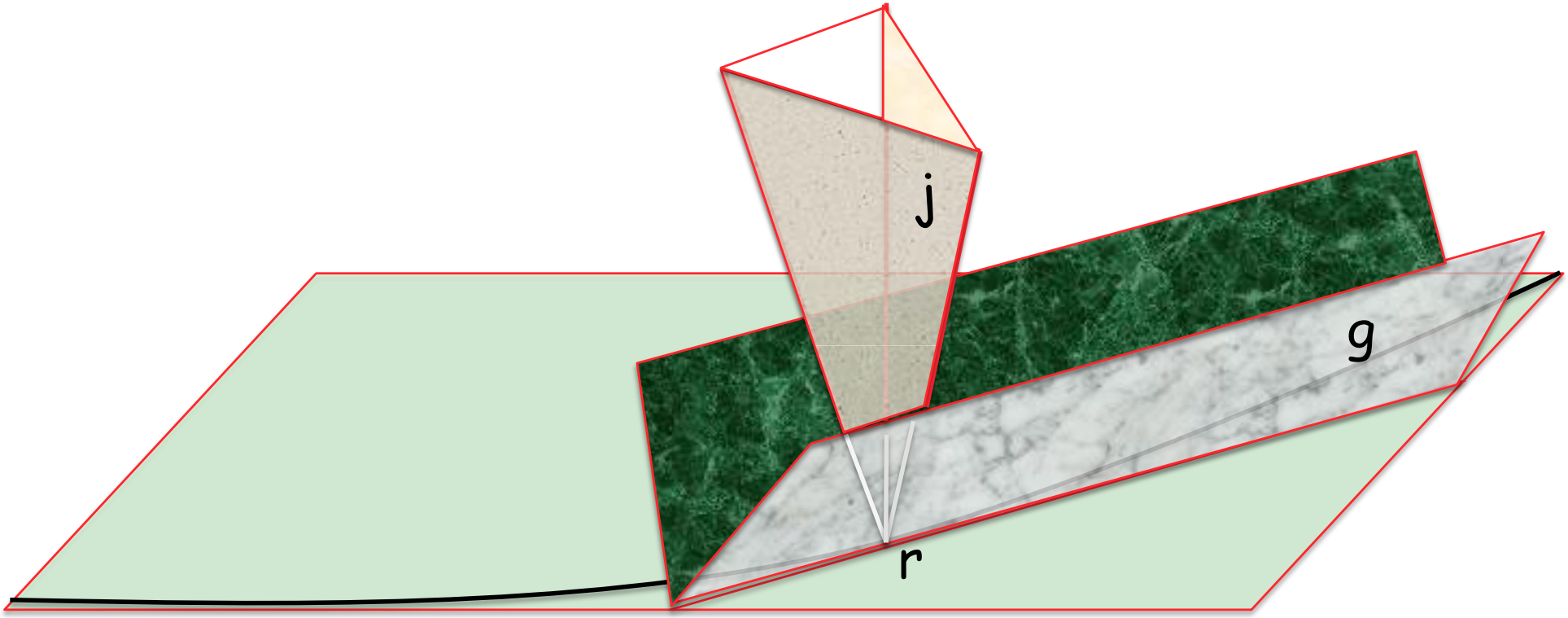
we now let <g> be the ideal of $FREEMV_2$ given by all functions f which are dominated on P by a multiple of g



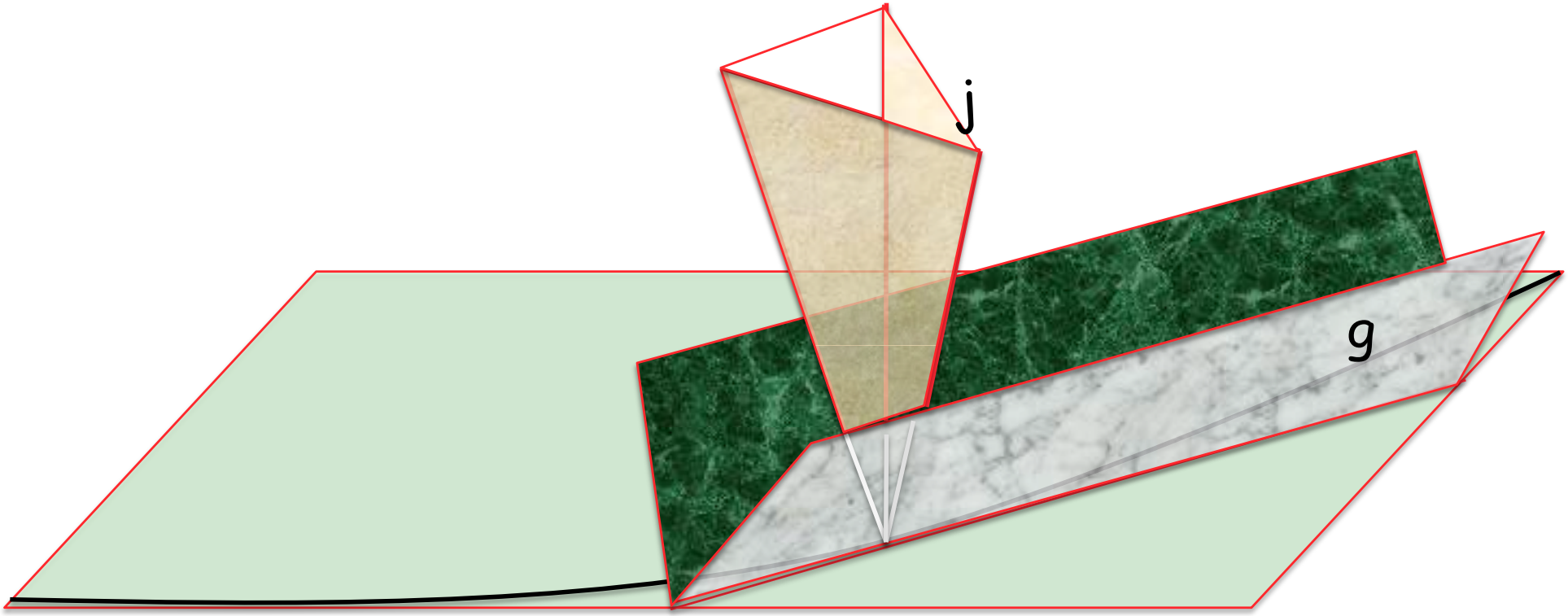
any function f in <g> will satisfy $\partial f(r)/\partial u \rightarrow 0$ as $u \rightarrow t$, because so does g, and directional derivatives of McNaughton functions are continuous



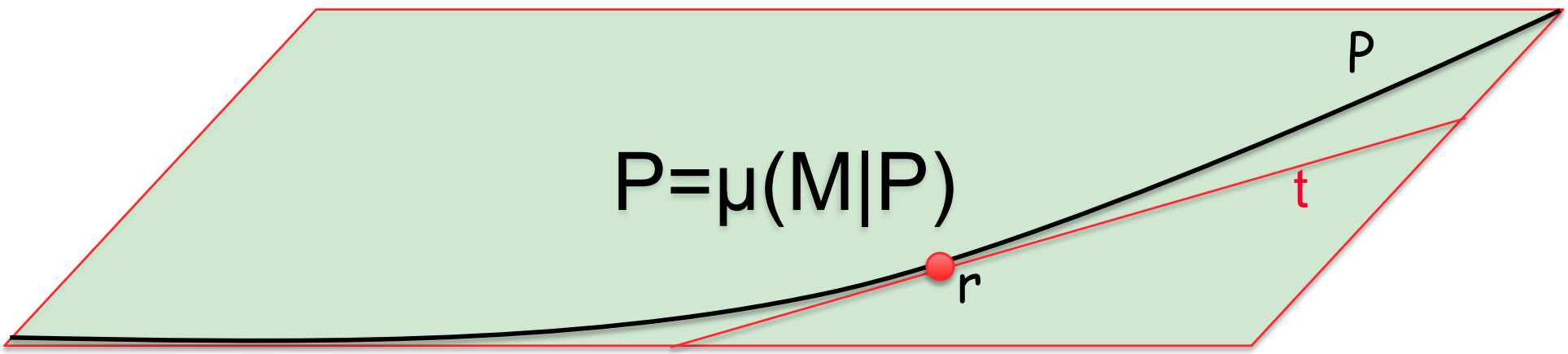
but the restriction to P of the function j does not satisfy this condition: as we see, $\partial j(r)/\partial u > 0$



thus <g> is a principal ideal of M|P different from the only maximal ideal <j> above <g>.
For, j belongs to <j> and does not belong to <g>



# M|P is not strongly semisimple: P has a **rational B-S tangent** t




we MUST use B-S tangents, for P can be a very general closed set

# MV-algebras and B-S tangents

**Theorem (M.Busaniche, D.M. 2012)**

***If an MV-algebra is semisimple but not strongly semisimple then its maximal spectral space has a Bouligand-Severi (B-S) tangent***

# low-dimensional MV-algebras

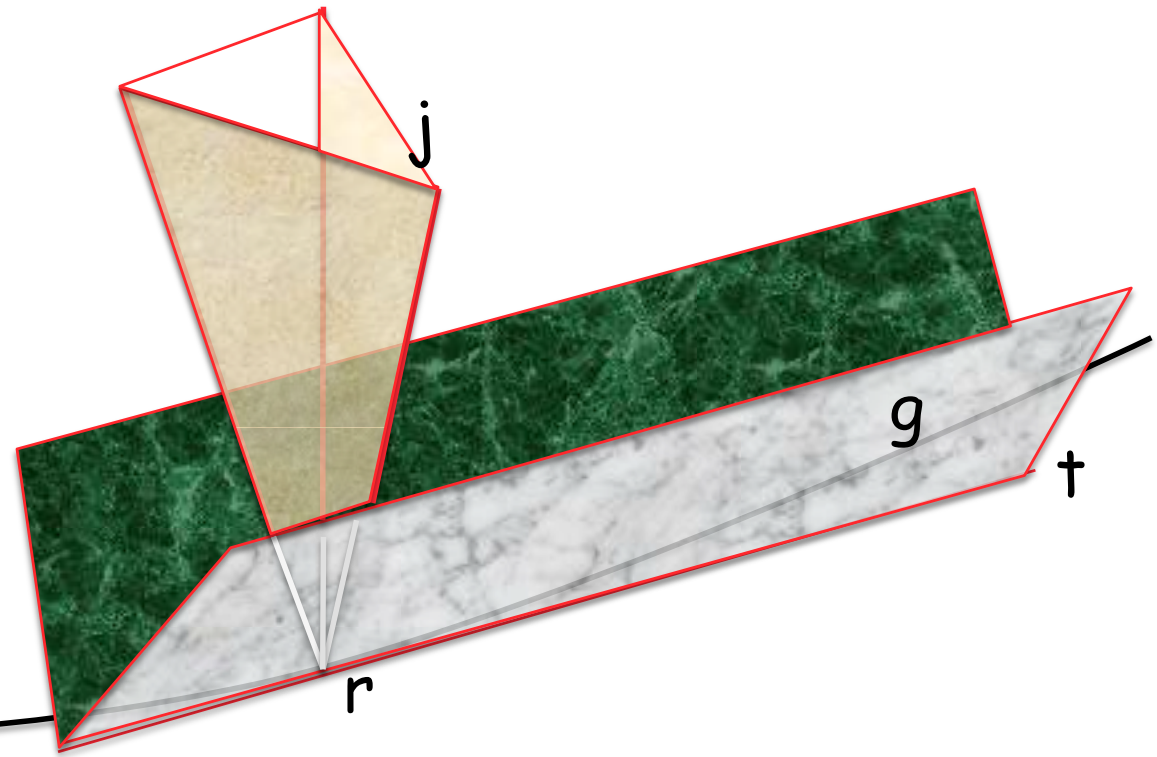
**Theorem**

*When $X$ is a closed nonempty subset of the unit square, the MV-algebra $M|X$ has the Dubuc-Poveda property if and only if $X$ has no rational tangents*

# what happens when $T^{|-} \neq T^{|=}$ ?

*there is a formula g such that the set of models of T+g has a B-S tangent t at some model r*

*every model v of T+g satisfies each formula f of T+g, and v satisfies f along direction t,* $\partial f(v)/\partial t = 0$



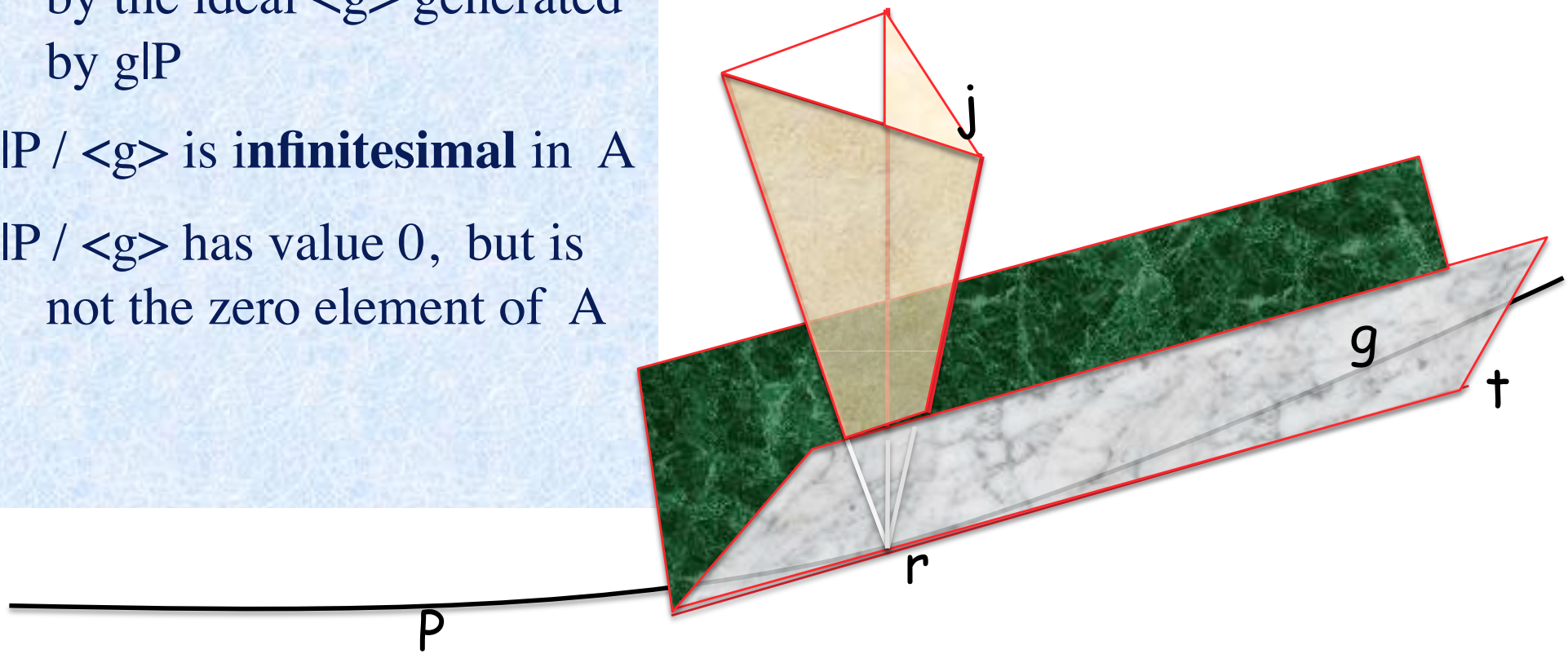

***but*** *some formula j satisfied by all models of T+g, has* $\partial j(v)/\partial t \neq 0$

# byproduct: a concrete representation of infinitesimals as directional derivatives

Let A be the quotient of M|P by the ideal <g> generated by g|P

j|P / <g> is **infinitesimal** in A

j|P / <g> has value 0, but is not the zero element of A



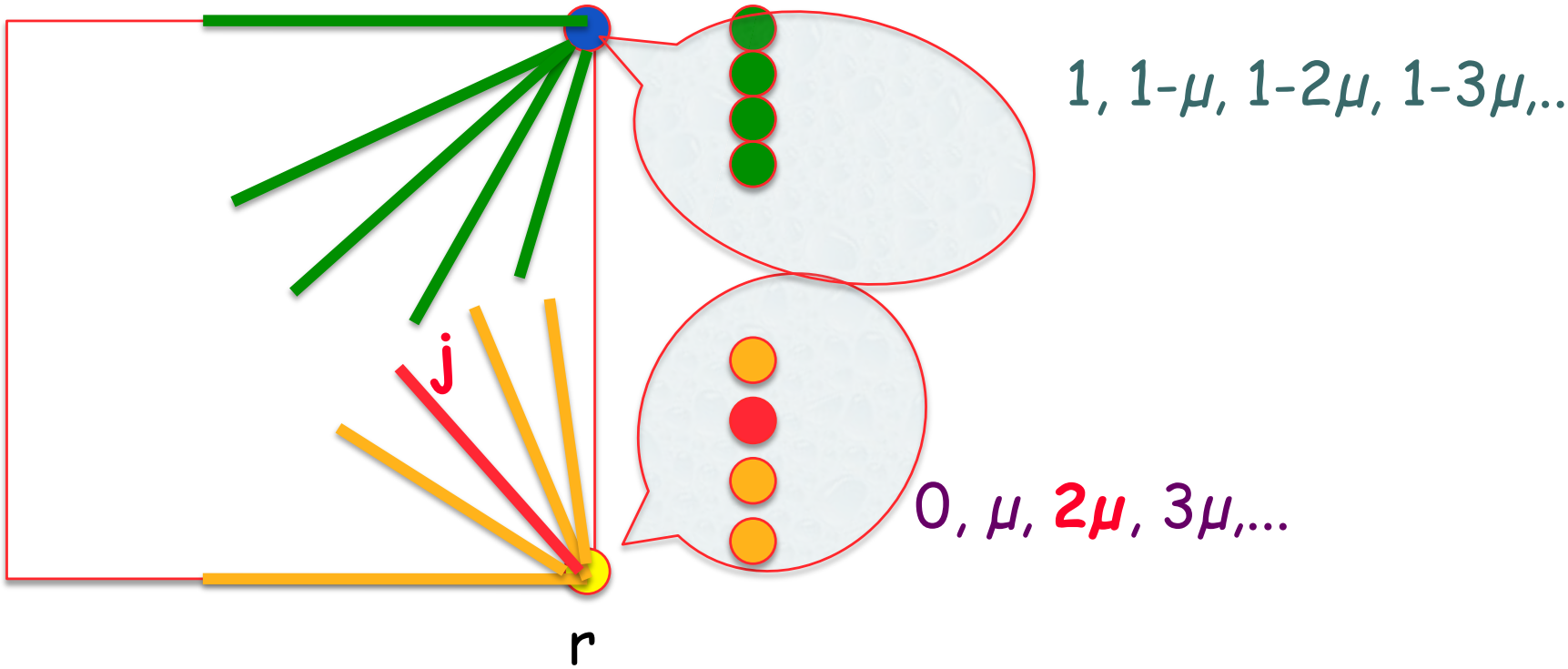
# the McNaughton function j is infinitesimal in A

let A be the quotient of M|P by the ideal <g> generated by g|P

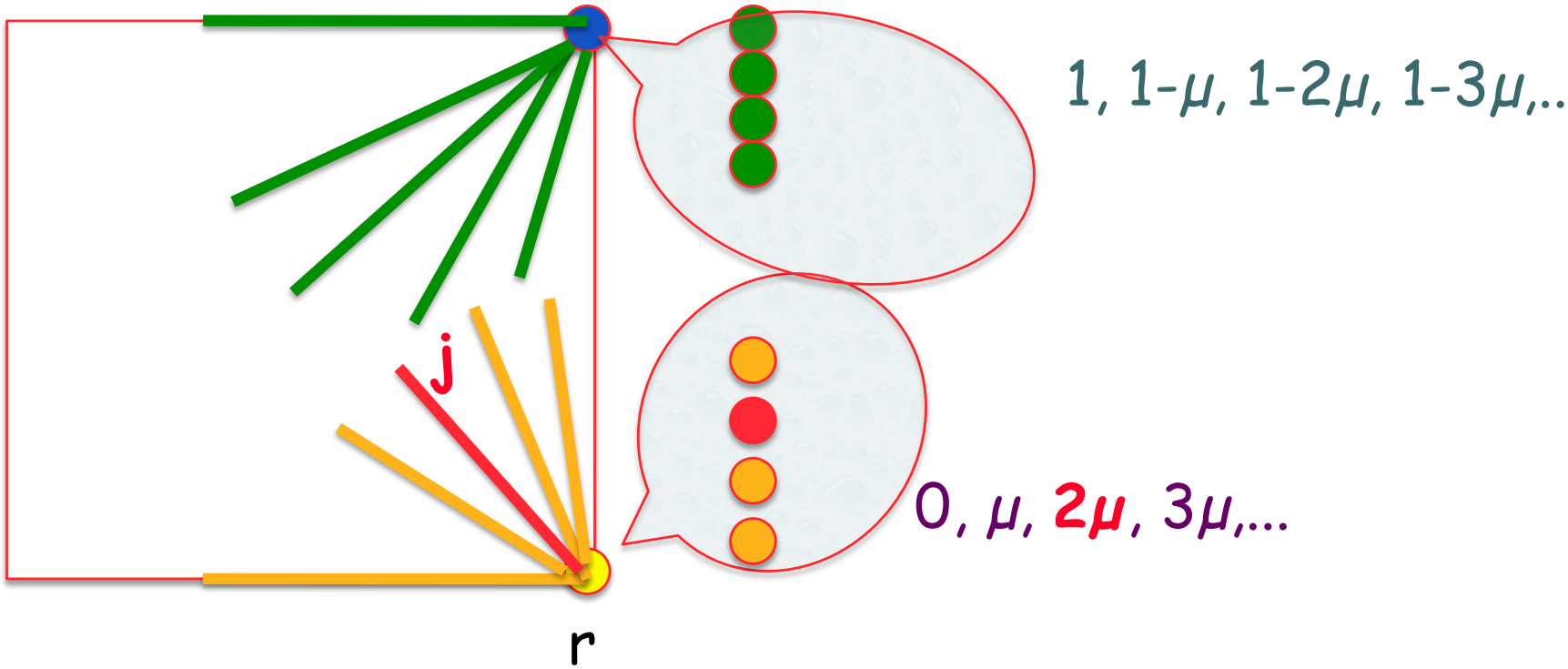
A is the MV-algebra of all possible behaviours (=germs) of McNaughton functions f at point r along direction t

these germs are determined by the value f(r) and its derivative ∂f(r)/∂t

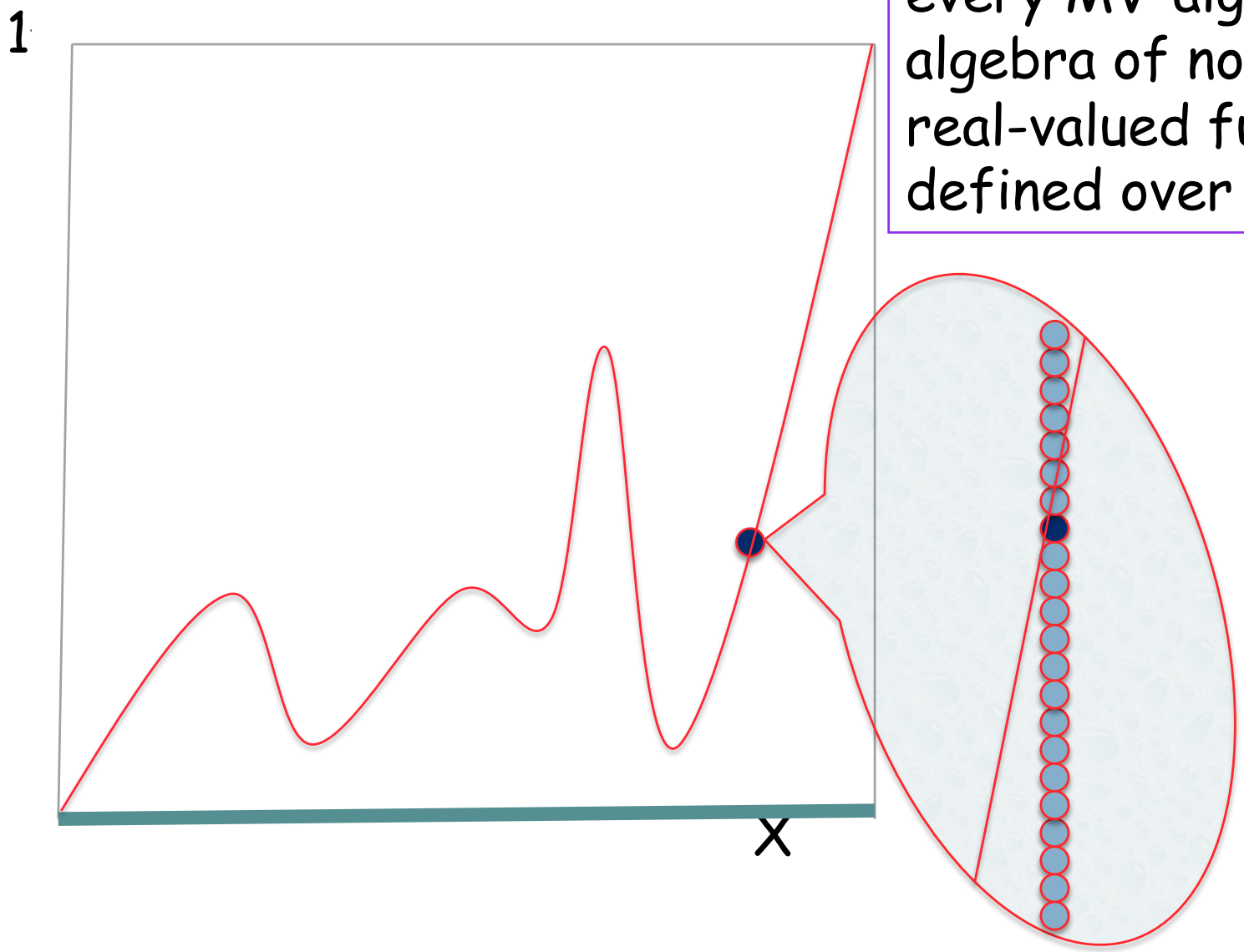
elements of A are GERMS, i.e., values of functions f(r) together with their directional derivatives $\partial f(r)/\partial t$



# j is an infinitesimal in A: j has value zero but nonzero derivative



# Di Nola's theorem




every MV-algebra is an algebra of nonstandard real-valued functions, defined over some set X

# $j$ is an MV-algebraic infinitesimal

*the consequences of S form a smaller set than the set of consequences in the sense of Bolzano*

$1-j$ *represents a formula with* $j(r)=1$ *but* $\partial j(r)/\partial x < 0$,

*while for every formula* $f$ *of* $S$, $f(r)=1$ *and* $\partial f(r)/\partial x = 0$

*The deduction rules are so that every* ***consequence*** *of S must have the same stability properties which are common to all formulas of S.*

$1-j$ *is not a consequence of S because it is unstable*

# conclusions

the Dubuc-Poveda property in MV-algebras is decisive in understanding the semantics of ∞-valued Lukasiewicz logic

the classical notion of “semantic (Bolzano-Tarski) consequence” does not coincide with provability, because it is insensitive to small perturbations of the models of a theory

a strongly complete semantics can be obtained by using valuations that take into account the differential behaviour of formulas, and the tangent spaces of their model sets

# thank you