MV-algebras freely generated by finite Kleene algebras

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Joint work with Leonardo Cabrer and Vincenzo Marra

Dedicated to Tonino Di Nola in the occasion of his 65^{th} birthday

Free $\mathbb{V}$-algebras over $\mathbb{W}$-algebras

Let $\mathbb{V}$ and $\mathbb{W}$ be two varieties such that each $\mathbb{V}$-algebra A has a reduct $U(A)$ in $\mathbb{W}$.

- Forgetful functor $U\colon \mathbb{V} \to \mathbb{W}$ (U identity on morphisms).
- U has a left-adjoint $F\colon \mathbb{W} \to \mathbb{V}$.

$F(B)$ is the free $\mathbb{V}$-algebra over the $\mathbb{W}$-algebra B.

$$B \in \mathbb{W} \qquad \Longleftrightarrow \qquad B \cong \mathcal{F}_\kappa^{\mathbb{W}}/\Theta \text{ for some congruence } \Theta$$

$$\Theta \subseteq \mathcal{F}_\kappa^{\mathbb{W}} \times \mathcal{F}_\kappa^{\mathbb{W}}.$$

$$\Theta \text{ generates a uniquely determined congruence } \widehat{\Theta} \subseteq \mathcal{F}_\kappa^{\mathbb{V}} \times \mathcal{F}_\kappa^{\mathbb{V}}.$$

$$F(B) \cong \mathcal{F}_\kappa^{\mathbb{V}}/\widehat{\Theta}.$$

Free MV-algebras over Kleene algebras

In this work we solve, for the varieties of MV-algebras and of Kleene algebras, and for finitely generated Kleene algebras, the two classical problems of:

1. *Description* – which consists in describing the MV-algebraic structure of $F(B)$ in terms of the finitely generated Kleene algebra B;

2. *Recognition* – which consists in finding conditions on the structure of an MV-algebra A that are necessary and sufficient for the existence of a finitely generated Kleene algebra B such that $A \cong F(B)$.

The proofs rely on the Davey-Werner natural duality for Kleene algebras, on the representation of finitely presented MV-algebras by compact rational polyhedra, and on the theory of bases of MV-algebras.

Similar (recent) results

- MV-algebras free over finite distributive lattices:
  [Marra, Archive for Mathematical Logic, 2008]

- Gödel algebras free over finite distributive lattices:
  [Aguzzoli, Gerla, Marra, Annals of Pure and Applied Logic, 2008]

MV-algebras and Kleene algebras

- Variety $\mathbb{M}$ of MV-algebras:

  $$(M, \oplus, \neg, 0)$$

  such that $(M, \oplus, 0)$ is a commutative monoid, $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$ and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

  $\mathbb{M}$ is generated by $([0,1], \min\{1, x+y\}, 1-x, 0)$.

- Variety $\mathbb{K}$ of Kleene algebras:

  $$(K, \vee, \wedge, \neg, 0, 1)$$

  such that $(K, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, $\neg\neg x = x$, $\neg(x \wedge y) = \neg x \vee \neg y$ and $(x \wedge \neg x) \vee (y \vee \neg y) = (y \vee \neg y)$.

  $\mathbb{K}$ is generated by $(\{0, 1/2, 1\}, \max\{x, y\}, \min\{x, y\}, 1-x, 0, 1)$.

Upon defining $x \vee y := \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$ each MV-algebra M has a Kleene algebra reduct $U(M)$.

Kleene algebras Natural duality

Finite Kleene space:

$$(W, \leq, R, M)$$

such that $(W, \leq)$ is a finite poset, $M \subseteq \max W$, and $R \subseteq W^2$ satisfies

1. $(x, x) \in R$;
2. $(x, y) \in R$ and $x \in M$ imply $y \leq x$;
3. $(x, y) \in R$ and $z \leq y$ imply $(z, x) \in R$.

A morphism of Kleene spaces $f\colon (W, \leq, R, M) \to (W', \leq', R', M')$ is an order-preserving and R-preserving function $f\colon W \to W'$ such that $f(M) \subseteq M'$.

The category KS of finite Kleene spaces and their morphisms is dually equivalent to the category $\mathbb{K}_{fin}$ of finite Kleene algebras and homomorphisms.

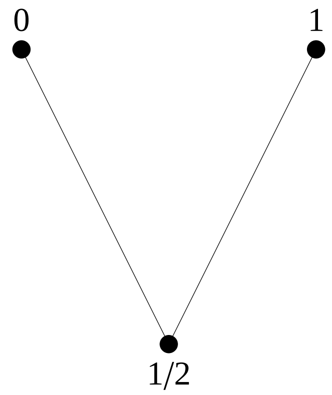
Kleene algebras $\mathsf{KS} \equiv \mathbb{K}_{fin}^{\mathrm{op}}$

Denote:

$$\mathbf{K} = (\{0, 1/2, 1\}, \max\{x, y\}, \min\{x, y\}, 1 - x, 0, 1)$$

$$\tilde{K} = (\{0, 1/2, 1\}, \preceq, \sim, \{0, 1\})\,,$$

where $\preceq$ is the following order:



and $\sim$ is the relation $\{0, 1/2, 1\}^2 \setminus \{(0, 1), (1, 0)\}$

Kleene algebras $\mathsf{KS} \equiv \mathbb{K}_{fin}^{\mathrm{op}}$

The equivalence $\mathsf{KS} \equiv \mathbb{K}_{fin}^{\mathrm{op}}$ is implemented by the functors:

$$D\colon \mathbb{K}_{fin} \to \mathsf{KS}\,, \qquad E\colon \mathsf{KS} \to \mathbb{K}_{fin}$$

For each finite Kleene algebra B:
$D(B) = \mathrm{Hom}\,(B, \mathbf{K}) \subseteq \tilde{K}^B$;

for every homomorphism $f\colon B \to C$:
$D(f)\colon D(C) \to D(B)$ is defined by $(D(f))(h) = h \circ f$ for each $h \in D(C)$.

For each finite Kleene space X:
$E(X) = \mathrm{Hom}\,(X, \tilde{K}) \subseteq \mathbf{K}^X$;

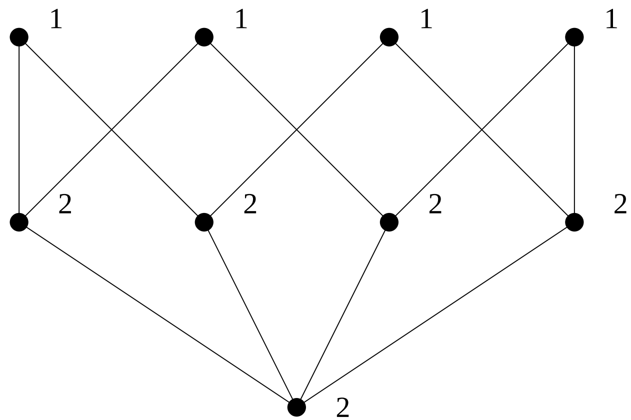
for each morphism $f\colon X \to Y$:
$E(f)\colon E(Y) \to E(X)$ is defined by $(E(f))(h) = h \circ f$ for each $h \in E(Y)$.

Kleene algebras Dual representation of free algebras

$$\tilde{K}^n = (\{0, 1/2, 1\}^n, \preceq_n, \sim_n, \{0,1\}^n)$$

where $\preceq_n$ and $\sim_n$ are defined componentwise from $\preceq$ and $\sim$.

Example: $\tilde{K}^2$:



For each $n \geq 1$, $E(\tilde{K}^n)$ is the free Kleene algebra over n generators.

Kleene algebras Dual representation of finite algebras

For any $\Theta \subseteq E(\tilde{K}^n)^2$ define:

$$\mathrm{Sol}_{\mathbb{K}}(\Theta) = \{v \in \{0, 1/2, 1\}^n \mid f(v) = g(v) \text{ for each } (f, g) \in \Theta\}.$$

Let $W \subseteq \{0, 1/2, 1\}^n$.

Then $(W, \preceq, \sim, M)$ is a subobject of $\tilde{K}^n$ if $\preceq$, $\sim$ and M are defined by restriction from $\preceq_n$, $\sim_n$ and $\{0, 1\}^n$, resp.

Considering the embedding $\iota\colon (W, \preceq, \sim, M) \hookrightarrow \tilde{K}^n$:

$$W = \mathrm{Sol}_{\mathbb{K}}(\{(f, g) \in E(\tilde{K}^n) \mid (E(\iota))(f) = (E(\iota))(g)\}).$$

The polyhedron associated with a Kleene space

Abstract Simplicial Complex over a finite set V: a family $\mathcal{S} \subseteq 2^V$, closed under taking subsets and including all singletons.

k-simplices := Elements of $\mathcal{S}$ of cardinality $k+1$; vertices of $\mathcal{S}$:= 0-simplices.

Weighted Abstract Simplicial Complex : a pair $(\mathcal{S}, \omega)$ where $\mathcal{S}$ is an abstract simplicial complex over V, and $\omega\colon V \to \mathbb{N}^+$.

Isomorphism of weighted abstract simplicial complexes $(\mathcal{S}, \omega)$ and $(\mathcal{S}', \omega')$ over V and V', resp.: a bijection $f\colon V \to V'$ such that:

- carries simplices to simplices: $\{v_1, \dots, v_u\} \in \mathcal{S}$ iff $\{f(v_1), \dots, f(v_u)\} \in \mathcal{S}'$
- preserves weights: $\omega' \circ f = \omega$.

Polytope associated to a weighted abstract simplex $S = \{v_{i_1}, \dots, v_{i_u}\} \in (\mathcal{S}, \omega)$: $\bar{S} := \operatorname{conv}\{e_{i_1}/\omega(v_{i_1}), \dots, e_{i_u}/\omega(v_{i_u})\} \subseteq \mathbb{R}^d$. ($e_{i_j}$: unit vector of $\mathbb{R}^d$)

Polyhedron associated with $(\mathcal{S}, \omega)$: $P^{\omega}_{\mathcal{S}} := \bigcup_{S \in \mathcal{S}} \bar{S}$.

The polyhedron associated with a Kleene space

$P^{\omega}_{\mathcal{S}}$ is called the geometric realisation of $(\mathcal{S}, \omega)$.

The nerve $\mathcal{N}(O)$ of a finite poset O:
family of all subsets of O that are chains under the order inherited by restriction from O.

$\mathcal{N}(O)$ is an abstract simplicial complex.

For any $(W, \leq, R, M) \in \mathsf{KS}$:

- its associated weighted abstract simplicial complex is defined as $(\mathcal{N}(W), \omega)$, where $\omega(v) = 1$ if $v \in M$; $\omega(v) = 2$ otherwise.
- its companion polyedron is the geometric realisation $P^{\omega}_{\mathcal{N}(W)}$ of $(\mathcal{N}(W), \omega)$. (Note $(\mathcal{N}(W), \omega)$ does not depend on R).

Regular triangulations

Rational n-simplex: $\sigma := \operatorname{conv} S$, for $S = \{v_0, v_1, \ldots, v_n\}$ set of affinely independent points in $\mathbb{Q}^d$. ver $\sigma := S$.

Denominator of $v = (p_1/q_1, \ldots, p_d/q_d) \in \mathbb{Q}^d$: $\operatorname{den} v := \operatorname{lcm}(q_1, q_2, \ldots, q_d)$.

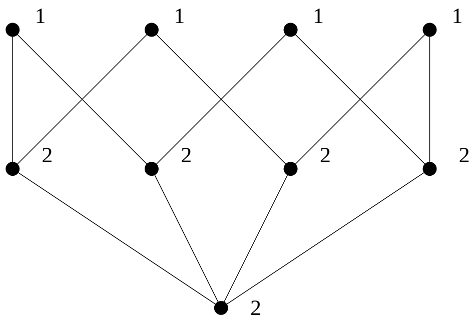
A rational simplex $\operatorname{conv}\{v_0, v_1, \ldots, v_d\}$ is regular if $\det((v_i, 1)\operatorname{den} v_i)_{i=0}^d = \pm 1$.

Regular triangulation Σ in $\mathbb{R}^d$: finite family of regular simplices in $\mathbb{R}^d$ such that any two of them intersect in a common face. Support of Σ: $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$.

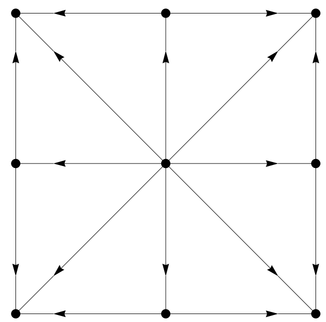
Kleene triangulation of $[0,1]^n$: $\mathcal{S}_n := \{\operatorname{conv} C \mid C \text{ chain of } (\{0, 1/2, 1\}^n, \preceq_n)\}$.

$\mathcal{S}_n$ is a regular triangulation of (*i.e.*, with support) $[0,1]^n$.

Example: $\tilde{K}_2$:



$\mathcal{S}_2$:



MV-algebras Functional representation

A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is McNaughton, or, a $\mathbb{Z}$-map if:

- f is continuous wrt. the Euclidean topology on $\mathbb{R}^d$;
- f is piecewise linear, that is, $\exists\, p_1, \ldots, p_u$ linear polynomials, such that $\forall x \in \mathbb{R}^d \; \exists i \in \{1, 2, \ldots, u\} : \; f(x) = p_i(x)$;
- each piece of f has integer coefficients: that is, all the coefficients of each p_i are integers.

For $X \subseteq [0,1]^d \subseteq \mathbb{R}^d$, a $\mathbb{Z}$-map on X is a function $f\colon X \to [0,1]$ which coincides with a $\mathbb{Z}$-map $\mathbb{R}^d \to \mathbb{R}$ over X.

$$\mathcal{M}(X) := \{f\colon X \to [0,1] \mid f \text{ is a } \mathbb{Z}\text{-map}\}$$

$\mathcal{M}(X)$ is an MV-algebra when equipped with operations defined pointwise from the standard MV-algebra $[0,1]$.

MV free over Kleene Description problem

For each finite Kleene algebra B:

Let $D(B) = (W, \preceq, R, M)$ denote the Kleene space dual to B;

Let $(\mathcal{N}(W), \omega)$ denote the weighted abstract simplicial complex associated with $D(B)$;

Let $P^{\omega}_{\mathcal{N}(W)}$ denote the companion polyhedron of $D(B)$.

Then:

$$F(B) \cong \mathcal{M}(P^{\omega}_{\mathcal{N}(W)})\,.$$

$F(B) \cong \mathcal{M}(P^{\omega}_{\mathcal{N}(W)})$ Tools for the proof

- $\mathcal{M}_n := \mathcal{M}([0,1]^n)$ is (isomorphic to) $\mathcal{F}_n^{\mathbb{M}}$.
- For any $\Theta \subseteq \mathcal{M}_n^2$ define $\mathrm{Sol}_{\mathbb{M}}(\Theta) := \{v \in [0,1]^n \mid f(v) = g(v) \text{ for each } (f,g) \in \Theta\}$. Then $\mathcal{M}_n/\widehat{\Theta} \cong \mathcal{M}(\mathrm{Sol}_{\mathbb{M}}(\Theta))$.
- For any $\Theta \subseteq E(\tilde{K}^n)^2$ it holds that $\mathrm{Sol}_{\mathbb{K}}(\Theta) = \mathrm{Sol}_{\mathbb{M}}(\Theta) \cap \{0, 1/2, 1\}^n$.
- For any $\Theta \subseteq E(\tilde{K}^n)^2$ the set $\Sigma_\Theta := \{\sigma \in \mathcal{S}_n \mid \mathrm{ver}\,\sigma \subseteq \mathrm{Sol}_{\mathbb{K}}(\Theta)\}$ is a regular triangulation in $[0,1]^n$ such that $\mathrm{Sol}_{\mathbb{M}}(\Theta) = |\Sigma_\Theta|$.
- For each regular triangulation Δ, $\mathcal{S}(\Delta) := \{\mathrm{ver}\,\sigma \mid \sigma \in \Sigma\}$ is an abstract simplicial complex.
- Let Σ and Δ be regular triangulations of $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^{d'}$. If $(\mathcal{S}(\Sigma), \mathrm{den}) \cong (\mathcal{S}(\Delta), \mathrm{den})$ then $\mathcal{M}(P) \cong \mathcal{M}(Q)$.

$F(B) \cong \mathcal{M}(P^{\omega}_{\mathcal{N}(W)})$ Sketch of proof

- $B \in \mathbb{K}_{fin} \Longrightarrow B \cong E(\tilde{K}^n)/\Theta$ for some congruence Θ;
- $E(\tilde{K}^n) \twoheadrightarrow B$ dualises to $D(B) \hookrightarrow \tilde{K}^n$, where $D(B) = (W, \preceq, R, M)$ with $W = \mathrm{Sol}_{\mathbb{K}}(\Theta)$.

- On the other hand $F(B) = \mathcal{M}_n/\widehat{\Theta} \cong \mathcal{M}(\mathrm{Sol}_{\mathbb{M}}(\Theta))$.
- But $\mathrm{Sol}_{\mathbb{M}}(\Theta) = |\Sigma_\Theta|$, for $\Sigma_\Theta = \{\sigma \in \mathcal{S}_n \mid \mathrm{ver}\,\sigma \subseteq \mathrm{Sol}_{\mathbb{K}}(\Theta)\}$.
- Then $(\mathcal{S}(\Sigma_\Theta), \mathrm{den}) \cong (\mathcal{N}(\mathrm{Sol}_{\mathbb{K}}(\Theta)), \omega)$.

- Hence, $\mathcal{M}(\mathrm{Sol}_{\mathbb{M}}(\Theta)) \cong \mathcal{M}(P^{\omega}_{\mathcal{N}(\mathrm{Sol}_{\mathbb{K}}(\Theta))})$ that yields the desired

$$F(B) \cong \mathcal{M}(P^{\omega}_{\mathcal{N}(W)}).$$

MV-algebras Bases

Let $\mathcal{B} = \{b_1, \ldots, b_t\} \subseteq A \in \mathbb{M}$, $b_i \neq 0$.

Pick $b_r \neq b_s \in \mathcal{B}$ such that $b_r \wedge b_s \neq 0$.

The stellar subdivision of $\mathcal{B}$ at $\{b_r, b_s\}$ is

$$\mathcal{B}_{b_r, b_s} := \{b'_1, \ldots, b'_t, b'_{t+1}\} \setminus \{0\},$$

where:

$$\begin{aligned} b'_r &:= b_r \odot \neg(b_r \wedge b_s) \\ b'_s &:= b_s \odot \neg(b_r \wedge b_s) \\ b'_{t+1} &:= b_r \wedge b_s \\ b'_i &:= b_i \ \text{otherwise.} \end{aligned}$$

MV-algebras Bases

Let $\mathcal{B} = \{b_1, \ldots, b_t\} \subseteq A \in \mathbb{M}$, $b_i \neq 0$.

- $\mathcal{B}$ is 1-regular if for each stellar subdivision $\mathcal{B}_{b_r,b_s}$ it holds that for any $1 \leq i_1 < \cdots < i_k \leq t$: if $(b_r \wedge b_s) \wedge b_{i_1} \wedge \cdots \wedge b_{i_k} > 0$ holds in A then for every $\emptyset \neq J \subseteq \{i_1, \ldots, i_k\}$, with $\{r, s\} \not\subseteq J$

$$(b_r \wedge b_s) \wedge \bigwedge_{j \in J} b'_j > 0 \quad \text{holds in } A\,.$$

- $\mathcal{B}$ is regular if it is 1-regular, and each one of its stellar subdivisions is 1-regular, too.

- $\mathcal{B}$ is a basis of A, if it generates A, it is regular, and there are integers (multipliers) $m_1, \ldots, m_t \geq 1$ such that for each $i \in \{1, \ldots, t\}$:

$$\neg b_i = (m_i - 1)b_i \oplus \bigoplus_{i \neq j} m_j b_j\,.$$

Missing faces and comparabilities

Let $\mathcal{S}$ be an abstract simplicial complex over the vertex set V.

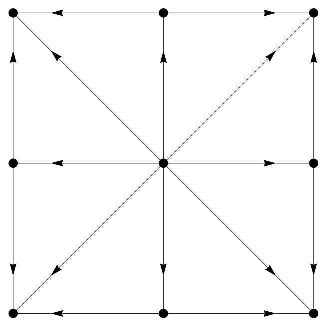
Non-face of $\mathcal{S}$: a subset $N \subseteq V$ such that $N \notin \mathcal{S}$;

Missing face of $\mathcal{S}$: a non-face that is inclusion-minimal.

Write $\mathcal{S}^{(2)}$ for the 2-skeleton of $\mathcal{S}$, that is $\mathcal{S}^{(2)} := \{S \in \mathcal{S} \mid |S| = 2\}$.

There is a comparability over the graph $\mathcal{S}^{(2)}$ if its edges can be transitively oriented, that is:

whenever $\{p, r_1\}, \{r_1, r_2\}, \ldots, \{r_{u-1}, r_u\}, \{r_u, q\} \in \mathcal{S}^{(2)}$ are oriented as $(p, r_1), (r_1, r_2), \ldots, (r_{u-1}, r_u), (r_u, q)$ then there is $\{p, q\} \in \mathcal{S}^{(2)}$ oriented as (p, q).

Example: $\mathcal{S}_2^{(2)}$: 

MV free over Kleene Recognition problem

Any basis $\mathcal{B}$ of an MV-algebra A determines an abstract simplicial complex

$$\mathcal{B}^{\bowtie} := \{C \subseteq \mathcal{B} \mid \bigwedge C > 0 \text{ holds in } A\}.$$

A basis $\mathcal{B}$ of an MV-algebra A is a Kleene basis if

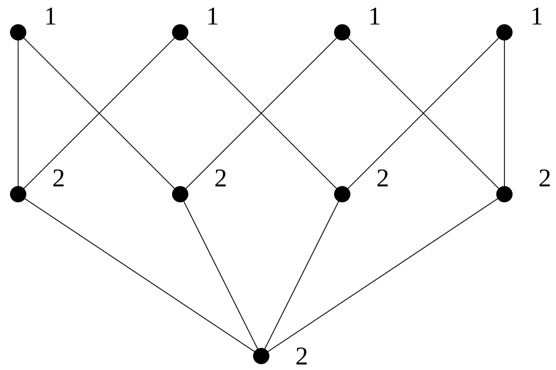
- The multiplier of each $b \in \mathcal{B}$ is either 1 or 2;
- The abstract simplicial complex $\mathcal{B}^{\bowtie}$ has no missing faces of cardinality ≥ 3;
- There is a comparability over $(\mathcal{B}^{\bowtie})^{(2)}$ such that each $b \in \mathcal{B}$ with multiplier 1 is a sink (it never occurs as first member of an edge of the comparability).

Let A be any MV-algebra.

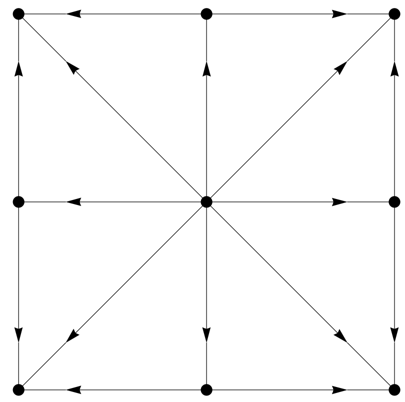
Then A is free over some finite Kleene algebra iff A has a Kleene basis.

Examples $\mathcal{F}^2_{\mathbb{M}}$ and $\mathcal{F}^2_{\mathbb{K}}$

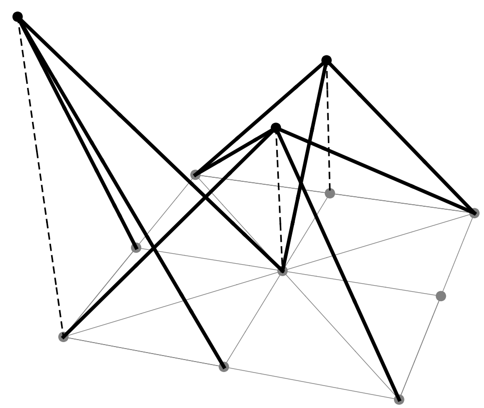
$\tilde{K}_2$:



comparability over $\mathcal{S}(\mathcal{S}_2)$:



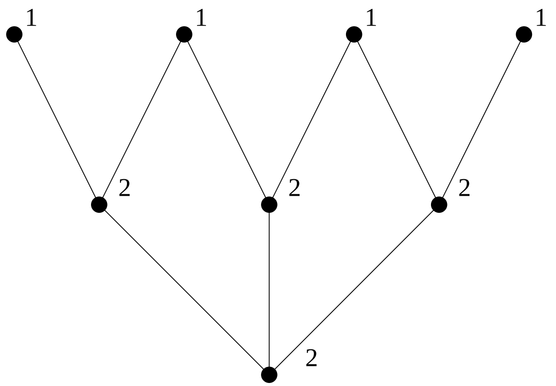
Some Schauder hats belonging to a Kleene basis for $\mathcal{F}^2_{\mathbb{M}}$:



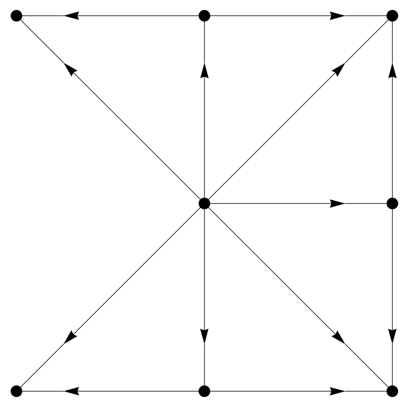
Examples $x = x \vee (y \wedge \neg y)$

Let $\Theta \subseteq \mathcal{F}^2_{\mathbb{K}} \times \mathcal{F}^2_{\mathbb{K}}$ be the congruence determined by $x = x \vee (y \wedge \neg y)$.

$D(\mathcal{F}^2_{\mathbb{K}}/\Theta)$:

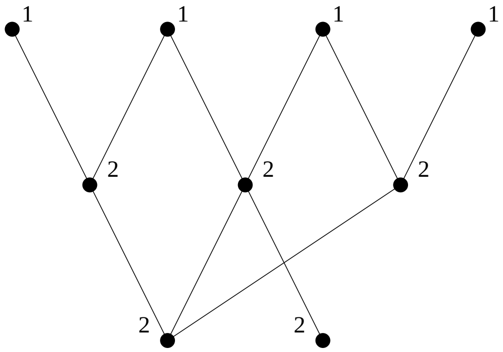


comparability over $\mathcal{S}(\Sigma_\Theta)$:

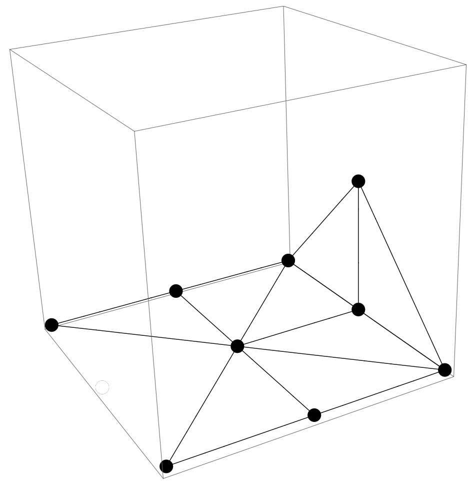


Examples $\mathcal{F}^2_{\mathbb{M}}$ free over a non-free Kleene algebra

$D(B)$:



Σ_Θ shows B is 3-generated:



$F(B) \cong \mathcal{F}^2_{\mathbb{M}}$: