MV-modules

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In honor of Antonio Di Nola's 65th birthday

Happy Birthday!

Part I: MV-modules

MV-algebras

- MV-algebras are the algebraic structures of the ∞-valued Łukasiewicz logic.
- (C.C.Chang, 1958) *MV-algebra* $(A, \oplus, ^*, 0)$
  1. $(A, \oplus, 0)$ abelian monoid,
  2. $(x^*)^* = x$,
  3. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
  4. $0^* \oplus x = 0^*$.
- $1 := 0^*$, $x \odot y := (x^* \oplus y^*)^*$, $x \vee y := (y^* \oplus x)^* \oplus x$, $x \wedge y := (x^* \vee y)^*$
- Standard model: $([0,1], \oplus, ^*, 0)$
  $x \oplus y = min(x + y, 1)$, $x \odot y = max(x + y - 1, 0)$,
  $x^* = 1 - x$, $1 = 0^*$, $x \vee y = max(x, y)$, $x \wedge y = min(x, y)$
- A Boolean algebra iff $\oplus = \vee$ and $\odot = \wedge$.

MV-algebras and lattice-ordered groups

$(G, +, 0, \leq)$ is a lattice-ordered group if
$(G, +, 0)$ group, $(G, \leq)$ lattice,
$x \leq y$ implies $x + z \leq y + z$ for any x, y, $z \in G$.

- u strong unit: $u \geq 0$, for any $x \in G$ there is $n \geq 1$ s.t. $x \leq nu$.

Mundici, 1986

For any MV-algebra A there exists an abelian lattice-ordered group with strong unit (G, u) such that $A \simeq [0, u]_G$.
The category of MV-algebras is equivalent with the category of lattice-ordered groups with strong unit.

- $\Gamma(G, u) = ([0, u]_G, \oplus, ^*, 0)$: $x \oplus y = (x + y) \wedge 1$, $x^* = 1 - x$.

Adding a product

to a lattice-ordered group, one gets

- lattice-ordered rings

  or

- lattice-ordered modules

PMV-algebras

are unit intervals in lattice-ordered rings with strong unit.

Di Nola, Dvurečenskij, 2001

PMV-algebra $(P, \cdot)$, where P is an MV-algebra and $\cdot : P \times P \to P$

- For any PMV-algebra $(P, \cdot)$ there exists a lattice-ordered ring with strong unit $(R, \cdot, v)$ such that $P \simeq [0, v]_R$.
  The category of PMV-algebras is equivalent with the category of lattice-ordered rings with strong unit.
- The class of PMV-algebras is equational.

Montagna, 2005

As a PMV-algebra, $[0, 1]$ generates the quasi-variety defined by $x \cdot x = 0 \Rightarrow x = 0$.

Lattice-ordered modules

Let $(R, \cdot, v)$ be a unital lattice-ordered ring. A pair $(Q, \circ)$ is a **lattice-ordered module** over R if Q is an abelian lattice-ordered group and $\varphi : R \times Q \to Q$ such that:

(ℓM1) $r \circ (x + y) = r \circ x + r \circ y$,
(ℓM2) $(r + t) \circ x = r \circ x + t \circ x$,
(ℓM3) $(r \cdot t) \circ x = r \circ (t \circ x)$,
(ℓM4) $r \geq 0$, $x \geq 0 \Rightarrow r \circ x \geq 0$.
(ℓM5) $v \circ x = x$,
whenever r, $t \in R$ and x, $y \in Q$.

Steinberg, Lattice-ordered rings and modules, Springer, 2010, 632 pages

Lattice-ordered modules over $\mathbb{R}$ are called **Riesz spaces**.

MV-module

Di Nola, Flondor, I.L., 2003

Main example: $[0,u]_Q/[0,v]_R$

$(R,\cdot,v)$ unital lattice-ordered ring, v is strong unit
(Q,u) lattice-ordered module over (R,v), u is strong unit

Let $(P,\cdot)$ be a PMV-algebra. An **MV-module over** P is a pair $(M,\circ)$, where M is an MV-algebra and

$$\circ\colon P\times M\to M$$

such that, whenever x, $y\in M$ and t, $r\in P$:
(MvM1) if $x\leq y^*$ then $(r\circ x)\leq(r\circ y)^*$ and
$r\circ(x\oplus y)=(r\circ x)\oplus(r\circ y)$,
(MvM2) if $r\leq t^*$ then $(r\circ x)\leq(t\circ x)^*$ and
$(r\oplus t)\circ x=(r\circ x)\oplus(t\circ x)$,
(MvM3) $(r\cdot t)\circ x=r\circ(t\circ x)$,
(MvM4) $1_P\circ x=x$.

MV-modules

Extension of Mundici's equivalence

$(R, \cdot, v)$ lattice-ordered ring, v strong unit, $P = [0, v]_R$

If $(M, \circ)$ is an MV-module over P then there exists a lattice-ordered module with strong unit (Q, u) such that $M \simeq [0, u]_Q$.

The category of MV-modules over P is equivalent with the category of lattice-ordered modules with strong unit over R.

$(R, \cdot, v) = (\mathbb{R}, \cdot, 1)$
Riesz MV-algebras are MV-modules over $[0, 1]$.

MV-modules

Equational characterization

$(M, \circ)$ is an MV-module over P iff the following identities are satisfied for any r, $t \in P$ and x,$y \in M$:

(M1) $(r \circ x) \odot ((r \vee t) \circ x)^* = 0$,
(M2) $(r \odot t^*) \circ x = (r \circ x) \odot ((r \wedge t) \circ x)^*$,
(M3) $(r \cdot t) \circ x = r \circ (t \circ x)$,
(M4) $r \circ (x \odot y^*) = (r \circ x) \odot (r \circ y)^*$,
(M5) $1 \circ x = x$.

Examples

- The lexicographic product
- The tensor product

Examples

- The lexicographic product

$(R, \cdot, v)$ lattice-ordered ring, $P = [0, v]_R$, G lattice-ordered group

$M = [(0,0),(v,0)]_{R \times_{lex} G} = \Gamma_R(R \times_{lex} G, (v,0))$

$\circ : [0, v]_R \times M \to M$, $r \circ (q, x) := (r \cdot q, x)$
for r, $q \in [0, v]_R$, $x \in M$

$(M, \circ)$ MV-module over P

Problems

- Characterize the class $\mathcal{K}$ of MV-modules obtained in this way.
- Investigate the functor $\Delta_P : \mathcal{ALG} \to \mathcal{K}$.

Remark
If $R = \mathbb{Z}$, then we get Di Nola's functor $\Delta : \mathcal{ALG} \to \mathcal{P}erf$.
If $R = \mathbb{R}$, then we get "Riesz MV-algebras", as defined by
Di Nola and Lettieri in 1996.

Bilinear functions and bimorphisms

A, B, C MV-algebras

- $\omega : A \to B$ is **linear** if
  $\omega(x \oplus y) = \omega(x) + \omega(y)$ whenever $x \leq y^*$.

- $\beta : A \times B \to C$ is **bilinear** if it is linear in each argument.

- $\beta : A \times B \to C$ is a **bimorphism** if it is bilinear and

$$\begin{aligned}
\beta(a, b_1 \vee b_2) &= \beta(a, b_1) \vee \beta(a, b_2),\\
\beta(a_1 \vee a_2, b) &= \beta(a_1, b) \vee \beta(a_2, b),\\
\beta(a, b_1 \wedge b_2) &= \beta(a, b_1) \wedge \beta(a, b_2),\\
\beta(a_1 \wedge a_2, b) &= \beta(a_1, b) \wedge \beta(a_2, b),
\end{aligned}$$

  whenever a, a_1, $a2 \in A$ and b, b_1, $b_2 \in B$.

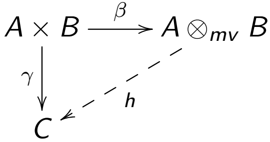
The tensor product $\otimes_{mv}$ of MV-algebras

A, B are MV-algebras

Mundici, 1999

The tensor product of A and B is an MV-algebra $A \otimes_{mv} B$ together with a **bimorphism** $\beta : A \times B \to A \otimes B$ satisfying the following universal property:

for any MV-algebra C and **bimorphism** $\gamma : A \times B \to C$ there exists a unique morphism $h : A \otimes B \to C$ such that $h \circ \beta \to \lambda$.



Flondor, I.L., 2003

If P is a PMV-algebra then $P \otimes_{mv} A$ is an MV-module over P for any MV-algebra A.

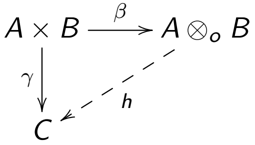
The tensor product $\otimes_o$ of MV-algebras

A, B are MV-algebras

Flondor, I.L, 2003 (following Martinez, 1972)

The $\otimes_o$ tensor product of A and B is an MV-algebra $A \otimes_o B$ together with a **bilinear** function $\beta : A \times B \to A \otimes B$ satisfying the following universal property:

for any MV-algebra C and **bilinear** $\gamma : A \times B \to C$ there exists a unique morphism $h : A \otimes B \to C$ such that $h \circ \beta \to \gamma$.



Scalar extension property: P PMV-algebra

$P \otimes_o A$ is MV-module over P for any MV-algebra A.

P (unital) PMV-chain

$(\mathbf{T}, \mathbf{U})$ adjoint functors

$\mathbf{T} : \mathcal{MV} \to \mathcal{MV}Mod_P$, $T(M) := P \otimes_{mv} M$
$\mathbf{U} : \mathcal{MV}Mod_P \to \mathcal{MV}$ the forgetful functor

$$Free_{\mathcal{MV}Mod_P}(X) \simeq P \otimes_{mv} Free_{\mathcal{MV}}(X)$$

$P \otimes_{mv} M$ is the MV-module over P generated by M.

Logic approach

The propositional calculus of MV-modules over P has completeness w.r.t. chains.

If $P = [0, 1]$ then we also have standard completeness, i.e. an identity holds in any Riesz MV-algebra iff it holds in $[0, 1]$.

Conclusions

For MV-modules one has:

- extension of Mundici's equivalence,
- ideal theory and representation w.r.t. linearly ordered structure,
- equational characterization,
- general methods for obtaining modules from algebras,
- characterization of free structures,
- logical approach.

Part II: a problem

Work in progress!

Stochastic independence

Riečan and Mundici, 2002

Given two σ-complete MV-algebras M and N, let us agree to define their σ-tensor product by the following procedure:

. . .

Assuming M and N to be probability MV-algebras, generalize the classical theory of "stochastically independent" σ-subalgebras as defined in Fremlin's treatise [Measure Theory, 325L].

States. Probability MV-algebras.

A MV-algebra

A **state** is a linear function $s : A \to [0,1]$ such that $s(1_A) = 1$.

- s is faithful if $s(x) = 0$ implies $x = 0$.
- If A is σ-complete then s is a σ-state if $s(x_n) \nearrow s(x)$ whenever $\{x_n\}_n \subseteq A$ and $x_n \nearrow x$.
- If (A, s_A), (B, s_B) MV-algebras with states then $s_A \times s_B : A \times B \to [0,1]$, $s_A \times s_B(a,b) := s_A(a) \cdot s_B(b)$ for any $a \in A$, $b \in B$.
- A probability MV-algebra is a pair (A, s), where A is a σ-complete MV-algebra and s is a faithful σ-state.

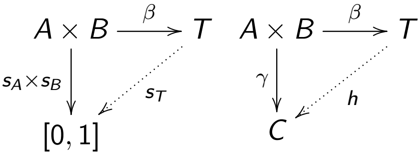
Independence for MV-algebras

I.L., ManyVal 2008, JLC 2011

Assume $\mathcal{K}$ is a class of MV-algebras with states.

Given (A, s_A), (B, s_B) in $\mathcal{K}$ there exists (T, s_T) in $\mathcal{K}$ and $\beta : A \times B \to T$ bilinear such that:

- $s_T \circ \beta = s_A \times s_B$,
- universal property
  for any MV-algebra C and **bilinear** $\gamma : A \times B \to C$ there exists a unique morphism $h : A \otimes B \to C$ such that $h \circ \beta \to \gamma$.



Solutions

- If A and B are Boolean algebras then $T = A \otimes B$.
- $T = A \otimes_{mv} B$ or $T = A \otimes_{o} B$, depending on $\mathcal{K}$
- s_T is extremal state, i.e. $s_T : T \to [0,1]$ MV-algebra homomorphism

We obtained solutions for:

- $\mathcal{K}$ = MV-algebras with states,
- $\mathcal{K}$ = MV-algebras with extremal states,
- $\mathcal{K}$ = semisimple MV-algebras.

No solution for probability MV-algebras!

The problem

(A, s_A), (B, s_B) probability MV-algebras

Define a probability MV-algebra (T, s_T) and a bilinear function $\beta : A \times B \to T$ such that:

- $s_T \circ \beta = s_A \times s_B$,
- universal property

Main ingredients

- 2-divisible MV-algebras
- state-completion of an MV-algebra
- concrete representation of state-complete Riesz MV-algebras
- measure theoretical construction

Step 1: the 2-divisible hull

Fedel, Keimel, Montagna, Roth, 2012

- An MV-algebra A is 2-divisible if
  for any $x \in A$ there exists $y \in A$ such that $y \oplus y = x$.
- Any MV-algebra A has a 2-divisible hull A^d.
- For any (faithful) state $s : A \to [0,1]$ there exists a unique (faithful) state $s^d : A^d \to [0,1]$ such that $s^d|_A = s$.

Step 2: the state-completion

(A, s) probability MV-algebra

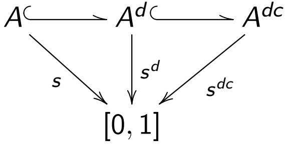
- A^d the 2-divisible hull of A, $s^d : A^d \to [0,1]$

  $\rho^d : A^d \times A^d \to [0,1]$
  $\rho^d(x,y) := s^d(d(x,y))$ metric on A^d

  A^{dc} the Cauchy completion of A^d w.r.t. ρ^d
  $A^{dc} = \{[\{a_n\}_n] \mid \{a_n\}_n \text{ Cauchy sequence in } A^d\}$

  $s^{dc}([\{a_n\}_n]) = \lim_n s^d(a_n)$
- I.L., ManyVal 2008, JLC 2011
  (A^{dc}, s^{dc}) is a probability MV-algebra

- $A \hookrightarrow A^{dc}, \quad a \mapsto [(a)]$
  embedding of σ-continuous MV-algebras.



Remark

A^{dc} 2-divisible σ-complete MV-algebra, hence

there exists a Riesz space with strong unit (V, u)
such that $A^{dc} \simeq [0, u]_V$, i.e.

A^{dc} is a Riesz MV-algebra.

Concrete representation state-complete Riesz MV-algebras

(X, Ω, μ) measure space

- $L_1(\mu)$ the Riesz space of Lebesgue integrable functions on X, provided we identify any two that are equal almost everywhere.
- $L_1(\mu)_u = [\mathbf{0}, \mathbf{1}]_{L_1(\mu)}$

I.L., ManyVal 2010, IPMU 2012

For any state-complete Riesz MV-algebra (M, s) there exists a measure space (X, Ω, μ) such that $M \simeq L_1(\mu)_u$.

This result is an MV-algebraic version of Kakutani's concrete representation of abstract L-spaces.

$$A \hookrightarrow A^{dc} \simeq L_1(\mu_A)_u \qquad B \hookrightarrow B^{dc} \simeq L_1(\mu_B)_u$$

$$A \ni a \mapsto f_a \in L_1(\mu_A)_u \qquad B \ni b \mapsto f_b \in L_1(\mu_B)_u$$

The product measure

(X_A, Ω_A, μ_A), (X_B, Ω_B, μ_B) measure spaces

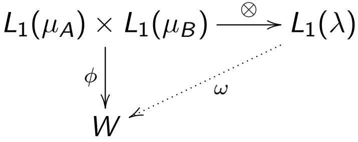
Fremlin, MT [253F,253G]

There is a measure space $(X_A \times X_B, \Lambda, \lambda)$ such that

- $\otimes : L_1(\mu_A) \times L_1(\mu_B) \to L_1(\lambda)$ bimorphism
  $(f, g) \mapsto f \otimes g$
- $\int (f \otimes g) d\lambda = \int f d\mu_A \int g d\mu_B$
  whenever $f \in L_1(\mu_A)$, $g \in L_1(\mu_B)$
- $f \otimes g \geq 0$ in $L_1(\lambda)$ whenever $f \geq 0$ and $g \geq 0$.

The product measure

- Universal property
  For any Banach lattice W (norm complete Riesz space) and bilinear function ϕ there exists a unique linear function ω such that $\omega \circ \otimes = \phi$.



$(X_A \times X_B, \Lambda, \lambda)$ **the product space**

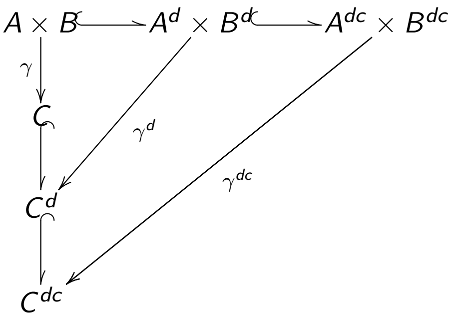
(A, s_A), (B, s_B) probability MV-algebras

- (X_A, Ω_A, μ_A) L-measure space, $A^{dc} \simeq L_1(\mu_A)_u$, $a \mapsto f_a$
- (X_B, Ω_B, μ_B) L-measure space, $B^{dc} \simeq L_1(\mu_B)_u$, $b \mapsto f_b$
- $(X_A \times X_B, \Lambda, \lambda)$ **the product space**
- $T_{A,B} = L_1(\lambda)_u$
- $s_T : T \to [0, 1]$, $s_T(f) = \int f d\lambda$
- $\beta : A \times B \to T$, $\beta(a, b) = f_a \otimes f_b$ bimorphism
- $s_T(\beta(a, b)) = \int f_a \otimes f_b d\lambda = \int f d\mu_A \int g d\mu_B = s_A(a)s_B(b)$ for any $a \in A$, $b \in B$
- (T, s_T) is a probability MV-algebra

Universal property for (T, s_T)

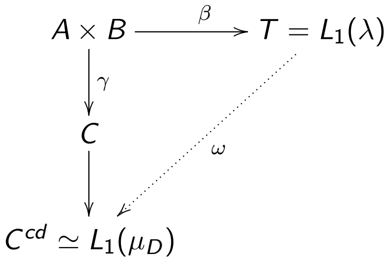
(C, s_C) probability MV-algebra, $\gamma : A \times B \to C$ bilinear function

- Any bilinear function $\gamma : A \times B \to C$ admits a unique bilinear extension $\gamma^d : A^d \times B^d \to C^d$.
- Any bilinear function $\gamma^d : A^d \times B^d \to C^d$ admits a unique bilinear extension $\gamma^{dc} : A^{dc} \times B^{dc} \to C^{dc}$ by $\gamma^{dc}([\{a_n\}_n], [\{b_n\}_n]) = [\{\gamma^d(a_n, b_n)\}_n]$



Universal property for (T, s_T)

$$\beta(a,b) = f_a \otimes f_b$$



- For any probability MV-algebra (C, s_C) and
  any bilinear function $\gamma : A \times B \to C$
  there exists a unique linear function $\omega : T \to C^{cd}$ such that
  $\omega(\beta(a,b)) = \gamma(a,b)$ whenever $a \in A$, $b \in B$, i.e.
  $\omega(f_a \otimes f_b) = f_{\gamma(a,b)}$ whenever $a \in A$, $b \in B$.

Extension result

If (X, Ω, μ) is a measure space, then $\mathbf{1}$ is a **weak unit** of $L_1(\mu)$.

Weak unit

If V is a Riesz space, an element $e \geq 0$ in V is a **weak unit** if $(x \wedge ne) \nearrow x$ for any $x \geq 0$ in V.

Proposition.

Assume V_1 and V_2 are σ-complete Riesz spaces,
$e_1 \in V_1$ and $e_2 \in V_2$ are weak units.

Then any linear function $\omega : [0, e_1]_{V_1} \to [0, e_2]_{V_2}$ can be extended to a linear function $\widetilde{\omega} : V_1 \to V_2$.

Thank you for your attention!