O-Minimal and Weakly O-Minimal MV-Chains

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O-MINIMALITY

Definition (van Den Dries; Pillay, Steinhorn)

A linearly ordered structure **A** is said to be *o-minimal* if any parametrically definable subset of A is a finite union of open intervals and points in A.

- Examples: ordered divisible Abelian groups, real closed fields, and their expansions.
- An ordered Abelian group (OAG) **G** is:

  o-minimal IFF it is divisible IFF $\mathsf{Th}(\mathbf{G})$ has quantifier elimination in $\langle +, -, 0, < \rangle$

  (Pillay, Steinhorn; Lenski).

MV-CHAINS AND O-MINIMALITY

- Each MV-chain is isomorphic to a structure over the unit interval of a unique (up to isomorphism) OAG with strong unit (Chang; Mundici).
- Is there an algebraic and model-theoretic characterization of o-minimal MV-chains (similar to the one for OAG)?
- We give a purely algebraic characterization. However, not all MV-chains with QE in $\mathcal{L}_{MV} = \langle \oplus, ^*, 0, < \rangle$ are o-minimal.
- A model-theoretic characterization of QE is achieved through the concept of weak o-minimality.

Weak o-minimality

Definition (Dickmann; Macpherson, Marker, Steinhorn)

A linearly ordered structure **A** is said to be *weakly o-minimal* if any parametrically definable subset of A is a finite union of convex sets in A.

- If a structure **A** is o-minimal then it is weakly o-minimal, the converse is not generally true:
- Let α be a real transcendental number and let the unary predicate symbol P be interpreted by the convex set $S := (-\alpha, \alpha) \cap \mathbb{R}_{alg}$. Then, the ordered field $\langle \mathbb{R}_{alg}, S \rangle$ is weakly o-minimal.
- An OAG is o-minimal IFF it is weakly o-minimal.

MV-Chains and Weak o-Minimality

- We show that there exist natural examples of weakly o-minimal MV-chains that are not o-minimal.
- We show that an MV-chain $\mathbf{A}$ is weakly o-minimal IFF $\mathsf{Th}(\mathbf{A})$ has QE in $\mathcal{L}_{\text{MV}} = \langle \oplus, ^*, 0, < \rangle$.
- We also give a full algebraic characterization of the class of weakly o-minimal MV-chains.

MV-CHAINS: A REMINDER (I)

- We assume basic knowledge of the theory of MV-algebras and simply point out some facts we will make use of.
- An MV-chain $\mathbf{A}$ is called *divisible* when $\mathbf{A} \cong \Gamma(\mathbf{G}, u)$ and $\mathbf{G}$ is a divisible OAG.
- The *radical* $Rad(\mathbf{A})$ of an MV-algebra $\mathbf{A}$ is the intersection of the maximal ideals of $\mathbf{A}$.
- $Rad(\mathbf{A})$ is called *divisible* when $\langle Rad(\mathbf{A}), \oplus, 0\rangle$ is divisible as a monoid.

MV-CHAINS: A REMINDER (II)

- The *order* of an MV-chain $\mathbf{A}$ is defined by $ord(\mathbf{A}) = n$ IFF $\mathbf{A} \cong \mathbf{S}_n$.
  Whenever $\mathbf{A} \ncong \mathbf{S}_n$, $ord(\mathbf{A}) = \infty$.
- The *rank* of $\mathbf{A}$ is defined by $rank(\mathbf{A}) = ord(\mathbf{A}/Rad(\mathbf{A}))$.
- Let $\mathbf{A}$ be a simple MV-chain of rank n. Then $\mathbf{A} \cong \mathbf{S}_n$.
- $\mathbf{A}$ is a non-simple MV-chain of rank n iff $\mathbf{A} \cong \Gamma(\mathbb{Z} \vec{\times} \mathbf{G}, (n, g))$, for some OAG $\mathbf{G}$.

Classes with QE (I)

Lemma

Let $\mathbf{A}$ *be an* MV*-chain.*

(1) *If* $Rad(\mathbf{A})$ *is divisible and* $\mathbf{A}/Rad(\mathbf{A})$ *is finite, then* $\mathbf{A}$ *has finite rank.*

(2) *If* $Rad(\mathbf{A})$ *is divisible and* $\mathbf{A}/Rad(\mathbf{A})$ *is divisible, then* $\mathbf{A}$ *is divisible.*

Proof.

(1) Obvious.

(2) $\mathbf{A}/Rad(\mathbf{A})$ is divisible, and so is $\Gamma^{-1}(\mathbf{A}/Rad(\mathbf{A}))$. $\Gamma^{-1}(\mathbf{A})$ is divisible and so is $\mathbf{A}$.

□

Classes with QE (II)

Consequently, any MV-chain satisfying the hypotheses of the lemma is either:

(*a*) a finite MV-chain;

(*b*) a non-simple MV-chain of finite rank $\Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, g))$, where $\mathbf{G}$ is a divisible OAG;

(*c*) a divisible MV-chain.

QE: FINITE MV-CHAINS

Lemma

Let $\mathbf{A}$ *be any finite* MV-*chain , then* $\mathsf{Th}(\mathbf{A})$ *has quantifier elimination in* $\mathcal{L}_{\mathrm{MV}}$.

Proof.

Every singleton is definable without quantifiers since $x = a$ amounts to $x \leq a \leq x$, so every subset is definable without quantifiers. □

QE: $\mathbf{A} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, g))$

Lemma

Let $\mathbf{A}$ *be an* MV*-chain of finite rank such that* $\mathbf{A} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, g))$ *where* $\mathbf{G}$ *is a ordered divisible Abelian group.* $\mathsf{Th}(\mathbf{A})$ *has quantifier elimination in* $\mathcal{L}_{\mathrm{MV}}$.

Proof.

$\mathsf{Th}(\mathbf{A})_\forall$ has the Amalgamation Property.
$\mathsf{Th}(\mathbf{A})$ is model-complete. This is obtained by showing that there is an interpretation of $\mathsf{Th}(\mathbf{A})$ into the theory of $\mathbb{Z}\vec{\times}\mathbf{G}$, where $\mathbf{G}$ is a divisible OAG, which is model-complete in the language

$$\langle +, -, <, 0, 1, \{m|\}_{m\in\mathbb{N}}\rangle,$$

where each $m|$ is a unary predicate denoting the elements divisible by m, and 1 is interpreted as the element $(1, 0)$ (Komori). □

QE: DIVISIBLE MV-CHAINS

Lemma

Let $\mathbf{A}$ *be any divisible* MV*-chain. Then* $\mathsf{Th}(\mathbf{A})$ *has quantifier elimination in* $\mathcal{L}_{\mathrm{MV}}$*.*

Proof.

Well-known result (M.; Baaz, Veith). □

Classes with QE

Theorem
Let $\mathbf{A}$ *be an* MV-*chain, and suppose that one of the following conditions holds:*

(1) $Rad(\mathbf{A})$ *is divisible and* $\mathbf{A}/Rad(\mathbf{A})$ *is finite.*

(2) $Rad(\mathbf{A})$ *is divisible and* $\mathbf{A}/Rad(\mathbf{A})$ *is divisible.*

Then $\mathsf{Th}(\mathbf{A})$ *has quantifier elimination in* $\mathcal{L}_{\mathrm{MV}}$*.*

FROM QE TO WEAK O-MINIMALITY

Theorem

Let $\mathbf{A}$ *be an* MV*-chain. If* $\mathsf{Th}(\mathbf{A})$ *has quantifier elimination in* $\mathcal{L}_{\text{MV}}$*, then* $\mathbf{A}$ *is weakly o-minimal.*

Proof.

- If $\mathbf{A}$ is not weakly o-minimal, there is a formula $\phi(x)$ such that $\{a \mid \mathbf{A} \models \phi(a)\}$ is not a finite union of convex sets.
- If $\mathsf{Th}(\mathbf{A})$ had QE in $\mathcal{L}_{\text{MV}}$, $\phi(x)$ would be equivalent to a quantifier-free formula $\psi(x)$.
- The set defined over A by a quantifier-free formula in one variable in $\mathcal{L}_{\text{MV}}$ is always a finite union of convex sets.

□

Weak o-minimality (I)

Lemma

Let $\mathbf{A}$ *be an* MV*-chain. If* $\mathbf{A}$ *is weakly o-minimal, then* $Rad(\mathbf{A})$ *is divisible.*

Proof

- Suppose that $Rad(\mathbf{A})$ is not n-divisible for some n, and let x be an element not divisible by n.
- Then, $0 < x < nx < (n+1)x < 2nx < (2n+1)x < \ldots < knx < (kn+1)x < \ldots$ is an infinite alternating sequence of n-divisible and non-n-divisible elements.

WEAK O-MINIMALITY (II)

...continued.

- $\phi_n(y) := \exists x\ y = nz$ defines over A the set of n-divisible elements.
- If **A** were weakly o-minimal, $\phi_n(y)$ would define a finite union of convex sets $\bigcup_i X_i$. So there would be a set X_j, containing infinitely many n-divisible and non-n-divisible elements.
- Therefore, **A** cannot be weakly o-minimal.

□

WEAK O-MINIMALITY (III)

Lemma

Let $\mathbf{A}$ *be an* MV*-chain. If* $\mathbf{A}$ *is weakly o-minimal, then* $\mathbf{A}/Rad(\mathbf{A})$ *is finite or divisible.*

Proof.

- If $\mathbf{A}$ is weakly o-minimal, then, for all $x \in A$, x is n-divisible if and only if $\mathfrak{p}(x) \in A/Rad(A)$ is n-divisible as well.
- Suppose that $\mathbf{A}/Rad(\mathbf{A})$ is infinite and not n-divisible for some n.
- Up to isomorphism, $\mathbf{A}/Rad(\mathbf{A})$ is a dense subalgebra of $[0,1]_{\text{MV}}$.
- Both n-divisible elements and non-n-divisible elements are dense in $\mathbf{A}/Rad(\mathbf{A})$.

WEAK O-MINIMALITY (IV)

...continued.

- $\phi_n(y) := \exists x\; y = nz$ defines over A the set of n-divisible elements.
- If **A** were weakly o-minimal, both $\phi_n(y)$ and $\neg\phi_n(y)$ would define finite unions of convex sets.
- But if **A** were weakly o-minimal, there would be a one-to-one correspondence between n-divisible elements of **A** and $\mathbf{A}/Rad(\mathbf{A})$.
- Then there would be a definable subset of $A/Rad(A)$ containing only elements that are either all n-divisible or all non-n-divisible: i.e. a contradiction.

□

WEAK O-MINIMALITY: FULL CHARACTERIZATION

Theorem

Let $\mathbf{A}$ *be any* MV*-chain, and let* $\mathsf{Th}(\mathbf{A})$ *be the first-order theory of* $\mathbf{A}$ *in the language* $\mathcal{L}_{\mathrm{MV}} = \langle \oplus, ^*, 0, < \rangle$*. Then the following are equivalent:*

(1) $\mathbf{A}$ *is weakly o-minimal.*

(2) $\mathsf{Th}(\mathbf{A})$ *has elimination of quantifiers in* $\mathcal{L}_{\mathrm{MV}}$*.*

(3) $Rad(\mathbf{A})$ *is divisible, and* $\mathbf{A}/Rad(\mathbf{A})$ *is finite or divisible.*

WEAKLY O-MINIMAL MV-CHAINS THAT ARE NOT O-MINIMAL

- Let $\mathbf{A} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, g))$, where $\mathbf{G}$ is a divisible OAG.
- The formula $(n+1)x < 1$ defines a set that exactly coincides with $Rad(\mathbf{A})$.
- $Rad(\mathbf{A})$ is a convex set but does not have an endpoint in A.
- Therefore, every non-simple MV-chain of finite rank $\mathbf{A} \cong \Gamma(\mathbb{Z}\vec{\times}\mathbf{G}, (n, g))$ cannot be o-minimal.

Weakly o-minimal MV-chains that are not o-minimal

Theorem

Let $\mathbf{A}$ *be any* MV*-chain in the language* $\mathcal{L}_{\mathrm{MV}} = \langle \oplus, ^*, 0, < \rangle$*. Then the following are equivalent:*

(1) $\mathbf{A}$ *is o-minimal.*

(2) $\mathbf{A}$ *is finite or divisible.*

Proof.

- If $\mathbf{A}$ is o-minimal, then $\mathbf{A}$ is weakly o-minimal.
- So, $\mathbf{A}$ is either divisible, or finite, or is a non-simple MV-chain of finite rank (which is not possible).
- The converse is obvious.

□

THANKS!