Layers of zero probability for conditional states of many-valued events

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Outline

1. Introduction
   - Betting on conditional events
   - Lexicographic and non-standard probability
2. The case of Łukasiewicz Events
3. Conditional probability (states) on MV-algebras
   - Defining layers of zero-probability for Łukasiewicz events
4. Future work

Conditional probability on Boolean algebras

Let $B = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$ be a Boolean algebra, and consider $C = B \times B \setminus \{\bot\}$.
A *conditional probability* on B is a map $P(\cdot \mid \cdot) : C \to [0, 1]$ satisfying the following axioms:

1. $P(H \mid H) = 1$ for every $H \in B \setminus \{\bot\}$;
2. $P(\cdot \mid H)$ is, for every $H \in B \setminus \{\bot\}$, a (finitely additive) probability measure on B;
3. $P((E \wedge A) \mid H) = P(E \mid H) \cdot P(A \mid E \wedge H)$, for every $A \in B$, and each $E, H \in B \setminus \{\bot\}$ such that $E \wedge H \neq \bot$.

Outline

Conditional probability can also be characterized in terms of coherence (not-sure loss principle).

Betting $\lambda \in \mathbb{R}$ on $E \mid H = \alpha$ means that we accept to pay $\alpha\lambda$ to the bookmaker in order to receive, in the possible world V:

- λ if both $V(E) = 1$ and $V(H) = 1$,
- 0 if $V(E) = 0$ and $V(H) = 1$,
- $\alpha\lambda$, if $V(H) = 0$.

Then an assignment

$$\chi : E_i \mid H_i \to \alpha_i.$$

is coherent (it does not ensure a sure loss), iff there is no way of betting on χ ensuring a sure loss for the bookmaker, i.e. a sure win for the gambler.

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Coherence criterion for conditional assignments, actually characterizes conditional probability by the following:

Theorem ([1])

Let $E_1 \mid H_1, \ldots, E_n \mid H_n$ be conditional events, let $\chi : E_i \mid H_i \mapsto \alpha_i$ an assignment, and let B be the Boolean algebra generated by the unconditional events E_i, H_i.
Then the following are equivalent:

1. *χ is coherent;*
2. *There exists a conditional probability $P(\cdot \mid \cdot)$ on B such that for each $1 \leq i \leq n$,*

$$P(E_i \mid H_i) = \chi(E_i \mid H_i) = \alpha_i.$$

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Non-standard and lexicographic probability (cf. [1, 2])

Coherent conditional probability assignments can be characterized in terms of *non-standard probability*, and *lexicographic probability*

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1. *χ is coherent (i.e. it does not allow surely winning strategies);*
2. *There exists a non-standard probability $P^* : B \to {}^*[0,1]$ such that*
   - *For every i, $P^*(H_i) > 0$;*
   - *For every i, $\alpha_i = St\left(\frac{P^*(E_i \wedge H_i)}{P^*(H_i)}\right)$.*
3. *There exists a $r \in \mathbb{N}$, and a lexicographic probability space $\langle P_0, \ldots, P_r \rangle$ such that*
   - *For every i there exists a $1 \leq \ell(i) \leq r$ such that $P_{\ell(i)}(H_i) > 0$, and*
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Idea of the proof

The class $\mathcal{A} = \{a_1, \ldots, a_m\}$ of atoms generated by the events E_i, H_i can be stratified in a hierarchy defined by $r+1$ probability distributions $p_0, \ldots, p_r$, as follows:

- for each j, $p_j : \mathcal{A} \to [0,1]$,
- if $\mathcal{A}_j = \{a_t \in \mathcal{A} : p_j(a_t) = 0\}$, then p_{j+1} is 0 on $\mathcal{A} \setminus \mathcal{A}_j$.
- for each H_i, there exists a minimum j such that $P_j(H_i) = \sum_{a \in H_i} p_j(a) > 0$. This minimum level is $\ell(i)$: the zero-layer of H_i.
- The nonstandard probability P^* respects zero-layers. In fact P^* is defined such that, for each H_i, H_z,

$$\text{if } \ell(i) < \ell(z), \text{ then } P^*(H_z) << P^*(H_i),$$

i.e. $P^*(H_z)/P^*(H_i)$ is a positive infinitesimal.

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Łukasiewicz events and states

An MV-algebra is a structure $A = (A, \oplus, \neg, \bot)$ of type $(2, 1, 0)$ such that $(A, \oplus, \bot)$ is a commutative monoid with neutral element $\bot$, and the following equations hold:

- $\neg(\neg x) = x$
- $x \oplus \top = \top$, where $\top = \neg\bot$
- $x \oplus \neg(x \oplus \neg y) = y \oplus \neg(y \oplus \neg x)$

($\star$) The real unit interval $[0, 1]$ with operations $\oplus$ and $\neg$ defined by

$$x \oplus y = \min\{1, x + y\}, \textit{ and } \neg x = 1 - x$$

is an MV-algebra denoted by $[0, 1]_{MV}$ and called the standard MV-algebra.

($\star$) Fix a $k \in \mathbb{N}$. Then the set of all functions from $[0, 1]^k$ into $[0, 1]$ that are continuous, piecewise linear, and such that each piece has integer coefficient, with the operations $\oplus$ and $\neg$ defined as a pointwise application of those of $[0, 1]_{MV}$ is an MV-algebra (actually the free MV-algebra over k generators).

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The free MV-algebra $Free(k)$ over k generators, is the Lindenbaum algebra of Łukasiewicz propositional logic.

A *Łukasiewicz event* will be, for us, any equivalence class of formulas $[\theta]$, that is, any McNaughton function $f \in Free(k)$.

A *state* [6] on an MV-algebra A is any map $s : A \to [0, 1]$ such that

- $s(\top) = 1$,
- whenever $x \odot y = \bot$, then $s(x \oplus y) = s(x) + s(y)$.

A state $s : A \to [0, 1]$ is said to be *faithful* if $s(x) = 0$, implies $x = \bot$.

A state $s : A \to {}^*[0, 1]$ is said to be a *hyperstate*.

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There are several attempts to generalize conditional probability (conditional states) on MV-algebras

- Gerla [1]: Axiomatic approach to conditional probability on MV-algebras. A conditional probability $s(\cdot \mid \cdot)$ is a primitive notion.
- Kroupa [2]: Conditional probability is definable from a simple probability:

$$s(f \mid g) = \frac{s(f \cdot g)}{s(g)},$$

  whenever $s(g) > 0$. (Montagna [3] provided a characterization of Kroupa's approach in terms of a not-sure loss principle)
- Mundici [7, 8]: Conditional probability as a state on a quotient algebra (the quotient is obtained by forcing an antecedent), and Rényi conditional probability on MV-algebras.
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Montagna's result

A complete real-valued assignment

$$\Lambda : f_1 \mid g_1 \mapsto \alpha_1, \ldots, f_n \mid g_n \mapsto \alpha_n, g_1 \mapsto \beta_1, \ldots, g_n \mapsto \beta_n$$

is *stably coherent* if there is another assessment

$$\Lambda' : f_1 \mid g_1 \mapsto \alpha'_1, \ldots, f_n \mid g_n \mapsto \alpha'_n, g_1 \mapsto \beta'_1, \ldots, g_n \mapsto \beta'_n$$

such that $\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_n$ belong to a nonstandard extension $^*[0,1]$ of $[0,1]$, and in addition:

(i) Λ and Λ' differ by an infinitesimal, that is, for $i = 1, \ldots, n$, $|\alpha'_i - \alpha_i|$ and $|\beta'_i - \beta_i|$ are infinitesimal;

(ii) for $i = 1, \ldots n$, $\beta'_i > 0$

(iii) Λ' avoids sure loss.

Montagna's result

Theorem

Let $f_1 \dots f_n, g_1, \dots g_n$ be Łukasiewicz events, and let

$$\Lambda : f_i \mid g_i \mapsto \alpha_i,\ g_i \mapsto \beta_i \ \ (i = 1, \dots, n)$$

be a complete assignment. Then the following are equivalent:

- *Λ is stably coherent;*
- *There exists a faithful hyperstate s^* such that*
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Let $f_1 \dots f_n, g_1, \dots g_n$ be Łukasiewicz events, and let

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Outline

- Let $f_1, g_1, \ldots, f_n, g_n$ be *Łukasiewicz events* in $Free(k)$.
- Let Δ be a minimal unimodular triangulation of the hypercube $[0,1]^k$ that linearizes each f_i and g_i. Also let

$$Ver(\Delta) = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$$

be the set of vertices of Δ.

- Let $\mathbf{h}_1, \ldots, \mathbf{h}_m$ be the normalized Shauder hats corresponding to the vertices in $Ver(\Delta)$. Each $\mathbf{h}_i$ is a McNaughton function, and hence $\mathbf{h}_i \in Free(k)$.
  - For distinct $\mathbf{h}_i, \mathbf{h}_j \in \mathcal{H}$, $\mathbf{h}_i \odot \mathbf{h}_j = 0$, and $\bigoplus_{t=1}^{m} \mathbf{h}_t = 1$;
  - For each $i = 1, \ldots, n$,

$$f_i = \bigoplus_{t=1}^{m} \mathbf{h}_t \cdot f_i(\mathbf{x}_t), \text{ and } g_i = \bigoplus_{t=1}^{m} \mathbf{h}_t \cdot g_i(\mathbf{x}_t).$$

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- Let $Free(k)^+$ be the MV-algebra generated by

$$Free(k) \cup \{f_i \cdot g_i : i = 1, \ldots, n\}.$$

- Every state s on $Free(k)$ can be extended to a state $(s)^+$ on $Free(k)^+$ by stipulating: for every $p \in Free(k)^+$

$$(s)^+(p) = \sum_{t=1}^{m} s(\mathbf{h}_t) \cdot p(\mathbf{x}_t).$$

In a similar way, every hyperstate s^* on $Free(k)$ extends to a hyperstate $(s^*)^+$ on $Free(k)^+$.

- Given a class $\{d_0, \ldots, d_r\}$ of mappings from $\{\mathbf{h}_1, \ldots, \mathbf{h}_m\}$ into $[0, 1]$ satisfying, for each j, $\sum_{t=1}^{m} d_j(\mathbf{h}_t) = 1$ (we will henceforth call them *distributions*), we define the *zero-layer* of a function p in $Free(k)^+$ as

$$\ell(p) = \min\{j : \exists t \leq m, d_j(\mathbf{h}_t) > 0,\ p(\mathbf{x}_t) > 0\}$$

if such a j exists, and $\ell(p) = \infty$ otherwise.

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Theorem

Let $f_1 \mid g_1, \ldots, f_n \mid g_n$ be as above, and let $\chi : f_i \mid g_i \mapsto \alpha_i, g_i \mapsto \beta_i$ (for $i = 1, \ldots, n$) be a real-valued complete assignment. Then the following are equivalent:

(i) *There exists a faithful hyperstate $s^* : Free(k) \to {}^*[0, 1]$ such that for every g_i, $St(s^*(g_i)) = \beta_i$, and for every $i = 1, \ldots, n$,*

$$\alpha_i = St\left(\frac{(s^*)^+(f_i \cdot g_i)}{s^*(g_i)}\right).$$

(ii) *There exist states $s_0, \ldots, s_r$ over $Free(k)$ such that, for every $i = 1, \ldots, n$, there exists $\ell(i) \in \{0, \ldots, r\}$ such that $s_{\ell(i)}(g_i) > 0$. Moreover, if $\beta_x > 0$, $\ell(g_x) = 0$, $s_0(g_x) = \beta_x$; and for every i,*

$$\alpha_i = \frac{(s_{\ell(i)})^+(f_i \cdot g_i)}{s_{\ell(i)}(g_i)}.$$

($\Rightarrow$). Let $s^* : Free(k) \to {}^*[0,1]$ be a faithful hyperstate, and let $\mathcal{H}$ the class of normalized Schauder hats given by f_i, g_i.

(1) Define:

- $\mathcal{H}_0 = \mathcal{H}$, and $d_0 : \mathcal{H} \to [0,1]$ as $d_0(\mathbf{h}) = St(s^*(\mathbf{h}))$;
- $\mathcal{H}_{i+1} = \{\mathbf{h} \in \mathcal{H}_i : d_i(\mathbf{h}) = 0\}$.
  If $\mathcal{H}_{i+1} = \emptyset$, then we stop;
  If $\mathcal{H}_{i+1} \neq \emptyset$, define $\Phi_{i+1} = \bigoplus\{\mathbf{h} \in \mathcal{H}_{i+1}\}$, and

$$d_{i+1}(\mathbf{h}) = St\left(\frac{s^*(\mathbf{h})}{s^*(\Phi_{i+1})}\right).$$

(2) The process stops in finitely many steps, giving a class of distributions $d_0, \ldots, d_r$ such that, for each g_i, $\ell(g_i) < \infty$.

(3) It holds

$$\alpha_i \cdot \sum_{t=1}^{m} d_{\ell(g_i)}(\mathbf{h}_t) \cdot g_i(\mathbf{x}_t) = \sum_{t=1}^{m} d_{\ell(g_i)}(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t).$$

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Let $\varepsilon > 0$ be a positive infinitesimal, and define $s^* : Free(k) \to {}^*[0,1]$ as:

$$s^*(f) = K \cdot \sum_{t=1}^{m} \varepsilon^{\ell(t)} \cdot d_{\ell(t)}(\mathbf{h}_t) \cdot f(\mathbf{x}_t)$$

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Future work

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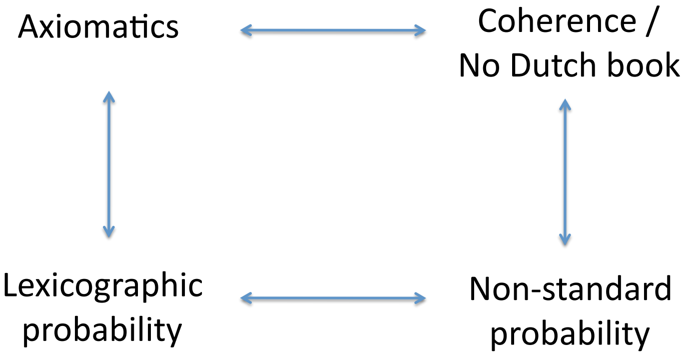
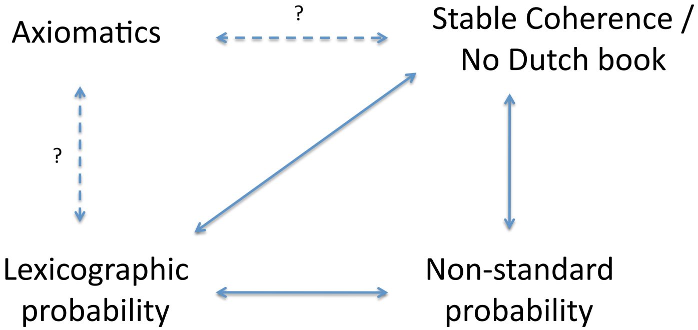
CONDITIONAL PROBABILITY
-- CLASSICAL EVENTS--



CONDITIONAL PROBABILITY
-- MANY-VALUED EVENTS --

$$P(A \cdot B) = P(A \mid B) \cdot P(B)$$



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