Schematic Extensions of psMTL Logic

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In honour of Antonio Di Nola's 65th birthday
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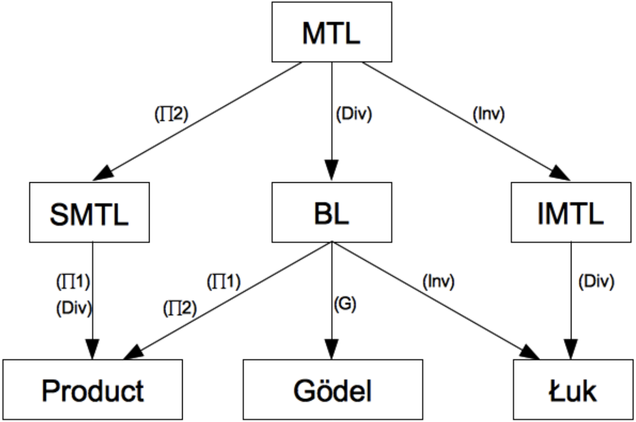
Outline

MTL logic and extensions



(Div) $(\varphi \wedge \psi) \to ((\varphi \to \psi) \& \varphi)$

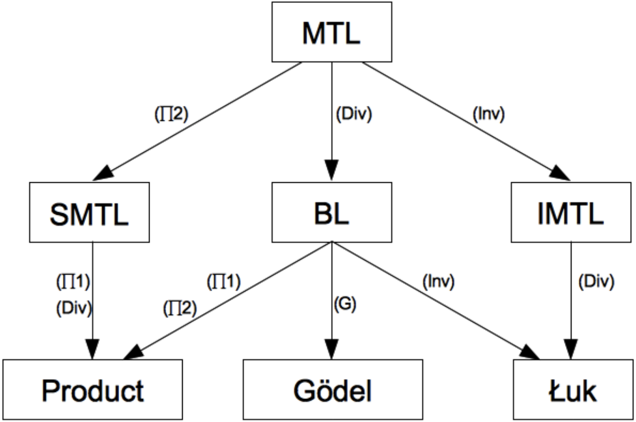
(Inv) $\neg\neg\varphi \to \varphi$

(G) $\varphi \to (\varphi \& \varphi)$

(Π1) $\neg\neg\psi \to (((\varphi \& \psi) \to (\chi \& \psi)) \to (\varphi \to \chi))$

(Π2) $\varphi \wedge \neg\varphi \to \overline{0}$

MTL logic and extensions



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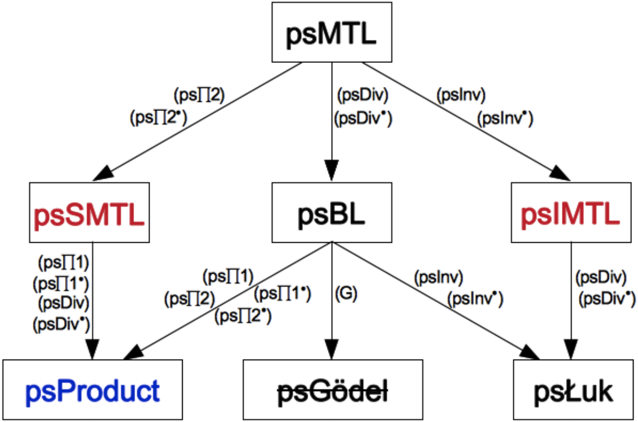
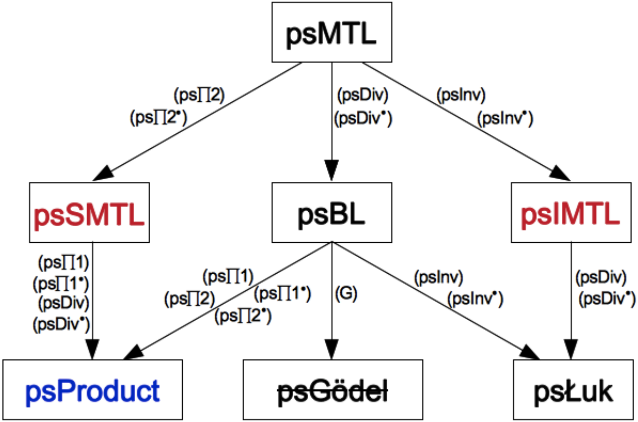
psMTL logic and extensions



(psDiv)	$(\varphi \wedge \psi) \rightarrow ((\varphi \rightarrow \psi) \& \varphi)$	(psDiv$^\bullet$)	$(\varphi \wedge \psi) \rightsquigarrow (\varphi \& (\varphi \rightsquigarrow \psi))$
(psInv)	$\sim \neg \varphi \rightarrow \varphi$	(psInv$^\bullet$)	$\neg \sim \varphi \rightsquigarrow \varphi$
(psΠ1)	$\sim \neg \psi \rightarrow (((\varphi \& \psi) \rightarrow (\chi \& \psi)) \rightarrow (\varphi \rightarrow \chi))$	(psΠ1$^\bullet$)	$\neg \sim \psi \rightsquigarrow (((\psi \& \varphi) \rightsquigarrow (\psi \& \chi)) \rightsquigarrow (\varphi \rightsquigarrow \chi))$
(psΠ2)	$\varphi \wedge \neg \varphi \rightarrow \overline{0}$	(psΠ2$^\bullet$)	$\varphi \wedge \sim \varphi \rightsquigarrow \overline{0}$

psMTL logic and extensions



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- The language:
  - the primitive connectives: $\vee$, $\wedge$, $\&$, $\rightarrow$, $\rightsquigarrow$
  - the constant: $\overline{0}$
- For any formula φ, we define the formula $\varphi^{\bullet}$ that:
  - reverses the arguments of $\&$
  - interchanges the implications $\rightarrow$ and $\rightsquigarrow$
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Syntax

The axioms of psMTL logic are:

I. any formula which has one of the following forms is an axiom:

$$
\begin{array}{ll}
(A1) & (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \\
(A2) & (\varphi \& \psi) \to \varphi \\
(A3) & (\varphi \wedge \psi) \to \varphi \\
(A4) & (\varphi \wedge \psi) \to (\psi \wedge \varphi) \\
(A5) & ((\varphi \to \psi) \& \varphi) \to (\varphi \wedge \psi) \\
(A6a) & (\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi) \\
(A6b) & ((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi)) \\
(A7) & ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \\
(A8a) & (\varphi \vee \psi) \to (((\varphi \rightsquigarrow \psi) \to \psi) \wedge ((\psi \rightsquigarrow \varphi) \to \varphi)) \\
(A8b) & (((\varphi \rightsquigarrow \psi) \to \psi) \wedge ((\psi \rightsquigarrow \varphi) \to \varphi)) \to (\varphi \vee \psi) \\
(A9) & \overline{0} \to \varphi
\end{array}
$$

II. if φ is an axiom of the form $(A1)$, $(A2)$, $(A5)$, $(A6a)$, $(A6b)$, $(A7)$, $(A8a)$ or $(A8b)$, then $\varphi^{\bullet}$ is an axiom.

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The deduction rules of psMTL logic are:

(MP1) $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$ (Impl1) $\dfrac{\varphi \to \psi}{\varphi \rightsquigarrow \psi}$

(MP2) $\dfrac{\varphi,\ \varphi \rightsquigarrow \psi}{\psi}$ (Impl2) $\dfrac{\varphi \rightsquigarrow \psi}{\varphi \to \psi}$

Algebraic semantics

Definition

A psMTL-algebra is a structure of the form

$$\mathcal{A} = (A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$$

satisfying the following conditions:

(RL1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice

(RL2) $(A, \odot, 1)$ is a monoid

(pPR) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ (adjoindness property)

(pprel) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ (prelinearity condition)

Equivalent definitions for a psMTL-algebra:

- a residuated lattice $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ satisfying condition (pprel);
- a bounded psBCK(pPR)-lattice $(A, \vee, \wedge, \rightarrow, \rightsquigarrow, \odot, 0, 1)$ satisfying condition (pprel).

$$x^- \stackrel{def}{=} x \rightarrow 0 \qquad x^\sim \stackrel{def}{=} x \rightsquigarrow 0$$

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Pseudo-t-norms

- A pseudo-t-norm $\otimes$ is a binary operation on the real unit interval that is associative, non-decreasing in both arguments and $x \otimes 1 = 1 \otimes x = x$.
- If $\otimes$ is a left-continuous pseudo-t-norm, then we define the left residuum and the right residuum by:

$$a \to b = \sup\{c \mid c \otimes a \leq b\}$$

$$a \rightsquigarrow b = \sup\{c \mid a \otimes c \leq b\}$$

- Any continuous pseudo-t-norm is commutative.
- There are left-continuous non-commutative pseudo-t-norms.

Let $0 < a_1 < a_2 < b_2 < 1$ and $T_{1,2} : [0,1] \times [0,1] \to [0,1]$ be

$$T_{1,2}(x,y) = \begin{cases} a_1, & \text{if } a_1 < x \leq a_2 \text{ and } a_1 < y \leq b_2 \\ \min(x,y), & \text{otherwise} \end{cases}$$

- standard psMTL-algebra

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$\mathrm{psMTL^r}$ logic

- The variety of psMTL-algebras does not have subdirect representation property.
- Representable psMTL-algebras ($\mathrm{psMTL^r}$-algebras) are obtained by adding Kühr's axioms:

  (R1) $(y \to x) \vee (z \rightsquigarrow ((x \to y) \odot z)) = 1$
  (R2) $(y \rightsquigarrow x) \vee (z \to (z \odot (x \rightsquigarrow y))) = 1$

- The logic $\mathrm{psMTL^r}$ is the extension of psMTL by the axioms:

  $(A10)$ $(\varphi \to \psi) \vee (\chi \rightsquigarrow ((\psi \to \varphi)\&\chi))$
  $(A10^\bullet)$ $(\varphi \rightsquigarrow \psi) \vee (\chi \to (\chi\&(\psi \rightsquigarrow \varphi)))$

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Completeness results

- strong completeness for psMTL logic - P. Hájek
- strong chain completeness for psMTL^r logic - P. Hájek
- standard completeness for psMTL^r logic - S. Jenei, F. Montagna
- finite strong standard completeness for psMTL^r logic

Outline

Syntax

- Predicate language: $\mathrm{J} = (\mathrm{Pred_J}, \mathrm{Const_J})$
- The axioms of $\mathrm{psMTL}\forall$ logic are:
  I. the axioms of the propositional calculus psMTL;
  II. a formula which has one of the following forms is an axiom:

$$
\begin{array}{lll}
(\forall 1) & (\forall x)\varphi(x) \to \varphi(t) & (t \text{ is substitutable for } x \text{ in } \varphi(x)) \\
(\exists 1) & \varphi(t) \to (\exists x)\varphi(x) & (t \text{ is substitutable for } x \text{ in } \varphi(x)) \\
(\forall 2) & (\forall x)(\varphi \to \psi) \to (\varphi \to (\forall x)\psi) & (x \text{ not free in } \varphi) \\
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\end{array}
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  III. if φ is an axiom of the form $(\forall 2)$ or $(\exists 2)$, then $\varphi^{\bullet}$ is an axiom.

- The deduction rules of $\mathrm{psMTL}\forall$ are those of psMTL and the rule:

$$
(\mathrm{G})\ \frac{\varphi}{(\forall x)\varphi}
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- The logic $\mathrm{psMTL^r}\forall$ has the axioms of $\mathrm{psMTL^r}$ logic, the above axioms and:

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(\forall 3)\ (\forall x)(\varphi \vee \psi) \to ((\forall x)\varphi \vee \psi) \qquad (x \text{ not free in } \psi)
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Outline

Syntax

- The logic psSMTL is the extension psMTL logic by the non-commutative counterpart of the pseudo-complementation axiom:

$$(\text{ps}\Pi 2) \quad \varphi \wedge \neg\varphi \to \overline{0}$$
$$(\text{ps}\Pi 2^{\bullet}) \quad \varphi \wedge \sim \varphi \rightsquigarrow \overline{0}$$

- The logic psWMTL is the extension of psMTL logic with the non-commutative counterpart of the weak contraction axiom:

$$(\text{WCon}) \quad (\varphi \to \neg\varphi) \to \neg\varphi$$
$$(\text{WCon}^{\bullet}) \quad (\varphi \rightsquigarrow \sim \varphi) \rightsquigarrow \sim \varphi.$$

Theorem

The logics psSMTL and psWMTL are equivalent.

- psSMTL^{r} logic, psSMTL$\forall$ logic and psSMTL$^{r}\forall$ logic

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$$(WCon) \quad (\varphi \to \neg\varphi) \to \neg\varphi$$
$$(WCon^{\bullet}) \quad (\varphi \rightsquigarrow \sim\varphi) \rightsquigarrow \sim\varphi.$$

Theorem

The logics psSMTL and psWMTL are equivalent.

- psSMTL^{r} logic, psSMTL$\forall$ logic and psSMTL$^{r}\forall$ logic

Syntax

- The logic psSMTL is the extension psMTL logic by the non-commutative counterpart of the pseudo-complementation axiom:

$$\begin{array}{ll} (\mathrm{ps}\Pi 2) & \varphi \wedge \neg\varphi \to \bar{0} \\ (\mathrm{ps}\Pi 2^{\bullet}) & \varphi \wedge \sim \varphi \rightsquigarrow \bar{0} \end{array}$$

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Theorem

The logics psSMTL and psWMTL are equivalent.

- $\mathrm{psSMTL}^{\mathrm{r}}$ logic, $\mathrm{psSMTL}\forall$ logic and $\mathrm{psSMTL}^{\mathrm{r}}\forall$ logic

Algebraic semantics

Definition

A psMTL-algebra $\mathcal{A}$ is called strict (or psSMTL-algebra, for short) if it satisfies:

$$\text{(S) } (x \odot y)^- = x^- \vee y^- \text{ and } (x \odot y)^\sim = x^\sim \vee y^\sim .$$

Theorem

Let $\mathcal{A}$ be a psMTL-chain. The following are equivalent:

(1) *$\mathcal{A}$ is a psSMTL-algebra.*

(2) *$\mathcal{A}$ satisfies the condition: $x \odot y = 0$ iff $x = 0$ or $y = 0$.*

(3) *The negations of $\mathcal{A}$ are Gödel negations, i.e.*

$$x^- = x^\sim = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases} .$$

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Algebraic semantics

- psSMTL^r-algebra
- strict pseudo-t-norm, i.e. whose corresponding negations are Gödel negations

  Let $0 < a_1 < a_2 < b_2 < 1$ and $T_{1,2} : [0,1] \times [0,1] \to [0,1]$ be

  $$T_{1,2}(x,y) = \begin{cases} a_1, & \text{if } a_1 < x \leq a_2 \text{ and } a_1 < y \leq b_2 \\ \min(x,y), & \text{otherwise} \end{cases}$$

- standard psSMTL-algebra

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Completeness results

- strong completeness for psSMTL
- strong chain completeness for psSMTL^{r}
- strong completeness for $\mathrm{psSMTL}\forall$
- strong chain completeness for $\mathrm{psSMTL}^{r}\forall$

Theorem (Standard completeness for psSMTL^{r})

The logic $psSMTL^{r}$ is complete with respect to standard psSMTL-algebras.

Completeness results

- strong completeness for psSMTL
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Theorem (Standard completeness for $\mathrm{psSMTL}^{\mathrm{r}}$)

The logic $psSMTL^{r}$ is complete with respect to standard psSMTL-algebras.

Outline

Syntax

- The logic psIMTL is the extension of psMTL logic by the non-commutative counterpart of the double negation axiom:

$$\begin{array}{ll} (\text{psInv}) & \sim \neg\varphi \rightarrow \varphi \\ (\text{psInv}^\bullet) & \neg \sim \varphi \rightsquigarrow \varphi. \end{array}$$

Theorem

The non-commutative Łukasiewicz logic is the extension of psIMTL logic by the non-commutative counterpart of the divisibility axiom:

$$\begin{array}{ll} (\text{psDiv}) & (\varphi \wedge \psi) \rightarrow ((\varphi \rightarrow \psi)\&\varphi) \\ (\text{psDiv}^\bullet) & (\varphi \wedge \psi) \rightsquigarrow (\varphi\&(\varphi \rightsquigarrow \psi)) \end{array}$$

- psIMTL^r logic, $\text{psIMTL}\forall$ logic and $\text{psIMTL}^r\forall$ logic
- $\vdash_{\text{psIMTL}\forall} (\exists x)\varphi \leftrightarrow \neg(\forall x) \sim \varphi$
- The axioms ($\exists$1), ($\exists$2) and ($\forall$3) are redundant for $\text{psIMTL}^r\forall$ logic.

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Algebraic semantics

Definition

A psIMTL-algebra $\mathcal{A}$ is a psMTL-algebra satisfying the condition:

$$\text{(pDN)}\ (x^{-})^{\sim} = (x^{\sim})^{-} = x.$$

Example

Let $(G, \vee, \wedge, +, -, 0)$ be a linearly ordered l-group and let $u \in G$, $u \leq 0$.
Define the non-commutative generalization of Fodor's t-norm and Fodor's implication:

$$x \odot^{L} y = \begin{cases} u, & \text{if } x + y \leq u \\ x \wedge y, & \text{if } x + y > u \end{cases},$$

$$x \to y = \begin{cases} 0, & \text{if } x \leq y \\ (u - x) \vee y, & \text{if } x > y \end{cases}, \quad x \rightsquigarrow y = \begin{cases} 0, & \text{if } x \leq y \\ (-x + u) \vee y, & \text{if } x > y \end{cases},$$

The structure $([u, 0], \vee, \wedge, \odot^{L}, \to, \rightsquigarrow, u, 0)$ is a psIMTL-algebra.

- standard psIMTL-algebra

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Completeness results

- strong completeness for psIMTL
- strong chain completeness for $\mathrm{psIMTL}^{\mathrm{r}}$
- strong completeness for $\mathrm{psIMTL}\forall$
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Theorem (Standard completeness for $\mathrm{psIMTL}^{\mathrm{r}}$)

The logic psIMTL^r is complete with respect to standard psIMTL-algebras.

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Theorem (Standard completeness for $\mathrm{psIMTL}^{\mathrm{r}}$)

The logic $psIMTL^r$ is complete with respect to standard psIMTL-algebras.

Outline

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Propositional case

- A propositional pseudo-Kripke frame is a structure of the form

$$\mathcal{M} = (M, \leq, \odot, 0, 1)$$

1) $(M, \leq, 0, 1)$ such that $\leq$ is a linear order on M
2) $(M, \odot, 1)$ is a monoid
3) $\odot$ is non-decreasing in both arguments
4) $x \odot (\bigvee_{i\in I} y_i) = \bigvee_{i\in I}(x \odot y_i)$ and $(\bigvee_{i\in I} y_i) \odot x = \bigvee_{i\in I}(y_i \odot x)$.

- A propositional pseudo-Kripke frame is called residuated if there exist

$$y \to z \stackrel{not}{=} \max\{x \mid x \odot y \leq z\} \text{ and } x \rightsquigarrow z \stackrel{not}{=} \max\{y \mid x \odot y \leq z\}.$$

- A propositional pseudo-Kripke frame is called complete if $\leq$ is a complete order.

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Propositional case

- A forcing relation on a propositional pseudo-Kripke frame $\mathcal{M}$ is a binary relation $\Vdash \subseteq \mathcal{M} \times \mathrm{Var}$ such that
  - (a) if $a \Vdash p$ and $b \leq a$, then $b \Vdash p$
  - (b) $0 \Vdash p$
- A forcing relation $\Vdash$ on a propositional pseudo-Kripke frame $\mathcal{M}$ can be uniquely extended to a relation $\Vdash \subseteq \mathcal{M} \times \mathrm{Form}_{\mathrm{psMTL}}$ by the following:
  - (1) $a \Vdash \overline{0}$ iff $a = 0$
  - (2) $a \Vdash \varphi \wedge \psi$ iff $a \Vdash \varphi$ and $a \Vdash \psi$
  - (3) $a \Vdash \varphi \vee \psi$ iff either $a \Vdash \varphi$ or $a \Vdash \psi$
  - (4) $a \Vdash \varphi \& \psi$ iff there are b, c such that $b \Vdash \varphi$, $c \Vdash \psi$ and $a \leq b \odot c$
  - (5) $a \Vdash \varphi \to \psi$ iff for all b, if $b \Vdash \varphi$, then $a \odot b \Vdash \psi$
  - (6) $a \Vdash \varphi \rightsquigarrow \psi$ iff for all b, if $b \Vdash \varphi$, then $b \odot a \Vdash \psi$

  If $a \Vdash \varphi$, we say that a forces φ.

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  If $a \Vdash \varphi$, we say that a forces φ.

Propositional case

- A forcing relation $\Vdash$ on a propositional pseudo-Kripke frame $\mathcal{M}$ is called r-forcing relation if the set $\{x \in M \mid x \Vdash p\}$ has a maximum.
- A propositional pseudo-Kripke model is a pair $(\mathcal{M}, \Vdash)$, where $\mathcal{M}$ is a propositional pseudo-Kripke frame and $\Vdash$ is a forcing relation on $\mathcal{M}$.
- A propositional pseudo-Kripke model is called residuated if $\mathcal{M}$ is residuated and $\Vdash$ is an r-forcing relation on $\mathcal{M}$.
- A propositional pseudo-Kripke model is called complete if $\mathcal{M}$ is complete and $\Vdash$ is an r-forcing relation on $\mathcal{M}$.
- We say that a formula φ of psMTL logic is valid in a propositional pseudo-Kripke model $(\mathcal{M}, \Vdash)$ if $1 \Vdash \varphi$.
- We have the same definitions for $\mathrm{psMTL}^{\mathrm{r}}$ logic.

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Predicate case

- A predicate pseudo-Kripke frame is a pair $(\mathcal{M}, \mathcal{U})$, where $\mathcal{M}$ is a complete propositional pseudo-Kripke frame and $\mathcal{U} = (U, (U_P)_{P \in \mathrm{Pred}}, (u_c)_{c \in \mathrm{Cont}})$ is an $\mathcal{M}$-structure for J.
- A forcing relation on a predicate pseudo-Kripke frame $(\mathcal{M}, \mathcal{U})$ is an r-forcing relation $\Vdash$ between $\mathcal{M}$ and the closed atomic formulas of $\mathrm{psMTL}\forall$ logic, defined as above.
- A forcing relation $\Vdash$ on a predicate pseudo-Kripke frame $(\mathcal{M}, \mathcal{U})$ can be uniquely extended to a relation between $\mathcal{M}$ and the formulas $\mathrm{Form}_{\mathrm{psMTL}\forall}$ of $\mathrm{psMTL}\forall$ logic by means of the above clauses and by the following clauses for quantifiers:
  - (7) $a \Vdash (\forall x)\varphi(x)$ iff for all $u \in U$, $a \Vdash \varphi(u)$,
  - (8) $a \Vdash (\exists x)\varphi(x)$ iff for all $b < a$, there are $c > b$ and $u \in U$ such that $c \Vdash \varphi(u)$.
- A predicate pseudo-Kripke model is a triple $(\mathcal{M}, \mathcal{U}, \Vdash)$, where $(\mathcal{M}, \mathcal{U})$ is a predicate pseudo-Kripke frame and $\Vdash$ is a forcing relation on $(\mathcal{M}, \mathcal{U})$.
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- A forcing relation $\Vdash$ on a predicate pseudo-Kripke frame $(\mathcal{M},\mathcal{U})$ can be uniquely extended to a relation between $\mathcal{M}$ and the formulas $\mathrm{Form}_{\mathrm{psMTL}\forall}$ of $\mathrm{psMTL}\forall$ logic by means of the above clauses and by the following clauses for quantifiers:
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  - (8) $a \Vdash (\exists x)\varphi(x)$ iff for all $b < a$, there are $c > b$ and $u \in U$ such that $c \Vdash \varphi(u)$.
- A predicate pseudo-Kripke model is a triple $(\mathcal{M},\mathcal{U},\Vdash)$, where $(\mathcal{M},\mathcal{U})$ is a predicate pseudo-Kripke frame and $\Vdash$ is a forcing relation on $(\mathcal{M},\mathcal{U})$.
- We have the same definitions for $\mathrm{psMTL}^{\mathrm{r}}\forall$ logic.

Predicate case

- A predicate pseudo-Kripke frame is a pair $(\mathcal{M},\mathcal{U})$, where $\mathcal{M}$ is a complete propositional pseudo-Kripke frame and $\mathcal{U} = (U, (U_P)_{P\in\text{Pred}}, (u_c)_{c\in\text{Cont}})$ is an $\mathcal{M}$-structure for J.

- A forcing relation on a predicate pseudo-Kripke frame $(\mathcal{M},\mathcal{U})$ is an r-forcing relation $\Vdash$ between $\mathcal{M}$ and the closed atomic formulas of psMTL∀ logic, defined as above.

- A forcing relation $\Vdash$ on a predicate pseudo-Kripke frame $(\mathcal{M},\mathcal{U})$ can be uniquely extended to a relation between $\mathcal{M}$ and the formulas $\text{Form}_{\text{psMTL}\forall}$ of psMTL∀ logic by means of the above clauses and by the following clauses for quantifiers:
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- We have the same definitions for $\mathrm{psMTL^r}\forall$ logic.

Kripke and standard completeness

- The logic psMTL^r is complete with respect to propositional pseudo-Kripke models.
- The logic $\mathrm{psMTL}^r\forall$ is complete with respect to predicate pseudo-Kripke models.

Theorem (Standard completeness for $\mathrm{psMTL}^r\forall$)

Let φ be a closed formula of psMTL$^r\forall$ logic. The following are equivalent:

(1) $\vdash_{psMTL^r\forall} \varphi$;

(2) φ *is valid in every predicate pseudo-Kripke model of the form*

$$(([0,1], \leq, \hat{*}, 0, 1), \mathcal{U}, \Vdash),$$

where $\hat{*}$ is a left-continuous pseudo-t-norm, $\mathcal{U}$ is any structure on the standard psMTL-algebra induced by $\hat{*}$ and $\Vdash$ is any forcing relation.

(3) φ *is a tautology with respect to any standard psMTL-algebra.*

Kripke and standard completeness

- The logic $\mathrm{psMTL^r}$ is complete with respect to propositional pseudo-Kripke models.
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(3) φ *is a tautology with respect to any standard psMTL-algebra.*

Outline

- We introduce the following property of a propositional pseudo-Kripke frame $\mathcal{M} = (M, \leq, \odot, 0, 1)$:

(nn) for all $x > 0$, $x \odot x > 0$

Theorem

(psΠ2) and (psΠ2•) are valid in every propositional pseudo-Kripke model $(\mathcal{M}, \Vdash)$ iff $\mathcal{M}$ satisfies condition (nn).

- A propositional psSMTL-frame is just a propositional pseudo-Kripke frame that satisfies (nn).
- A propositional psSMTL-model is a pair $(\mathcal{M}, \Vdash)$, where $\mathcal{M}$ is a propositional psSMTL-frame and $\Vdash$ is a forcing relation on $\mathcal{M}$.
- A predicate psSMTL-frame is just a predicate pseudo-Kripke frame that satisfies (nn).
- A predicate psSMTL-model is a triple $(\mathcal{M}, \mathcal{U}, \Vdash)$, where $(\mathcal{M}, \mathcal{U})$ is a predicate psSMTL-frame and $\Vdash$ is a forcing relation on $(\mathcal{M}, \mathcal{U})$.
- We have the same definitions for psSMTL^r logic $\mathrm{psSMTL}^r\forall$ logic.

- We introduce the following property of a propositional pseudo-Kripke frame $\mathcal{M} = (M, \leq, \odot, 0, 1)$:

$$\text{(nn) for all } x > 0,\ x \odot x > 0$$

Theorem

$(ps\Pi 2)$ and $(ps\Pi 2^{\bullet})$ are valid in every propositional pseudo-Kripke model $(\mathcal{M}, \Vdash)$ iff $\mathcal{M}$ satisfies condition (nn).

- A propositional psSMTL-frame is just a propositional pseudo-Kripke frame that satisfies (nn).
- A propositional psSMTL-model is a pair $(\mathcal{M}, \Vdash)$, where $\mathcal{M}$ is a propositional psSMTL-frame and $\Vdash$ is a forcing relation on $\mathcal{M}$.
- A predicate psSMTL-frame is just a predicate pseudo-Kripke frame that satisfies (nn).
- A predicate psSMTL-model is a triple $(\mathcal{M}, \mathcal{U}, \Vdash)$, where $(\mathcal{M}, \mathcal{U})$ is a predicate psSMTL-frame and $\Vdash$ is a forcing relation on $(\mathcal{M}, \mathcal{U})$.
- We have the same definitions for psSMTL^{r} logic $\mathrm{psSMTL}^{r}\forall$ logic.

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- We have the same definitions for psSMTL^r logic $\mathrm{psSMTL}^r\forall$ logic.

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(psΠ2) and (psΠ2•) are valid in every propositional pseudo-Kripke model $(\mathcal{M}, \Vdash)$ iff $\mathcal{M}$ satisfies condition (nn).

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- A propositional psSMTL-model is a pair $(\mathcal{M}, \Vdash)$, where $\mathcal{M}$ is a propositional psSMTL-frame and $\Vdash$ is a forcing relation on $\mathcal{M}$.
- A predicate psSMTL-frame is just a predicate pseudo-Kripke frame that satisfies (nn).
- A predicate psSMTL-model is a triple $(\mathcal{M}, \mathcal{U}, \Vdash)$, where $(\mathcal{M}, \mathcal{U})$ is a predicate psSMTL-frame and $\Vdash$ is a forcing relation on $(\mathcal{M}, \mathcal{U})$.
- We have the same definitions for psSMTL^r logic $\mathrm{psSMTL}^r\forall$ logic.

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- A predicate psSMTL-frame is just a predicate pseudo-Kripke frame that satisfies (nn).
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- We have the same definitions for psSMTL^{r} logic $\mathrm{psSMTL}^{r}\forall$ logic.

- The logic psSMTL^r is complete with respect to propositional psSMTL-models.
- The logic $\mathrm{psSMTL}^r\forall$ is complete with respect to predicate psSMTL-models.

Theorem (Standard completeness for $\mathrm{psSMTL}^r\forall$)

Let φ be a closed formula of psSMTL$^r\forall$ logic. The following are equivalent:

(1) $\vdash_{psSMTL^r\forall} \varphi$;

(2) φ *is valid in every predicate psSMTL-model of the form*

$$(([0,1], \leq, \hat{*}, 0, 1), \mathcal{U}, \Vdash),$$

where $\hat{*}$ is a left-continuous strict pseudo-t-norm, $\mathcal{U}$ is any structure on the standard psSMTL-algebra induced by $\hat{*}$ and $\Vdash$ is any forcing relation.

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Outline

- We introduce the following property of a propositional pseudo-Kripke frame $\mathcal{M} = (M, \leq, \odot, 0, 1)$:

  (inv1) for all $x, y \in M$, if $x < y$, then there is $z \in M$ such that $z \odot x = 0$ and $z \odot y \neq 0$

  (inv2) for all $x, y \in M$, if $x < y$, then there is $z \in M$ such that $x \odot z = 0$ and $y \odot z \neq 0$

Theorem

(psInv) and (psInv$^\bullet$) are valid in every residuated propositional pseudo-Kripke model $(\mathcal{M}, \Vdash)$ iff $\mathcal{M}$ satisfies conditions (inv1) and (inv2).

- A propositional psIMTL-frame is just a propositional pseudo-Kripke frame that satisfies (inv1) and (inv2).
- A propositional psIMTL-model is a pair $(\mathcal{M}, \Vdash)$, where $\mathcal{M}$ is a propositional psIMTL-frame and $\Vdash$ is a forcing relation on $\mathcal{M}$.
- A predicate psIMTL-frame is just a predicate pseudo-Kripke frame that satisfies (inv1) and (inv2).
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($psInv$) and ($psInv^{\bullet}$) are valid in every residuated propositional pseudo-Kripke model $(\mathcal{M}, \Vdash)$ iff $\mathcal{M}$ satisfies conditions (inv1) and (inv2).

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Let φ be a closed formula of psIMTL$^r\forall$ logic. The following are equivalent:

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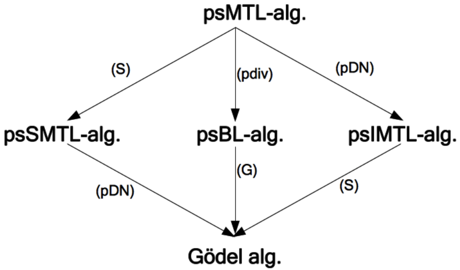
Further research



psSMTL logic + $(\mathrm{psInv}) + (\mathrm{psInv}^{\bullet}) \stackrel{?}{=}$ Gödel logic

psIMTL logic + $(\mathrm{ps\Pi2}) + (\mathrm{ps\Pi2}^{\bullet}) \stackrel{?}{=}$ Gödel logic

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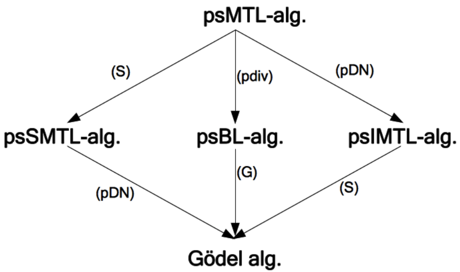
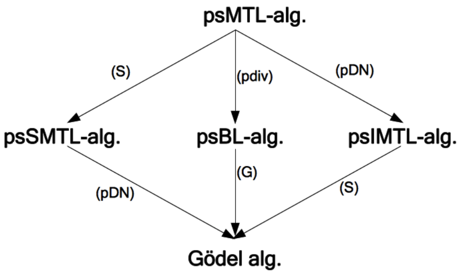
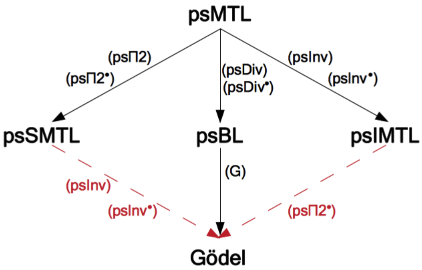


psSMTL logic + (psInv) + (psInv$^\bullet$) $\stackrel{?}{=}$ Gödel logic
psIMTL logic + (psΠ2) + (psΠ2$^\bullet$) $\stackrel{?}{=}$ Gödel logic

Further research





psSMTL logic + $(\text{psInv}) + (\text{psInv}^{\bullet}) \stackrel{?}{=}$ Gödel logic

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Thank you for your attention!