Proof theory for many valued logics – some applications

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The beginning of the story

(Cost Action "Many-valued Logics for CS Applications")
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This talk

Many-valued logics have semantic origins. But as logics they also have something to do with proofs.
This talk

Many-valued logics have semantic origins. But as logics they also have something to do with proofs.

My aim is to

- introduce some recent developments in proof theory for non-classical logics (especially many-valued logics)
- use proof theory for uniform (and automated) proofs of standard completeness
Analytic Calculi

"Praedicatum Inest Subjecto"

Calculi in which proof search proceeds by step-wise decomposition of the formulas to be proved
Analytic Calculi

"Praedicatum Inest Subjecto"

Calculi in which proof search proceeds by step-wise decomposition of the formulas to be proved

Sequent, hypersequent calculi, labelled calculi, many-placed sequents, sequents-of-relations, display logic, CoS...
Introducing such calculi

- Semantic-based approach
  (Baaz, Fermüller, Montagna, ...)
- Syntactic approach
  (Avron, Baaz, Baldi, Galatos, Terui, Metcalfe, Spendier, ...)

Proof theory for many valued logics – some applications – p.5/33
Introducing such calculi

- Semantic-based approach
  - E.g. decidability, complexity of validity ..
    (Baaz, Fermüller, Montagna, ...)
- Syntactic approach
  
  (Avron, Baaz, Baldi, Galatos, Terui, Metcalfe, Spendier, ...)

Proof theory for many valued logics – some applications – p.5/33
Introducing such calculi

- Semantic-based approach

  E.g. decidability, complexity of validity ..
  (Baaz, Fermüller, Montagna, ...)

- Syntactic approach: uniform and systematic

  E.g.
  
  - Herbrand theorem
  - order theoretic completions
  - standard completeness
  - ...

  (Avron, Baaz, Baldi, Galatos, Terui, Metcalfe, Spendier, ...)
Semantic-based Approach
Semantic-based Approach

Finite-valued logics: (Baaz, Zach...)

\[ S_1 \mid \ldots \mid S_n \quad \text{(Ex. } A \mid B \mid C') \]

MULTLOG
Semantic-based Approach

- Finite-valued logics: (Baaz, Zach...)

\[ S_1 \mid \ldots \mid S_n \quad (\text{Ex. } A \mid B \mid C') \]

- Projective logics: (Baaz and Fermüller)

\[ \Box^M(x_1, \ldots, x_n) = \begin{cases} 
  t_1 & \text{if } \land \lor R_{jk} \\
  \vdots & \vdots \\
  t_m & \text{if } \land \lor R_{pq}
\end{cases} \]

\[ R_{i_1}(F^1_1, \ldots, F^1_{r_1}) \mid \ldots \mid R_{i_k}(F^k_1, \ldots, F^k_{r_k}) \quad (\text{Ex. } A \leq B \mid A < C') \]
Semantic-based Approach

- Finite-valued logics: (Baaz, Zach...)
  \[ S_1 \mid \ldots \mid S_n \quad (\text{Ex. } A \mid B \mid C') \]

- Projective logics: (Baaz and Fermüller)
  \[ \square^M(x_1, \ldots, x_n) = \begin{cases} 
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- Semi-Projective logics: (- and Montagna, new)
  \textbf{Ex.}: Nilpotent Minimum logic, \(n\)-contractive BL...
Syntactic Approach

From Hilbert calculi to analytic calculi.

Hilbert calculi consist of:

- many axioms
- few rules (MP, generalization,...)

Pro: easy to define logics
Contra: not suitable for

- finding proofs
- analyzing proofs
- establishing properties of logics
Sequent Calculi

Sequents

\[ A_1, \ldots, A_n \Rightarrow B \]

Intuitively a sequent is understood as “the conjunction of \( A_1, \ldots, A_n \) implies \( B \).
Sequent Calculi

Sequents

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Axioms

E.g., \( A \Rightarrow A \)

Rules

- Logical
- (cut)

\[
\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \text{ Cut}
\]

Structural
The system FLe

\[ \text{FLe} = \text{commutative Lambek calculus (} = \text{ intuitionistic Linear Logic or Monoidal Logic) } \]
The system FLε

\[
\begin{align*}
A, B, \Gamma \Rightarrow \Pi & \quad \rightarrow l \\
A \otimes B, \Gamma \Rightarrow \Pi & \quad \otimes l \\
\Gamma \Rightarrow A, \Delta \Rightarrow B & \quad \otimes r \\
\Gamma, \Delta \Rightarrow A \otimes B & \\
\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi & \quad \rightarrow l \\
\Gamma, A \rightarrow B, \Delta \Rightarrow \Pi & \quad \rightarrow r \\
\Gamma \Rightarrow A \rightarrow B & \\
\Gamma, A \Rightarrow B & \quad \rightarrow l \\
\end{align*}
\]
A (bounded pointed) commutative residuated lattice is

\[ P = \langle P, \& , \lor , \otimes , \to , \top , 0 , 1 , \bot \rangle \]

1. \( \langle P, \& , \lor , \top , 0 \rangle \) is a lattice with \( \top \) greatest and \( \bot \) least.

2. \( \langle P, \otimes , 1 \rangle \) is a commutative monoid.

3. For any \( x, y, z \in P \), \( x \otimes y \leq z \iff y \leq x \to z \)

4. \( 0 \in P \).

Many-valued logics = \( FL_e + \) axioms
Commutative Residuated Lattices

A (bounded pointed) commutative residuated lattice is

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Commutative Residuated Lattices

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Many-valued logics = \( \mathbf{FL} e \) + axioms

• Cut elimination is not preserved when axioms are added
• (Idea) Transform axioms into ‘good’ structural rules
On the structural rules

Example

- Contraction: \( \alpha \rightarrow \alpha \otimes \alpha \)
- Weakening l: \( \alpha \rightarrow 1 \)
- Weakening r: \( 0 \rightarrow \alpha \)
On the structural rules

Example

- **Contraction**: $\alpha \rightarrow \alpha \otimes \alpha$
  \[
  \frac{A, A, \Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi} \quad (c)
  \]

- **Weakening l**: $\alpha \rightarrow 1$
  \[
  \frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi} \quad (w, l)
  \]

- **Weakening r**: $0 \rightarrow \alpha$
  \[
  \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Pi} \quad (w, r)
  \]
On the structural rules

Example

- **Contraction:** $\alpha \rightarrow \alpha \otimes \alpha$

- **Weakening l:** $\alpha \rightarrow 1$

- **Weakening r:** $0 \rightarrow \alpha$

Equivalence between rules and axioms

\[ \vdash FLe+(axiom) = \vdash FLe+(rule) \]
Our preliminary results
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The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas defined by:

$\mathcal{P}_0, \mathcal{N}_0 := \text{Atomic formulas}$

$\mathcal{P}_{n+1} := \mathcal{N}_n \mid \mathcal{P}_{n+1} \otimes \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \lor \mathcal{P}_{n+1} \mid 1 \mid \bot$

$\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \land \mathcal{N}_{n+1} \mid 0 \mid \top$

$\mathcal{P}$ and $\mathcal{N}$

- Positive connectives $1, \bot, \otimes, \lor$ have invertible left rules:

- Negative connectives $\top, 0, \land, \rightarrow$ have invertible right rules:
Our preliminary results

The sets \( \mathcal{P}_n, \mathcal{N}_n \) of formulas defined by:

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\]

\[
\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \land \mathcal{N}_{n+1} \mid 0 \mid \top
\]

\( \mathcal{P} \) and \( \mathcal{N} \)

- **Positive connectives** \( 1, \bot, \otimes, \lor \) have *invertible* left rules:

- **Negative connectives** \( \top, 0, \land, \rightarrow \) have *invertible* right rules:
# Examples

<table>
<thead>
<tr>
<th>Class</th>
<th>Axiom</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N}_2$</td>
<td>$\alpha \rightarrow 1$, $\bot \rightarrow \alpha$</td>
<td>weakening</td>
</tr>
<tr>
<td></td>
<td>$\alpha \rightarrow \alpha \otimes \alpha$</td>
<td>contraction</td>
</tr>
<tr>
<td></td>
<td>$\alpha \otimes \alpha \rightarrow \alpha$</td>
<td>expansion</td>
</tr>
<tr>
<td></td>
<td>$\otimes \alpha^n \rightarrow \otimes \alpha^m$</td>
<td>knotted axioms $(n, m \geq 0)$</td>
</tr>
<tr>
<td></td>
<td>$\neg (\alpha &amp; \neg \alpha)$</td>
<td>weak contraction</td>
</tr>
<tr>
<td>$\mathcal{P}_2$</td>
<td>$\alpha \lor \neg \alpha$</td>
<td>excluded middle</td>
</tr>
<tr>
<td></td>
<td>$(\alpha \rightarrow \beta) \lor (\beta \rightarrow \alpha)$</td>
<td>prelinearity</td>
</tr>
<tr>
<td>$\mathcal{P}_3$</td>
<td>$\neg \alpha \lor \neg \neg \alpha$</td>
<td>weak excluded middle (wnm)</td>
</tr>
<tr>
<td></td>
<td>$\neg (\alpha \otimes \beta) \lor (\alpha \land \beta \rightarrow \alpha \otimes \beta)$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{N}_3$</td>
<td>$((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha)$</td>
<td>Łukasiewicz axiom</td>
</tr>
</tbody>
</table>
Our preliminary results

Algorithm to transform:

- axioms up to the class $\mathcal{N}_2$ into "good" structural rules in sequent calculus
- axioms up to the class $\mathcal{P}_3$ into "good" structural rules in hypersequent calculus

( -, N. Galatos and K. Terui. LICS 2008) and
( - , L. Strassburger and K. Terui. CSL 2009)
Our preliminary results

Algorithm to transform:

- axioms up to the class $\mathcal{N}_2$ into "good" structural rules in sequent calculus
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(-, N. Galatos and K. Terui. LICS 2008) and
( -, L. Strassburger and K. Terui. CSL 2009)

Prolog program: AxiomCalc

http://www.logic.at/people/lara/axiomcalc.html
Hypersequent calculus

It is obtained embedding sequents into hypersequents

\[ \Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n \]

where for all \( i = 1, \ldots n \), \( \Gamma_i \Rightarrow \Pi_i \) is an ordinary sequent.
Hypersequent calculus

\[ \frac{\Gamma \Rightarrow A}{\Gamma, \Delta \Rightarrow \Pi} \quad \frac{A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \quad \text{Cut} \quad \frac{A \Rightarrow A}{\text{Identity}} \]

\[ \frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{\Gamma, A \rightarrow B, \Delta \Rightarrow \Pi} \quad \rightarrow \quad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \quad \rightarrow \quad \frac{A \Rightarrow A \rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \]
Hypersequent calculus

\[
\frac{G | \Gamma \Rightarrow A \quad G | A, \Delta \Rightarrow \Pi}{G | \Gamma, \Delta \Rightarrow \Pi} \quad \text{Cut} \quad \frac{G | A \Rightarrow A}{Identity} \quad \frac{G | \Gamma \Rightarrow A \quad G | B, \Delta \Rightarrow \Pi}{G | \Gamma, A \rightarrow B, \Delta \Rightarrow \Pi} \quad \rightarrow l \quad \frac{G | A, \Gamma \Rightarrow B}{G | \Gamma \Rightarrow A \rightarrow B} \quad \rightarrow r
\]
Hypersequent calculus

\[
\begin{align*}
& G|\Gamma \Rightarrow A \quad G|A, \Delta \Rightarrow \Pi \\
& \quad \quad \quad \frac{\text{Cut}}{G|\Gamma, \Delta \Rightarrow \Pi} \\
& G|\Gamma \Rightarrow A \\
& \quad \quad \quad \frac{\text{Identity}}{G|A \Rightarrow A} \\
& G|\Gamma \Rightarrow A \quad G|B, \Delta \Rightarrow \Pi \\
& \quad \quad \quad \frac{\rightarrow l}{G|\Gamma, A \rightarrow B, \Delta \Rightarrow \Pi} \\
& G|A, \Gamma \Rightarrow B \\
& \quad \quad \quad \frac{\rightarrow r}{G|\Gamma \Rightarrow A \rightarrow B} \\
& \end{align*}
\]

and adding suitable rules to manipulate the additional layer of structure.

\[
\begin{align*}
& G \\
& \quad \quad \quad \frac{(ew)}{G|\Gamma \Rightarrow A} \\
& G|\Gamma \Rightarrow A \\
& \quad \quad \quad \frac{(ec)}{G|\Gamma \Rightarrow A | \Gamma \Rightarrow A} \\
& \end{align*}
\]
From axioms to analytic rules

- Step 1
- Step 2
From axioms to analytic rules

- **Step 1**
  Transformation of any $N_2 (P_3)$ axiom into an equivalent (set of) structural rule(s).

- **Step 2**
  Analytic *completion* of the generated rules
From axioms to analytic rules

1. **Step 1**
   Transformation of any $N_2 (P_3)$ axiom into an equivalent (set of) structural rule(s).

2. **Step 2**
   Analytic *completion* of the generated rules

**How? Using**

- the invertibility of the rules $(\lor, l), (\land, r), (\otimes, l), (\to, r)$.
- the Lemma: Any axiom $A \Rightarrow B$ is equivalent to

$$\begin{align*}
\alpha \Rightarrow A \\
\alpha \Rightarrow B \\
\frac{B \Rightarrow \beta}{A \Rightarrow \beta}
\end{align*}$$

for $\alpha, \beta$ fresh variables.
An example

\[(\alpha \rightarrow \beta) \lor (\beta \rightarrow \alpha)\]
An example

$$(\alpha \to \beta) \lor (\beta \to \alpha)$$

is equivalent to

$$G \models \alpha \to \beta \models \beta \to \alpha$$
An example

\[(\alpha \rightarrow \beta) \lor (\beta \rightarrow \alpha)\]

and to

\[G \mid \alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha\]
An example

\[(\alpha \rightarrow \beta) \lor (\beta \rightarrow \alpha)\]

\[G \mid \alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha\]

and by the Lemma: Any sequent \(\alpha' \Rightarrow \beta'\) is equivalent to

\[\frac{\Gamma \Rightarrow \alpha'}{\Gamma \Rightarrow \beta'}\]

and also to

\[\frac{\beta', \Gamma \Rightarrow \Delta}{\alpha', \Gamma \Rightarrow \Delta}\]

(for \(\Gamma, \Delta\) fresh meta-variables)
An example

\[(\alpha \rightarrow \beta) \lor (\beta \rightarrow \alpha)\]

\[G | \alpha \Rightarrow \beta | \beta \Rightarrow \alpha\]

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and also to

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(for \(\Gamma, \Delta\) fresh meta-variables) is equivalent to

\[G | \Gamma \Rightarrow \alpha\]

\[G | \Gamma \Rightarrow \beta | \beta \Rightarrow \alpha\]
An example

\[(\alpha \rightarrow \beta) \lor (\beta \rightarrow \alpha)\]

\[G \mid \alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha\]

is equivalent to

\[G \mid \Gamma \Rightarrow \alpha \quad G \mid \Gamma' \Rightarrow \beta \quad G \mid \Sigma, \beta \Rightarrow \Delta \quad G \mid \Sigma', \alpha \Rightarrow \Delta'\]

\[G \mid \Gamma, \Sigma \Rightarrow \Delta \mid \Gamma', \Sigma' \Rightarrow \Delta'\]
An example

\[(\alpha \to \beta) \lor (\beta \to \alpha)\]

\[\begin{align*}
G \mid \alpha & \Rightarrow \beta \mid \beta \Rightarrow \alpha \\
G \mid \Gamma \Rightarrow \alpha & \quad G \mid \Gamma' \Rightarrow \beta & \quad G \mid \Sigma, \beta \Rightarrow \Delta & \quad G \mid \Sigma', \alpha \Rightarrow \Delta'
\end{align*}\]

is equivalent to

\[G \mid \Gamma, \Sigma \Rightarrow \Delta \mid \Gamma', \Sigma' \Rightarrow \Delta'\]

Some examples

\[ \otimes \alpha^n \rightarrow \otimes \alpha^m \]

\[ \{\Delta_{i_1}, \ldots, \Delta_{i_m}, \Gamma \Rightarrow \Pi\}_{i_1,\ldots,i_m \in \{1,\ldots,n\}} \]

\[ \Delta_1, \ldots, \Delta_n, \Gamma \Rightarrow \Pi \]

\[ (\text{knot}_m^n) \]

\[ \neg \alpha \lor \neg \neg \alpha \]

\[ \frac{G \mid \Gamma_1, \Gamma_2}{G \mid \Gamma_1 \Rightarrow \Gamma_2 \Rightarrow} \]

\[ (\text{lq}) \]

\[ \neg (\alpha \otimes \beta) \lor (\alpha \land \beta \rightarrow \alpha \otimes \beta) \]

\[ \begin{align*}
G \mid \Gamma_2, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \\
G \mid \Gamma_1, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \\
G \mid \Gamma_1, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \\
G \mid \Gamma_2, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \\
\end{align*} \]

\[ G \mid \Gamma_2, \Gamma_3 \Rightarrow \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \]

\[ (\text{wnm}) \]
Uniform cut-elimination

**Theorem**

The cut rule is admissible in (the hypersequent version of) $\mathsf{FL}_e$ extended with any completed rule.
Uniform cut-elimination

**Theorem**
The cut rule is admissible in (the hypersequent version of) FLc extended with any completed rule.

- **Syntactic argument:**
  
  elimination procedure

  Cut-ful Proofs $\implies$ Cut-free Proofs

- **Semantic argument:**
  
  Quasi-DM completion

  CRL $\iff$ ‘Intransitive’ CRL
Expressive powers of (hyper)sequents

Hilbert axioms = equations over CRL
Expressive powers of (hyper)sequents

Sequent *structural* rules: only equations

- that hold in Heyting algebras (IL)
- closed under DM completion

(-, N. Galatos and K. Terui. APAL 2012)
Expressive powers of (hyper)sequents

Sequent *structural* rules: only equations
- that hold in Heyting algebras (IL)
- closed under DM completion

Hypersequent *structural* rules: only equations
- closed under regular completions

(−, N. Galatos and K. Terui. Draft 2012)
An application

- A logic is **standard complete** when it is complete w.r.t. evaluations on \([0, 1]\).

- Algebraically:
  A logic is **standard complete** ⇔ it is complete w.r.t. algebras order-isomorphic to \([0, 1]\).
SC: algebraic approach

Given a logic $\mathcal{L}$:

1. Identify the algebraic semantics of $\mathcal{L}$ ($\mathcal{L}$-algebras)
2. Show completeness of $\mathcal{L}$ w.r.t. linear, countable $\mathcal{L}$-algebras
3. (Rational completeness): Find an embedding of countable $\mathcal{L}$-algebras into dense countable $\mathcal{L}$-algebras
4. Dedekind-Mac Neille style completion (embedding into $\mathcal{L}$-algebras with lattice reduct $[0, 1]$)
SC: algebraic approach

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- Step 1, 2, 4: routine
- Step 3: problematic (only ad hoc solutions)
(Metcalfe, Montagna JSL 2007) Given a logic $\mathcal{L}$:

- Define a suitable hypersequent calculus
- Add the density rule

$\frac{G \mid \Gamma \Rightarrow p \mid \Sigma, p \Rightarrow \Delta}{G \mid \Gamma, \Sigma \Rightarrow \Delta} \quad (density)$

($= \mathcal{L} + (density)$ is rational complete)

- Show that the addition of density produces no new theorems
- Dedekind-Mac Neille style completion
SC: proof-theoretic approach

(Metcalfe, Montagna JSL 2007) Given a logic \( \mathcal{L} \):

- Define a suitable hypersequent calculus
- Add the density rule

\[
\frac{G \mid \Gamma \Rightarrow p \mid \Sigma, p \Rightarrow \Delta}{G \mid \Gamma, \Sigma \Rightarrow \Delta} \quad (density)
\]

\((= \mathcal{L} + (density) \text{ is rational complete})\)

- Show that the addition of density produces no new theorems
- Dedekind-Mac Neille style completion

\[
\frac{A \Rightarrow p \mid p \Rightarrow B}{A \Rightarrow B} \quad (density)
\]
Density elimination

- Similar to cut-elimination
- Proof by induction on the length of derivations
Density elimination

- Similar to cut-elimination
- Proof by induction on the length of derivations

(-, Metcalfe TCS 2008)

Given a density-free derivation, ending in

\[ G \mid \Sigma, p \Rightarrow \Delta \mid \Gamma \Rightarrow p \]

\[ \frac{G \mid \Gamma, \Sigma \Rightarrow \Delta}{(D)} \]

- Asymmetric substitution: \( p \) is replaced
  - With \( \Sigma \Rightarrow \Delta \) when occurring on the right
  - With \( \Gamma \) when occurring on the left
Method for density elimination

\[
G \mid \Sigma, p \Rightarrow \Delta \mid \Gamma \Rightarrow p \\
\frac{G \mid \Gamma, \Sigma \Rightarrow \Delta}{(D)}
\]

becomes:

\[
G \mid \Gamma, \Sigma \Rightarrow \Delta \mid \Gamma, \Sigma \Rightarrow \Delta \\
\frac{G \mid \Gamma, \Sigma \Rightarrow \Delta}{(EC)}
\]
A problem

Asymmetric substitutions do not preserve derivability, as:

- A sequent $\Pi, p \Rightarrow p$ is derivable from $p \Rightarrow p +$ (weakening)
- $\Pi, \Gamma, \Sigma \Rightarrow \Delta$ is not derivable
A problem

Asymmetric substitutions do not preserve derivability, as:

- A sequent $\Pi, p \Rightarrow p$ is derivable from $p \Rightarrow p +$ (weakening)
- $\Pi, \Gamma, \Sigma \Rightarrow \Delta$ is not derivable

We can solve the problem with a suitable restructuring of the derivation...
Our Results

(Baldi, - , Spendier: Wollic 2012)

Theorem: Hypersequent calculus for $MTL +$ convergent rules admits density elimination
Our Results

(Baldi, - ,Spendier: Wollic 2012)

- **Theorem:** Hypersequent calculus for $MTL +$ convergent rules admits density elimination

  i.e. rules whose premises do not mix "too much" the conclusion

- **Example:**

  \[
  \begin{align*}
  G \mid \Gamma_2, \Gamma_1, \Delta_1 &\Rightarrow \Pi_1 & G \mid \Gamma_1, \Gamma_3, \Delta_1 &\Rightarrow \Pi_1 \\
  G \mid \Gamma_1, \Gamma_1, \Delta_1 &\Rightarrow \Pi_1 & G \mid \Gamma_2, \Gamma_3, \Delta_1 &\Rightarrow \Pi_1 \\
  \hline
  G \mid \Gamma_2, \Gamma_3 &\Rightarrow \Gamma_1, \Delta_1 &\Rightarrow \Pi_1
  \end{align*}
  \]

  \[\text{(wnm)}\]

  **Axiom:** $\neg(\alpha \otimes \beta) \lor (\alpha \land \beta \rightarrow \alpha \otimes \beta)$
Our Results

(Baldi, - , Spendier: Wollic 2012)

- Theorem: Hypersequent calculus for $MTL +$ convergent rules admits density elimination

(Baldi, - , Spendier: In Preparation)

- Theorem: Hypersequent calculus for $UL +$ sequent rules admits density elimination.
Let $\mathcal{L}$ be a suitable axiomatic extension of MTL (UL)
Let $\mathcal{L}$ be a suitable axiomatic extension of $\text{MTL (UL)}$

- define a hypersequent calculus for $\mathcal{L}$
SC: Automated Proofs

Let $\mathcal{L}$ be a *suitable* axiomatic extension of $\text{MTL}$ ($\text{UL}$)

- define a hypersequent calculus for $\mathcal{L}$
- check whether the calculus satisfies the condition for density elimination (*rational completeness*)
Let $\mathcal{L}$ be a suitable axiomatic extension of $\text{MTL (UL)}$

- define a hypersequent calculus for $\mathcal{L}$
- check whether the calculus satisfies the condition for density elimination (rational completeness)
- standard completeness follows by (-, Galatos, Terui Algebra Universalis 2011)
Let $\mathcal{L}$ be a suitable axiomatic extension of $\text{MTL (UL)}$

- define a hypersequent calculus for $\mathcal{L}$
- check whether the calculus satisfies the condition for density elimination ($\text{rational completeness}$)
- standard completeness follows by (-, Galatos, Terui Algebra Universalis 2011)

http://www.logic.at/people/lara/axiomcalc.html

**AxiomCalc Web Interface**

*Use AxiomCalc*

Axiom:

$(a \rightarrow b) \lor (b \rightarrow a)$

☑ Check for Standard Completeness  Submit
Example

Known Logics

\[ MTL + \neg(\alpha \otimes \beta) \lor ((\alpha \land \beta) \rightarrow (\alpha \otimes \beta)) \]

\[ MTL + \neg\alpha \lor \neg\neg\alpha \]

\[ MTL + \alpha^{n-1} \rightarrow \alpha^n \]

\[ UL + \alpha^{n-1} \rightarrow \alpha^n \]

\[ \ldots \]

New Fuzzy Logics
Example

Known Logics

\[ MTL + \neg (\alpha \otimes \beta) \lor ((\alpha \land \beta) \rightarrow (\alpha \otimes \beta)) \]

\[ MTL + \neg \alpha \lor \neg \neg \alpha \]

\[ MTL + \alpha^{n-1} \rightarrow \alpha^n \]

\[ UL + \alpha^{n-1} \rightarrow \alpha^n \]

... 

New Fuzzy Logics

\[ MTL + \neg (\alpha \otimes \beta)^n \lor ((\alpha \land \beta)^{n-1} \rightarrow (\alpha \otimes \beta)^n), \text{ for all } n > 1 \]

\[ UL + \neg \alpha \lor \neg \neg \alpha \]

\[ UL + \alpha^m \rightarrow \alpha^n \]

...
Open problems

- **Uniform** treatment of axioms beyond $P_3$
- **Systematic** introduction of analytic calculi

  - first-order logic
  - modal and temporal logic
  - logics with different connectives

- More on **standard completeness**

- Proving useful **properties** for classes

  - logics in a uniform and systematic way

  - ... many more ...

"Non-classical Proofs: Theory, Applications and Tools", research project 2012-2017
How far can we go?
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Analytic Calculi

- sequent, hypersequent calculi...
- display calculi
- nested sequents, deep inference, calculus of structures
- labelled systems
- ......
How far can we go?

Analytic Calculi
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- ......

(Strongly) Analytic calculus = weaker for of Herbrand theorem: If $\exists x B(x)$ is provable, where $B$ is quantifier-free, so is $\bigvee_{i=1}^{n} B(t_i)$, for some $n$. 

A negative result

Let $\mathcal{L}$ be a first-order logic satisfying:

1. $\vdash_{\mathcal{L}} A \rightarrow A$

2. $\vdash_{\mathcal{L}} \forall x A(x) \rightarrow B \rightarrow \exists x (A(x) \rightarrow B)$

3. $\vdash_{\mathcal{L}} (B \rightarrow \forall x A(x)) \rightarrow \forall x (B \rightarrow A(x))$

4. $\vdash_{\mathcal{L}} \forall x A(x) \rightarrow A(t)$, for any term $t$

5. there is an atomic formula $A$ in $\mathcal{L}$ such that for no $n$

$$\vdash_{\mathcal{L}} \bigvee_{i=1}^{n} A(x_i) \rightarrow A(x_{i+1})$$

then $\mathcal{L}$ does not admit any strongly analytic calculus.

(Baaz, -, Work in progress)
Corollary

The following logics do not admit any strongly analytic calculus:

- **witnessed logics** \((\forall = \text{min} \text{ and } \exists = \text{max})\), e.g. Gödel logic with truth values in \([0, 1]\), \(\wedge = \text{min}, \lor = \text{max}\),
  \[v(A) \rightarrow v(B) = 1 \text{ iff } v(A) \leq v(B), v(B) \text{ otherwise.} \forall = \text{min} \text{ and } \exists = \text{max}.\]

- (fragments of) **first-order Lukasiewicz logic**

- **Gödel logic** with set of truth values
  \[\{1 - 1/n : n \geq 1\} \cup \{1\}\]

- **first-order nilpotent minimum logic NM** with set of truth values
  \[\{1/n : n \geq 1\} \cup \{1 - 1/n : n \geq 1\}\]

- ....