HOW DO ℓ-GROUPS AND PO-GROUPS APPEAR IN ALGEBRAIC AND QUANTUM STRUCTURES ?

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Abstract. We show when ℓ-groups with strong unit or without strong unit and similarly po-groups are closely connected with some important algebraic structures like MV-algebras, pseudo MV-algebras, pseudo BL-algebras, and quantum structures, like effect algebras and pseudo effect algebras.

1. Introduction

Lattice ordered groups (= ℓ-groups) and partially ordered groups (= po-groups) met the last decade an increasing interest for their study in the framework of modern algebraic structures like MV-algebras or quantum structures. *MV-algebras* were introduced by Chang [Cha] at the end of the Fifties as an algebraic counterpart for many valued reasoning. A fundamental result for MV-algebras was given by Mundici [Mun] who proved that every MV-algebra is an interval in an Abelian ℓ-group G with strong unit u. We recall that an element u in a po-group G is a strong unit (or an order unit) if, given $g \in G$, there is an integer $n \geq 1$ such that $g \leq nu$, and a pair (G, u) is said to be a *unital po-group* (or a unital ℓ-group). Moreover, the variety of MV-algebras is categorically equivalent with the category of unital Abelian ℓ-groups. The non-commutativity of the basic MV-operation, conjunction operation $\oplus$ in MV-algebras, was skipped and in such a way, that *pseudo MV-algebras,* [GeIo], or equivalently *generalized MV-algebras*, [Rac], were defined as a non-commutative generalization of MV-algebras. The basic result on pseudo MV-algebras [Dvu1] says that every pseudo MV-algebra is an interval in a unital (not necessarily Abelian) ℓ-groups. In addition, there is a categorical equivalence between the variety of pseudo MV-algebras, which is also a variety, and the the variety of unital ℓ-groups which is not a variety. This result is a bridge or a vocabulary among different areas of mathematics. It allowed us to show that e.g. the lattice of pseudo MV-algebras varieties is uncountable whiles the lattice of MV-algebras varieties is countable, etc.

On the other hand, there is also a different area of mathematics, called *quantum structures.* They are motivated by actual problems of mathematical foundations

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of axiomatization of quantum mechanics, where as it is known neither Newtons classical mechanics neither Kolmogorovs probabilistic model does not work. They combine also in the last two decades a many-valued reasonings as well as fuzzy sets ideas. Nowadays, there is a whole hierarchy of quantum structures like Boolean algebras, orthomodular lattices, orthomodular posets, orthoalgebras, etc. In the beginning of the Nineties, *D-posets* were introduced by Kôpka and Chovanec [KoCh], where the primary notion is a partial operation difference of two comparable events. An alternative and equivalent structure with a D-poset is an *effect algebra* introduced by Foulis and Bennett, [FoBe], where the addition, $+$, is a partial basic operation. For a comprehensive source on effect algebras we recommend [DvPu]. An important property which we meet in theory of effect algebras is the *Riesz Decomposition Property*, (RDP). It means roughly speaking that every two finite decompositions of the unit admit a joint refinement. Ravindran [Rav] showed that every effect algebra with (RDP) is isomorphic to an interval in a unital po-group (G,u) such that G satisfies *interpolation* (i.e. if $g_1, g_2 \leqslant h_1, h_2$, there is a $d \in G$ such that $g_1, g_2 \leqslant d \leqslant h_1, h_2$), or equivalently (RDP); more for interpolation po-groups see [Goo]. Moreover, also a categorical equivalence between the category of effect algebras with (RDP) and unital po-groups with interpolation. In 2001, also a non-commutative generalization of effect algebras, called *pseudo effect algebras,* was introduced in [DvVe1, DvVe2], where the partial addition $+$ is not more necessarily commutative. For them it was necessary to introduce three types of the Riesz Decomposition Property, (RDP), $(\mathrm{RDP})_1$ and $(\mathrm{RDP})_2$. All these three properties are equivalent for effect algebras and Abelian po-groups. Finally, it was proved again, that every pseudo effect algebra with $(\mathrm{RDP})_1$ is an interval in a unital po-group with $(\mathrm{RDP})_1$. Similarly, the category of pseudo effect algebras with $(\mathrm{RDP})_1$ is categorical equivalent with the category of unital po-groups with $(\mathrm{RDP})_1$. Fuzzy logic can be studied in the framework of *BL-algebras,* and also they have a non-commutative generalization called *pseudo BL-algebras,* [DGI1]. In [Dvu2] it was shown that every linearly ordered pseudo BL-algebra is an ordinal sum of a system consisting of negative cones in ℓ-groups or of negative intervals in unital ℓ-groups.

We see that ℓ-groups and po-groups have a very close connection with many algebraic structures, like MV-algebras, pseudo MV-algebras, BL-algebras, pseudo BL-algebras, as well as with quantum structures, like effect algebras and pseudo effect algebras. In the proposed talk we would like to present some new unexpected relations among these structures and ℓ-groups and po-groups. We will study perfect effect algebras and MV-algebras, n-perfect pseudo MV-algebras and n-perfect pseudo effect algebras. We show an interesting construction of pseudo BL-algebras, called kites, which are based on ℓ-groups; we recall that starting even with an Abelian ℓ-group, we obtain a kite that is a non-commutative pseudo BL-algebra.

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