Assessing Simple Coalition Games in Many-valued Logics

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Motivation

- Simple games model yes/no voting in collective bodies
- Provided that partial membership degrees of players are allowed, players may form fuzzy coalitions
- Which class of games is obtained by replacing Boolean logic with Łukasiewicz logic in many-valued scheme of cooperation?

Coalition Game

Definition

- $N = \{1, \ldots, n\}$ is the set of players
- 2^N is the set of all coalitions
- $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is a coalition game

A coalition game is simple if v is a non-decreasing $\{0, 1\}$-valued function with $v(N) = 1$.

Examples of Simple Games

UNSC voting, U.S. presidential election etc.

Example (Majority voting)

Assume that $N = \{1, 2, 3\}$ is the set of players. The majority voting is captured by a simple game w such that

$$w(A) = \begin{cases} 1 & \text{if } |A| \geqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

Simple Games and Boolean Formulas

Each simple game v can be associated with a unique non-decreasing non-constant Boolean function

$$f_v(1_A) = v(A), \qquad A \subseteq N$$

Theorem

Let v: $2^N \to \{0, 1\}$ be a non-constant function. TFAE:

- *v is a simple game*
- *there is a negation-free formula φ such that f_v is the Boolean function corresponding to φ*

Cores of Simple Games

The core of v is the set of all efficient payoff vectors $x \in \mathbb{R}^n$ upon which no coalition A can improve:

$$\mathcal{C}(v) = \{\, x \in \mathbb{R}^n \mid \langle 1_N, x \rangle = v(N) \text{ and } \langle 1_A, x \rangle \geqslant v(A) \text{ for each } A \subseteq N \,\}$$

Theorem

A simple game has a non-empty core iff there is at least one veto player $i \in N$ (that is, $v(N \setminus \{i\}) = 0$) in the game.

- $\mathcal{C}(\text{UNSC voting game}) \neq \emptyset$
- $\mathcal{C}(\text{majority voting game}) = \emptyset$

From Boolean to Fuzzy Coalitions

In a coalition $A \subseteq N$, a player $i \in N$ participates in a degree 0 or 1.

Towards partial participation:

Definition

A fuzzy coalition is a vector $a = (a_1, \ldots, a_n) \in I^n$, where each a_i is a degree of membership of player i in a.

- a_i measures player's **degree of involvement** in some activity (Aubin)
- **large economy** argument: a fuzzy coalition in the finite economy becomes a coalition in the non-atomic economy made of a large number of homogeneous agents (Aumann; Azrieli and Lehrer; Husseinov)

Game with Fuzzy Coalitions

Definition (Aubin; 1974)

- $N = \{1, \dots, n\}$ is the set of players
- I^n is set of all fuzzy coalitions
- $v \colon I^n \to \mathbb{R}$ s.t. $v(1_\emptyset) = 0$ is a game with fuzzy coalitions

The core of v is the set of all efficient payoff vectors $x \in \mathbb{R}^n$ upon which no fuzzy coalition a can improve:

$$\mathcal{C}(v) = \{ x \in \mathbb{R}^n \mid \langle 1_N, x \rangle = v(1_N) \text{ and } \langle a, x \rangle \geqslant v(a) \text{ for each } a \in I^n \}$$

Changing the Logic

1. Replace the negation-free Boolean formulas with negation-free Łukasiewicz formulas
2. Observe: each negation-free formula

$$\varphi \in \mathrm{Form}_n$$

in Łuk. logic induces a non-decreasing McNaughton function

$$[\varphi] : I^n \to I$$

3. Which class of games with fuzzy coalitions is obtained?

Simple Łukasiewicz Games

Definition

A simple Łukasiewicz game is a McNaughton function $v\colon I^n \to I$ that is

- non-decreasing
- $v(1_\emptyset) = 0$ and $v(1_N) = 1$

By SŁG_n we denote the set of all simple Łukasiewicz games over I^n.

Theorem

Let $v\colon I^n \to I$ be a non-constant function. TFAE:

- $v \in \text{SŁG}_n$
- *there is a negation-free formula in Łuk. logic φ s.t. $v = [\varphi]$*

Cores of Simple Łukasiewicz Games

For each fuzzy coalition $a \in I^n$, put

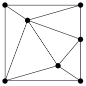
$$\mathcal{C}_a(v) = \begin{cases} \{\, x \in \mathbb{R}^n \mid \langle 1_N, x \rangle = v(1_N) \,\} & \text{if } a = 1_N, \\ \{\, x \in \mathbb{R}^n \mid \langle a, x \rangle \geqslant v(a) \,\} & \text{otherwise,} \end{cases}$$

The core of $v \in \mathrm{SŁG}_n$ is

$$\mathcal{C}(v) = \bigcap_{a \in I^n} \mathcal{C}_a(v)$$

1. Are there redundant sets $\mathcal{C}_a(v)$ in $\bigcap_{a \in I^n} \mathcal{C}_a(v)$?
2. Characterize non-emptiness of the core on $\mathrm{SŁG}_n$.

Shape of the Core



$\mathcal{N}(v)$ denotes the set of all nodes of v

Theorem

If $v \in \text{SŁG}_n$, *then*

$$\mathcal{C}(v) = \bigcap_{a \in \mathcal{N}(v)} \mathcal{C}_a(v).$$

Moreover, $\mathcal{C}(v)$ *is a (possibly empty) polytope included in the standard* $(n-1)$*-dimensional simplex*

$$\Delta_n = \{\, x \in \mathbb{R}^n \mid \langle 1_N, x \rangle = 1 \textit{ and } x_i \geqslant 0 \textit{ for every } i \in N \,\}.$$

Examples

Example ($v_\wedge(a) = a_1 \wedge \cdots \wedge a_n$)

Since $\mathcal{N}(v_\wedge) = \{0, 1\}^n$ we have the maximal core $\mathcal{C}(v_\wedge) = \Delta_n$

Example (Dictatorial game $v_i(a) = a_i$)

$\mathcal{C}(v_i) = \{1_i\}$

Example ($v_\oplus(a) = a_1 \oplus \cdots \oplus a_n$)

$\mathcal{C}(v_\oplus) = \emptyset$

Non-emptiness of Cores (1)

Let $\mathcal{B}$ be a finite set of fuzzy coalitions in I^n and $(\lambda_a)_{a\in\mathcal{B}}$ be a real vector with $\lambda_a \in I$. We say that a pair $(\mathcal{B}, (\lambda_a)_{a\in\mathcal{B}})$ is a balanced system if

$$\sum_{a\in\mathcal{B}} \lambda_a a = 1_N.$$

Theorem

Let $v \in \mathrm{SŁG}_n$. Then core $\mathcal{C}(v) \neq \emptyset$ iff the inequality

$$\sum_{a\in\mathcal{N}(v)} \lambda_a v(a) \leqslant 1$$

is true for every balanced system $(\mathcal{N}(v), (\lambda_a)_{a\in\mathcal{N}(v)})$.

Veto Players

Let $v \in \text{SŁG}_n$ and $i \in N$. The function $v^i(a)\colon I \to I$ defined by

$$v^i(a) = v(\underbrace{1, \ldots, 1}_{i-1}, a, 1, \ldots, 1), \quad a \in I.$$

measures a marginal influence of player i.

Definition

Let $v \in \text{SŁG}_n$. We say that player $i \in N$ is a

- null player if $v^i(0) = 1$,
- veto player if $v^i(0) = 0$.

Furthermore, we say that veto player $i \in N$ is a

- weak veto player if $v^i(c) = 1$ for some $c < 1$,
- strong veto player if $v^i(c) = 1$ implies $c = 1$.

Non-emptiness of Cores (2)

Lemma

Let $v \in \mathrm{S{\L}G}_n$. Then i is a strong veto player iff $v^i(a) \leqslant a$ for every $a \in I$.

Theorem

Any game $v \in \mathrm{S{\L}G}_n$ is a game with strong veto players iff it has a non-empty core.