Proof theory for many valued logics – some applications

Agata Ciabattoni

Vienna University of Technology

agata@logic.at

The beginning of the story

(Cost Action "Many-valued Logics for CS Applications")

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This talk

Many-valued logics have semantic origins. But as *logics* they also have something to do with proofs.

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My aim is to

- introduce some recent developments in proof theory for non-classical logics (especially many-valued logics)
- use proof theory for uniform (and automated) proofs of *standard completeness*

Analytic Calculi

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Sequent, hypersequent calculi, labelled calculi, many-placed sequents, sequents-of-relations, display logic, CoS ...

Introducing such calculi

- Semantic-based approach

(Baaz, Fermüller, Montagna, ...)

- Syntactic approach

(Avron, Baaz, Baldi, Galatos, Terui, Metcalfe, Spendier, ...)

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E.g. decidability, complexity of validity ..
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Introducing such calculi

- Semantic-based approach

E.g. decidability, complexity of validity ..
(Baaz, Fermüller, Montagna, ...)

- Syntactic approach : uniform and systematic

E.g.

- Herbrand theorem
- order theoretic completions
- *standard completeness*
- ...

(Avron, Baaz, Baldi, Galatos, Terui, Metcalfe, Spendier, ...)

Semantic-based Approach

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- Finite-valued logics: (Baaz, Zach...)

$$S_1 \mid \ldots \mid S_n \qquad (\text{Ex. } A \mid B \mid C)$$

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- Projective logics: (Baaz and Fermüller)

$$\Box^M(x_1, ..., x_n) = \begin{cases} t_1 & \text{if } \wedge \vee R_{j_k} \\ \vdots & \quad \vdots \\ t_m & \text{if } \wedge \vee R_{p_q} \end{cases}$$

$$R_{i_1}(F_1^1, \ldots, F_{r_1}^1) \mid \ldots \mid R_{i_k}(F_1^k, \ldots, F_{r_k}^k) \qquad (\text{Ex. } A \leq B \mid A < C)$$

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- Semi-Projective logics: (- and Montagna, new)
  Ex.: Nilpotent Minimum logic, n-contractive BL...

Syntactic Approach

From Hilbert calculi to analytic calculi.

Hilbert calculi consist of:

- many axioms
- few rules (MP, generalization,...)

Pro : easy to define logics
Contra : not suitable for

- finding proofs
- analyzing proofs
- establishing properties of logics

Sequent Calculi

Sequents

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Axioms

E.g., $A \Rightarrow A$

Rules

- Logical
- (cut)

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ Cut$$

- Structural

The system FLe

FLe = commutative Lambek calculus (= intuitionistic Linear Logic or Monoidal Logic)

The system FLe

$$\frac{A, B, \Gamma \Rightarrow \Pi}{A \otimes B, \Gamma \Rightarrow \Pi} \otimes l \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B} \otimes r$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{\Gamma, A \rightarrow B, \Delta \Rightarrow \Pi} \rightarrow l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \rightarrow r$$

$$\frac{A, \Gamma \Rightarrow \Pi \quad B, \Gamma \Rightarrow \Pi}{A \vee B, \Gamma \Rightarrow \Pi} \vee l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \vee r \qquad \frac{}{\mathbf{0} \Rightarrow} \mathbf{0} l$$

$$\frac{A_i, \Gamma \Rightarrow \Pi}{A_1 \,\&\, A_2, \Gamma \Rightarrow \Pi} \& l \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \,\&\, B} \& r \qquad \frac{}{\Gamma \Rightarrow \top} \top r$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathbf{0}} \mathbf{0} r \qquad \frac{}{\Rightarrow \mathbf{1}} \mathbf{1} r \qquad \frac{}{\bot, \Gamma \Rightarrow \Pi} \bot l \qquad \frac{\Gamma \Rightarrow \Pi}{\mathbf{1}, \Gamma \Rightarrow \Pi} \mathbf{1} l$$

Commutative Residuated Lattices

A (bounded pointed) commutative residuated lattice is

$$\mathbf{P} = \langle P, \&, \vee, \otimes, \rightarrow, \top, \mathbf{0}, \mathbf{1}, \bot \rangle$$

1. $\langle P, \&, \vee, \top, \mathbf{0} \rangle$ is a lattice with $\top$ greatest and $\bot$ least
2. $\langle P, \otimes, \mathbf{1} \rangle$ is a commutative monoid.
3. For any $x, y, z \in P$, $x \otimes y \leq z \Longleftrightarrow y \leq x \rightarrow z$
4. $\mathbf{0} \in P$.

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Many-valued logics= $\mathbf{FLe}$ + axioms

- Cut elimination is not preserved when axioms are added
- (Idea) Transform axioms into 'good' structural rules

On the structural rules

Example

- Contraction: $\alpha \to \alpha \otimes \alpha$
- Weakening l: $\alpha \to 1$
- Weakening r: $0 \to \alpha$

On the structural rules

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- Weakening l: $\alpha \to 1$ $$\frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (w, l)$$
- Weakening r: $0 \to \alpha$ $$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (w, r)$$

On the structural rules

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Equivalence between rules and axioms

$$\vdash_{FLe+(axiom)} \quad = \quad \vdash_{FLe+(rule)}$$

Our preliminary results

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$\mathcal{P}_0, \mathcal{N}_0 :=$ Atomic formulas

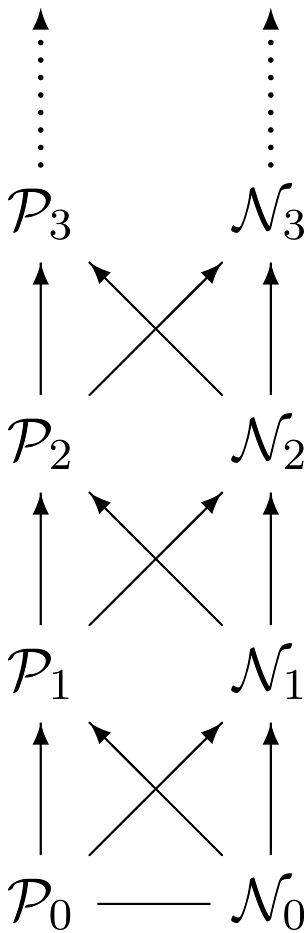
$$\mathcal{P}_{n+1} := \mathcal{N}_n \mid \mathcal{P}_{n+1} \otimes \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1 \mid \bot$$

$$\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0 \mid \top$$

$\mathcal{P}$ and $\mathcal{N}$

- Positive connectives $\mathbf{1}, \bot, \otimes, \vee$ have *invertible left rules*:
- Negative connectives $\top, \mathbf{0}, \wedge, \rightarrow$ have *invertible right rules*:

Our preliminary results



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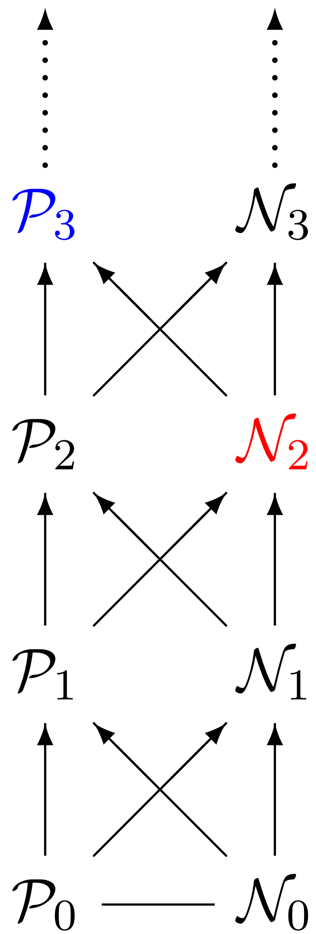
Examples

Class	Axiom	Name
$\mathcal{N}_2$	$\alpha \to \mathbf{1}, \perp \to \alpha$	weakening
	$\alpha \to \alpha \otimes \alpha$	contraction
	$\alpha \otimes \alpha \to \alpha$	expansion
	$\otimes \alpha^n \to \otimes \alpha^m$	knotted axioms ($n, m \geq 0$)
	$\neg(\alpha \,\&\, \neg\alpha)$	weak contraction
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	excluded middle
	$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\mathcal{P}_3$	$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
	$\neg(\alpha \otimes \beta) \vee (\alpha \wedge \beta \to \alpha \otimes \beta)$	(wnm)
$\mathcal{N}_3$	$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$	Łukasiewicz axiom

Our preliminary results



Algorithm to transform:

- axioms up to the class $\mathcal{N}_2$ into "good" structural rules in sequent calculus
- axioms up to the class $\mathcal{P}_3$ into "good" structural rules in hypersequent calculus

(-, N. Galatos and K. Terui. LICS 2008) and
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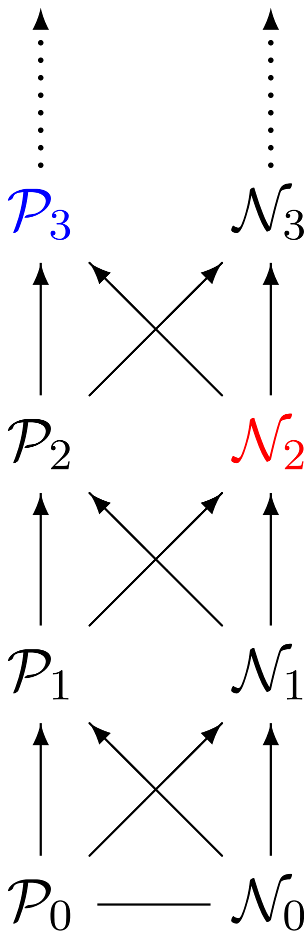

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Prolog program: *AxiomCalc*

http://www.logic.at/people/lara/axiomcalc.html

Hypersequent calculus

It is obtained embedding sequents into hypersequents

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \Rightarrow \Pi_i$ is an ordinary sequent.

Hypersequent calculus

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ Cut \qquad \frac{}{A \Rightarrow A}\ Identity$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{\Gamma, A \to B, \Delta \Rightarrow \Pi} \to l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B} \to r$$

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and adding suitable rules to manipulate the additional layer of structure.

$$\frac{G}{G \mid \Gamma \Rightarrow A}\ (\mathrm{ew}) \qquad \frac{G \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A}\ (\mathrm{ec})$$

From axioms to analytic rules

- Step 1

- Step 2

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  Transformation of any $\mathcal{N}_2$ ($\mathcal{P}_3$) axiom into an equivalent (set of) structural rule(s).
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  Analytic *completion* of the generated rules

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  Analytic *completion* of the generated rules

How? Using

- the invertibility of the rules $(\vee, l), (\&, r), (\otimes, l), (\rightarrow, r)$.
- the Lemma: Any axiom $A \Rightarrow B$ is equivalent to

$$\frac{\alpha \Rightarrow A}{\alpha \Rightarrow B} \quad \text{and also to} \quad \frac{B \Rightarrow \beta}{A \Rightarrow \beta}$$

for α, β fresh variables.

An example

$$(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$$

An example

$$(\alpha \to \beta) \vee (\beta \to \alpha)$$

is equivalent to

$$\frac{}{G \mid\ \Rightarrow \alpha \to \beta \mid\ \Rightarrow \beta \to \alpha}$$

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and by the Lemma: Any sequent $\alpha' \Rightarrow \beta'$ is equivalent to

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(for Γ, Δ fresh meta-variables)

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is equivalent to

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Gamma' \Rightarrow \beta \quad G \mid \Sigma, \beta \Rightarrow \Delta \quad G \mid \Sigma', \alpha \Rightarrow \Delta'}{G \mid \Gamma, \Sigma \Rightarrow \Delta \mid \Gamma', \Sigma' \Rightarrow \Delta'}$$

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is equivalent to

$$\frac{G \mid \Gamma, \Sigma' \Rightarrow \Delta' \quad G \mid \Gamma', \Sigma \Rightarrow \Delta}{G \mid \Gamma, \Sigma \Rightarrow \Delta \mid \Gamma', \Sigma' \Rightarrow \Delta'} \; (com)$$

(Avron, Annals of Math and art. Intell. 1991)

Some examples

- $\otimes\alpha^n \rightarrow \otimes\alpha^m$

$$\frac{\{\Delta_{i_1}, \ldots, \Delta_{i_m}, \Gamma \Rightarrow \Pi\}_{i_1,\ldots,i_m \in \{1,\ldots,n\}}}{\Delta_1, \ldots, \Delta_n, \Gamma \Rightarrow \Pi} \; (knot^n_m)$$

- $\neg\alpha \vee \neg\neg\alpha$

$$\frac{G \mid \Gamma_1, \Gamma_2}{G \mid \Gamma_1 \Rightarrow \;\mid \Gamma_2 \Rightarrow} \; \textit{(lq)}$$

- $\neg(\alpha \otimes \beta) \vee (\alpha \wedge \beta \rightarrow \alpha \otimes \beta)$

$$\frac{\begin{array}{cc} G \mid \Gamma_2, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \mid \Gamma_1, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \\ G \mid \Gamma_1, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \mid \Gamma_2, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \end{array}}{G \mid \Gamma_2, \Gamma_3 \Rightarrow \;\mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1} \; \textit{(wnm)}$$

Uniform cut-elimination

Theorem
The cut rule is admissible in (the hypersequent version of) **FLe** extended with any completed rule.

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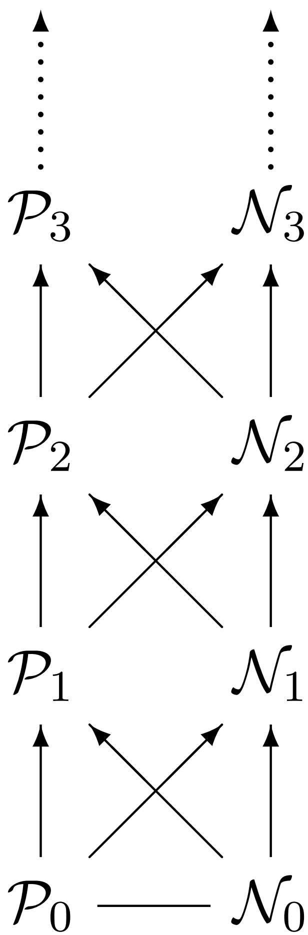
- Syntactic argument:

$$\text{Cut-ful Proofs} \quad \overset{\text{elimination procedure}}{\Longrightarrow} \quad \text{Cut-free Proofs}$$

- Semantic argument:

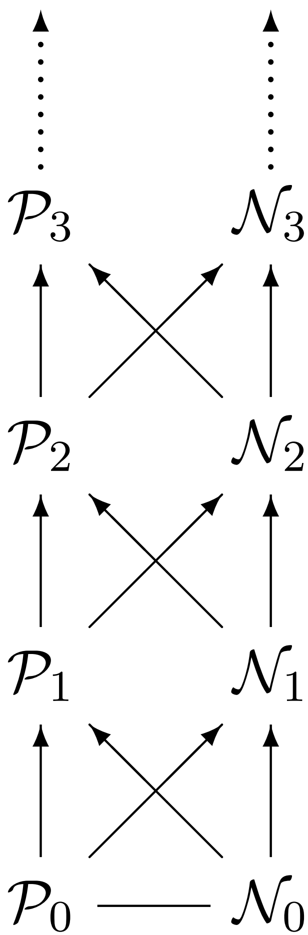
$$\text{CRL} \quad \overset{\text{Quasi-DM completion}}{\Longleftarrow} \quad \text{'Intransitive' CRL}$$

Expressive powers of (hyper)sequents



Hilbert axioms = equations over CRL

Expressive powers of (hyper)sequents

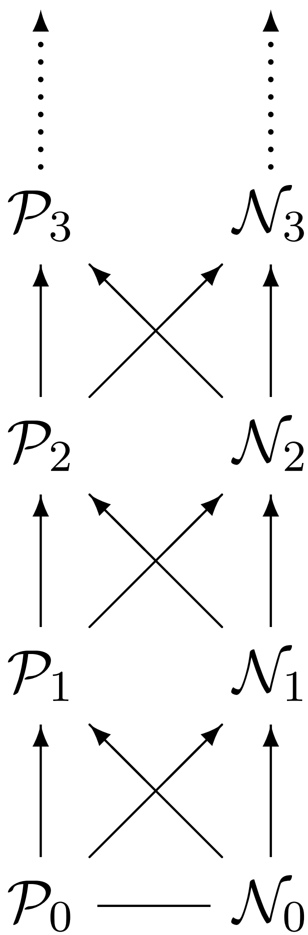


Sequent structural rules: only equations

- that hold in Heyting algebras (IL)
- closed under DM completion

(-, N. Galatos and K. Terui. APAL 2012)

Expressive powers of (hyper)sequents



Sequent structural rules: only equations

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Hypersequent structural rules: only equations

- closed under regular completions

(-, N. Galatos and K. Terui. Draft 2012)

An application

- A logic is standard complete when it is complete w.r.t. evaluations on $[0, 1]$
- Algebraically:

A logic is standard complete $\Leftrightarrow$ it is complete w.r.t algebras order-isomorphic to $[0, 1]$

SC: algebraic approach

Given a logic $\mathcal{L}$:

1. Identify the algebraic semantics of $\mathcal{L}$ ($\mathcal{L}$-algebras)
2. Show completeness of $\mathcal{L}$ w.r.t. linear, countable $\mathcal{L}$-algebras
3. (Rational completeness): Find an embedding of countable $\mathcal{L}$-algebras into dense countable $\mathcal{L}$-algebras
4. Dedekind-Mac Neille style completion (embedding into $\mathcal{L}$-algebras with lattice reduct $[0, 1]$)

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- Step $1, 2, 4$: routine
- Step 3: problematic (only ad hoc solutions)

SC: proof-theoretic approach

(Metcalfe, Montagna JSL 2007) Given a logic $\mathcal{L}$:

- Define a suitable hypersequent calculus
- Add the density rule

$$\frac{G \,|\, \Gamma \Rightarrow p \,|\, \Sigma, p \Rightarrow \Delta}{G \,|\, \Gamma, \Sigma \Rightarrow \Delta} \; (density)$$

  (= $\mathcal{L}$ + $(density)$ is rational complete)
- Show that the addition of density produces no new theorems
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- Show that the addition of density produces no new theorems
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$$\frac{A \Rightarrow p \mid p \Rightarrow B}{A \Rightarrow B} \ (density)$$

Density elimination

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- Proof by induction on the length of derivations

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(-, Metcalfe TCS 2008)

Given a density-free derivation, ending in

$$\dfrac{\begin{matrix}\vdots \\ G \mid \Sigma, p \Rightarrow \Delta \mid \Gamma \Rightarrow p\end{matrix}}{G \mid \Gamma, \Sigma \Rightarrow \Delta}\ (\mathrm{D})$$

- Asymmetric substitution: p is replaced
  - With $\Sigma \Rightarrow \Delta$ when occuring on the right
  - With Γ when occuring on the left

Method for density elimination

- $$\dfrac{\begin{array}{c}\vdots\\ G \mid \Sigma, p \Rightarrow \Delta \mid \Gamma \Rightarrow p\end{array}}{G \mid \Gamma, \Sigma \Rightarrow \Delta}\ (\mathrm{D})$$

  becomes:

- $$\dfrac{\begin{array}{c}\vdots\\ G \mid \Gamma, \Sigma \Rightarrow \Delta \mid \Gamma, \Sigma \Rightarrow \Delta\end{array}}{G \mid \Gamma, \Sigma \Rightarrow \Delta}\ (\mathrm{EC})$$

A problem

Asymmmetric substitutions do not preserve derivability, as :

- A sequent $\Pi, p \Rightarrow p$ is derivable from $p \Rightarrow p$ + *(weakening)*
- $\Pi, \Gamma, \Sigma \Rightarrow \Delta$ is not derivable

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- A sequent $\Pi, p \Rightarrow p$ is derivable from $p \Rightarrow p$ + *(weakening)*
- $\Pi, \Gamma, \Sigma \Rightarrow \Delta$ is not derivable

- We can solve the problem with a suitable restructuring of the derivation...

Our Results

(Baldi, - ,Spendier: Wollic 2012)

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i.e. rules whose premises do not mix "too much" the conclusion

- Example :

$$\frac{\begin{array}{cc} G \mid \Gamma_2, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \mid \Gamma_1, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \\ G \mid \Gamma_1, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \mid \Gamma_2, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \end{array}}{G \mid \Gamma_2, \Gamma_3 \Rightarrow \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1} \text{ (wnm)}$$

Axiom: $\neg(\alpha \otimes \beta) \vee (\alpha \wedge \beta \to \alpha \otimes \beta)$

Our Results

(Baldi, - ,Spendier: Wollic 2012)

- Theorem: Hypersequent calculus for MTL + **convergent** rules admits density elimination

(Baldi, - ,Spendier: In Preparation)

- Theorem: Hypersequent calculus for UL + sequent rules admits density elimination.

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- check whether the calculus satisfies the condition for density elimination (rational completeness)

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SC: Automated Proofs

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http://www.logic.at/people/lara/axiomcalc.html

Example

Known Logics

- MTL + $\neg(\alpha \otimes \beta) \vee ((\alpha \wedge \beta) \rightarrow (\alpha \otimes \beta))$
- MTL + $\neg\alpha \vee \neg\neg\alpha$
- MTL + $\alpha^{n-1} \rightarrow \alpha^{n}$
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New Fuzzy Logics

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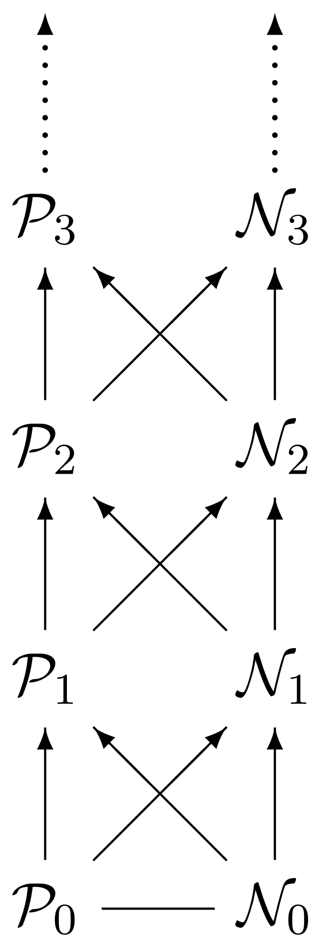
New Fuzzy Logics

- MTL + $\neg(\alpha \otimes \beta)^n \vee ((\alpha \wedge \beta)^{n-1} \rightarrow (\alpha \otimes \beta)^n)$, for all $n > 1$
- UL + $\neg\alpha \vee \neg\neg\alpha$
- UL + $\alpha^m \rightarrow \alpha^n$
- ...

Open problems



- Uniform treatment of axioms behond $\mathcal{P}_3$
- Systematic introduction of analytic calculi
  - first-order logic
  - modal and temporal logic
  - logics with different connectives
- more on standard completeness
- Proving useful properties for classes logics in a uniform and systematic way
- ... many more ...

"Non-classical Proofs: Theory, Applications and Tools", research project 2012-2017

How far can we go?

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Analytic Calculi

- sequent, hypersequent calculi...
- display calculi
- nested sequents, deep inference, calculus of structures
- labelled systems
- ......

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(Strongly) Analytic calculus = weaker for of Herbrand theorem: If $\exists x B(x)$ is provable, where B is quantifier-free, so is $\bigvee_{i=1}^{n} B(t_i)$, for some n.

A negative result

Let $\mathcal{L}$ be a first-order logic satisfying:

1. $\vdash_{\mathcal{L}} A \rightarrow A$
2. $\vdash_{\mathcal{L}} \forall x A(x) \rightarrow B \rightarrow \exists x(A(x) \rightarrow B)$
3. $\vdash_{\mathcal{L}} (B \rightarrow \forall x A(x)) \rightarrow \forall x(B \rightarrow A(x))$
4. $\vdash_{\mathcal{L}} \forall x A(x) \rightarrow A(t)$, for any term t
5. there is an atomic formula A in $\mathcal{L}$ such that for no n

$$\vdash_{\mathcal{L}} \bigvee_{i=1}^{n} A(x_i) \rightarrow A(x_{i+1})$$

then $\mathcal{L}$ does not admit any strongly analytic calculus.

(Baaz, -, Work in progress)

Corollary

The following logics do not admit any strongly analytic calculus:

- witnessed logics ($\forall$= min and $\exists$= max), e.g. Gödel logic with truth values in $[0,1]$, $\wedge$= min, $\vee$= max, $v(A) \to v(B) = 1$ iff $v(A) \leq v(B), v(B)$ otherwise. $\forall$= min and $\exists$= max.
- (fragments of) first-order Łukasiewicz logic
- Gödel logic with set of truth values $\{1 - 1/n : n \geq 1\} \cup \{1\}$
- first-order nilpotent minimum logic NM with set of truth values $\{1/n : n \geq 1\} \cup \{1 - 1/n : n \geq 1\}$
- ....