On some logical and algebraic properties of the monoidal t-norm based logic connected with single chain completeness

Matteo Bianchi

Department of Mathematics “Federigo Enriques”

Università degli Studi di Milano

matteo.bianchi@unimi.it

MTL Logic

The formulas of MTL are constructed by starting from the set of connectives $\{\&, \wedge, \rightarrow, \bot\}$, as follows

$$\varphi \& \psi, \quad \varphi \wedge \psi, \quad \varphi \rightarrow \psi, \quad \bot$$

Derived connectives:

$$\begin{array}{rcl}
\neg\varphi & := & \varphi \rightarrow \bot \\
\varphi \vee \psi & := & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\
\varphi \leftrightarrow \psi & := & (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi) \\
\top & := & \neg\bot
\end{array}$$

MTL is axiomatized as follows

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$(\varphi \& \psi) \to \varphi$
(A3)	$(\varphi \& \psi) \to (\psi \& \varphi)$
(A4)	$(\varphi \wedge \psi) \to \varphi$
(A5)	$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(A6)	$(\varphi \& (\varphi \to \psi)) \to (\psi \wedge \varphi)$
(A7a)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A7b)	$((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$\bot \to \varphi$

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MTL can be equivalently axiomatized as FL_{ew} plus $(\varphi \to \psi) \vee (\psi \to \varphi)$.

MTL-algebras

An MTL-algebra is an FL_{ew}-algebra satisfying prelinearity; that is, an algebraic structure of the form $\langle A, \sqcap, \sqcup, *, \Rightarrow, 0, 1 \rangle$ such that

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- $\langle *, \Rightarrow\rangle$ form a residuated pair, that is

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(prelinearity) $$(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1.$$

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Single-chain completeness properties

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Clearly the SSCC implies the SCC, but the vice-versa is an open problem.

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Proposition

An MTL-algebra is well-connected if and only if it is a chain.

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Corollary

For every axiomatic extension of MTL, the HC is equivalent to the SCC.

Deductive Maksimova variable separation property

Definition

A substructural logic L has the *deductive Maksimova's variable separation property* (DMVP), if for all sets of formulas $\Gamma \cup \{\varphi\}$ and $\Sigma \cup \{\psi\}$ that have no variables in common, $\Gamma, \Sigma \vdash_L \varphi \vee \psi$ implies $\Gamma \vdash_L \varphi$ or $\Sigma \vdash_L \psi$.

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Problem

Are there some examples of extensions of MTL enjoying the HC but not the DMVP ?

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Does the DMVP imply the CJEP ?

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- A logic L has the *deductive pseudo-relevance property* (DPRP), if for every theory Γ and formula ψ with no variables in common, $\Gamma \vdash_L \psi$ implies either $\Gamma \vdash_L \bot$ or $\vdash_L \psi$.

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- A logic L has the *strong deductive pseudo-relevance property* (SDPRP), if for every sets of formulas Γ and $\Sigma \cup \{\psi\}$ with no variables in common, $\Gamma, \Sigma \vdash_L \psi$ implies either $\Gamma \vdash_L \bot$ or $\Sigma \vdash_L \psi$.

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- *SDPRP implies DPRP for every L, and the converse holds also when the variety of L-algebras has the CEP (i.e. every pair of L-algebras $\mathcal{A}, \mathcal{B}$, with $\mathcal{A}$ being a subalgebra of $\mathcal{B}$, is such that for every congruence θ of $\mathcal{A}$ there is a congruence θ' of $\mathcal{B}$ such that $\theta = \theta' \cap \mathcal{A}^2$).*

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Lemma ([Nog06, page 42])

Every variety of MTL-algebras enjoys the CEP.

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Theorem

For every variety of MTL-algebras the SDPRP is equivalent to the DPRP.

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For every axiomatic extension of MTL, the PRP implies the DPRP.

The relation between PRP and DPRP

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Note that the DPRP however does not imply the PRP, in general: a counterexample is given by NM, that enjoys the CJEP (and hence the DPRP), whilst the PRP fails to hold.

Amalgamation and deductive interpolation

Definition

We say that a variety K of MTL-algebras has the *amalgamation property* (AP) if for every tuple $\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, i, j \rangle$, where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$ and $\mathcal{A} \overset{i}{\hookrightarrow} \mathcal{B}$, $\mathcal{A} \overset{j}{\hookrightarrow} \mathcal{C}$, there is a tuple $\langle \mathcal{D}, h, k \rangle$, with $\mathcal{D} \in K$, $\mathcal{B} \overset{h}{\hookrightarrow} \mathcal{D}$, $\mathcal{C} \overset{k}{\hookrightarrow} \mathcal{D}$, such that $h \circ i = k \circ j$.

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A logic L has the *deductive interpolation property* (DIP) if for any theory Γ and for any formula ψ of L, if $\Gamma \vdash_L \psi$, then there is a formula γ such that $\Gamma \vdash_L \gamma$, $\gamma \vdash_L \psi$ and every propositional variable occurring in γ occurs both in Γ and in ψ.

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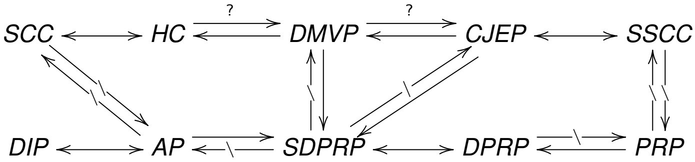
Theorem ([GJKO07])

An axiomatic extension of MTL enjoys the DIP iff the corresponding variety has the AP.

A general picture



Open problems

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Finally, another open question is the SCC for MTL: the results of this work can be useful to this aim.

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APPENDIX

Subdirectly irreducible single chain completeness. . .

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Corollary

The following n-contractive extensions of MTL enjoy the subSCC: WNM, NM, G, $Ł_n$, $SMTL^n$, SBL^n

In [MNH06, proposition 37] it is shown that every locally finite subvariety of MTL-algebras is n-contractive, for some n, then

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Theorem

The following non n-contractive extensions of MTL enjoy the subSCC: SMTL, BL, SBL, Ł, Π.

For the axiomatic extensions of MTL it holds the following form of deduction theorem (local deduction theorem)

Theorem ([Cin04])

Let L be an axiomatic extension of MTL and Γ, φ, ψ be a theory and two formulas. It holds that

$$\Gamma \cup \{\psi\} \vdash_L \varphi \quad \text{iff there exists } n \in \mathbb{N}^+ \text{ s.t.} \quad \Gamma \vdash_L \psi^n \to \varphi.$$

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Theorem ([HNP07, theorem 3.3])

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Let L be an axiomatic extension of MTL. We say that the *Craig interpolation theorem* holds for L iff for any two formulas φ and ψ of L, if $\vdash_L \varphi \to \psi$, then there is a formula γ such that $\vdash_L \varphi \to \gamma$, $\vdash_L \gamma \to \psi$ and every propositional variable occurring in γ occurs both in φ and in ψ.

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- Nevertheless, for the *n*-contractive axiomatic extensions of MTL that enjoys the DIP, we can obtain a weaker form of Craig's theorem (a generalization of the theorem given in [BM11] for some families of *n*-contractive extensions of BL).

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Theorem (Weak Craig interpolation theorem)

Let L be an n-contractive extension of MTL that enjoys the DIP. For every pair of formulas φ, ψ, if $\vdash_L \varphi^n \to \psi$, then there is a formula γ such that $\vdash_L \varphi^n \to \gamma$, $\vdash_L \gamma^n \to \psi$ and every propositional variable occurring in γ occurs both in φ and in ψ.