The structure of many-valued relations

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O. Ore, *Galois connexions*, Trans. AMS **55**(1944), 493513.
A. Tarski: *Sur quelques propriétés caractéristiques des images d'ensembles*, Annales de la Société Polonaise de Mathématique,**6**(1927), 127-128.

Definition

1. *Let A, B be two pre-ordered sets and $f : A \to B$, $g : B \to A$ be maps.*
   - *f and g form an antitone (or contravariant) Galois connection between A and B if*
     $$b \leq (a)f \Leftrightarrow a \leq (b)g.$$
   - *f and g form an isotone (or covariant) Galois connection or adjunction between A and B if*
     $$(a)f \leq b \Leftrightarrow a \leq (b)g.$$
2. *Let $A = (A, \vee, \wedge, 0, 1)$ be a Boolean algebra, Two functions $f, g : A \to A$ form a conjugated pair if*
   $$(a)f \wedge b = 0 \Leftrightarrow (b)g \wedge a = 0.$$

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It is well known the behavior of Galois connections and conjugated pair with respect the sup and inf; moreover, through these properties, it is possible to characterize them.

C. Guido, P. Toto: *Extended-order algebras*, Journal of Applied Logic, **6**(4) (2008), 609-626.
M.E.D.S., C. Guido: *Associativity, commutativity and symmetry in residuated structures*, Order, doi: 10.1007/s11083-012-9250-8.

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Definition

Let L be a set, $\rightarrow: L \times L \rightarrow L$ a binary operation, $\top \in L$ a fixed element. The triple $(L, \rightarrow, \top)$ is a complete distributive extended-order algebra, shortly cdeo algebra, if

1. *the relation $\leq$ in L defined by the equivalence $x \leq y \Leftrightarrow x \rightarrow y = \top, \forall x, y \in L$, is an order relation, called natural ordering in L (induced by $\rightarrow$);*
2. *$(L, \leq)$ is a complete lattice with maximum $\top$ and minimum, say, $\bot$;*
3. *$(\bigvee A) \rightarrow (\bigwedge B) = \bigwedge_{a \in A, b \in B} a \rightarrow b$, $\forall A, B \subseteq L$ (distributivity).*

The distributivity condition allows to consider the *adjoint product*

$$\otimes : L \times L \to L$$

defined by

$$a \otimes x = \bigwedge \{t \in L | \ x \leq a \to t\}.$$

Of course, $\otimes$ and $\to$ form an *adjoint pair*, i.e.

$$\forall x, y, z \in L : \ x \otimes y \leq z \Leftrightarrow y \leq x \to z.$$

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Proposition

1. *$a \otimes \top = a$;*
2. *if $b \leq c$, then $a \otimes b \leq a \otimes c$ and $b \otimes a \leq c \otimes a$;*
3. *$a \otimes (\bigvee B) = \bigvee_{b \in B}(a \otimes b)$; $(\bigvee A) \otimes b = \bigvee_{a \in A}(a \otimes b)$.*

The symmetry condition adds further properties on L, without assuming commutatitivity and associativity.

Definition

A cdeo algebra $(L, \to, \top)$ is called symmetrical if

- *$\exists$ a (dual) binary operation $\rightsquigarrow: L \times L \to L$ such that $(L, \rightsquigarrow, \top)$ is a cdeo algebra;*
- *$\to$ and $\rightsquigarrow$ induce the same order;*
- *$y \leq x \rightsquigarrow z \Leftrightarrow x \leq y \to z, \forall x, y, z \in L$, i.e. $[\to, \rightsquigarrow]$ is a Galois pair.*

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- $\to$ *and* $\rightsquigarrow$ *induce the same order;*
- $y \leq x \rightsquigarrow z \Leftrightarrow x \leq y \to z, \forall x, y, z \in L$*, i.e.* $[\to, \rightsquigarrow]$ *is a Galois pair.*

Remark

- *Under distributivity assumption, the adjoint products* $\otimes$ *and* $\tilde{\otimes}$ *of* $\to$ *and* $\rightsquigarrow$ *are related by* $a \tilde{\otimes} b = b \otimes a$*, for all* $a, b \in L$*.*
- *A cdeo algebra is symmetrical if and only if* $\top$ *is a unit for* $\otimes$*.*
- *A cdeo algebra with* $\otimes$ *commutative and associative is a complete integral commutative residuated lattice, according to notation of P. Jipsen, C. Tsinakis: A survey of residuated lattices, Ordered Algebraic Structures (J. Martinez, Editor), Kluwer Academic Publishers, Dordrecht 2002, 19-56.*

Now, we give the definitions of global and relative equivalence.
Let Z, L be two sets, $\equiv$ be an equivalence relation on L and let $h, k : Z \to L$ be two functions.
Obviously, h and k are equivalent with respect to $\equiv$ ($h \equiv k$) if $(z)h \equiv (z)k$, for every $z \in Z$.
Let $E \subseteq L$ and denote by

$$\langle E \rangle_{\equiv} = \{x \in L | \exists a \in E : x \equiv a\}$$

the saturation of E with respect $\equiv$.

Definition

h and k are E-equivalent, shortly $h \equiv_E k$, if for every $z \in Z$, the following equivalence holds:

$$(z)h \in \langle E \rangle_{\equiv} \Leftrightarrow (z)k \in \langle E \rangle_{\equiv}.$$

Obviously, $h \equiv k$ if and only if, for every $E \subseteq L, h \equiv_E k$.

We are interested in the case when $(L, \leq)$ is a (pre-)ordered set and consider the equivalence relation $\equiv$, induced by $\leq$ on L, i.e.

$$\forall a, b \in L : a \equiv b \Leftrightarrow a \leq b \wedge b \leq a,$$

which is the equality if $\leq$ is an order.
We call

1. the upper-hull of E the set $\uparrow E = \{x \in L |\ \exists e \in E : e \leq x\}$;
2. the lower-hull of E the set E: $\downarrow E = \{x \in L |\ \exists e \in E : x \leq e\}$.

With this notation, we give the following Definition.

Definition

The functions $h, k : Z \to L$ are upper (lower) E-equivalent with respect to $\equiv$, shortly $h \equiv_{\uparrow E} k$ $(h \equiv_{\downarrow E} k)$, if $\forall z \in Z$ one has

$$(z)h \in \uparrow E \Leftrightarrow (z)k \in \uparrow E \ ((z)h \in \downarrow E \Leftrightarrow (z)k \in \downarrow E).$$

Remark

Since $\uparrow E = \langle \uparrow E \rangle_{\equiv}$ and $\downarrow E = \langle \downarrow E \rangle_{\equiv}$, we can say that a relative upper (lower) equivalence is an equivalence relative to an upper (lower) set of L.

On the base of equivalences of functions with values in a (pre-)ordered set ($L \leq$), in particular of L-relations, we shall present our general approach to connections, including their classification in four types, related to each other by analogies and dualities, as in

J. Gutiérrez García, I. Mardones-Pérez, M.A. de Prada Vicente, D. Zhang: *Fuzzy Galois connections categorically*, Math. Log. Quart. **56** (2010), 131-147

Our general approach will allow also to include further the classical notion of conjugated pairs of functions introduced by Tarski and developed in

B. Jónsson and A. Tarski: *Boolean algebras with operators*, I, Amer. J. Math. **73** (1951), 891-939; **74**, (1952), 127-162.

We note that not only the notion of conjugated pairs, also considered recently in

G. Georgescu, A. Popescu: *Non-dual fuzzy connections*, Archive Math. Logic **43** (8) (2004), 1009-1039

will be extended, but it will be also classified into four types (we shall call them Tarski connections) that are perfectly order-dual to Galois connections.

We propose a quite general framework where Galois connections and Tarski connections (our terminology for conjugated pairs) are special instances of (global) connections, which are generalizations of fuzzy connections. All these notions are, in fact, determined by suitable pairs of relations which are (globally or relatively) equivalent.

Let $(L, \leq)$ be a (pre-)ordered set, (X, α), (Y, β) be two possibly structured sets, each equipped with a fixed binary L-relation on it and $f : X \to Y$, $g : Y \to X$ be two functions. Consider the L-relations from X to Y defined for all $x \in X, y \in Y$ by:

$$\mathcal{R}_{\beta f} : X \times Y \to L,\ (x, y) \mapsto (x, y)\mathcal{R}_{\beta f} = (y, (x)f)\beta;$$

$$\mathcal{R}_{f\beta} : X \times Y \to L,\ (x, y) \mapsto (x, y)\mathcal{R}_{f\beta} = ((x)f, y)\beta;$$

$$\mathcal{R}_{\alpha g} : X \times Y \to L,\ (x, y) \mapsto (x, y)\mathcal{R}_{\alpha g} = (x, (y)g)\alpha;$$

$$\mathcal{R}_{g\alpha} : X \times Y \to L,\ (x, y) \mapsto (x, y)\mathcal{R}_{g\alpha} = ((y)g, x)\alpha.$$

With the above notations, we state the following.

Definition

The pair of maps f and g is called

- *a type I (global) connection between X and Y, denoted by $[f\text{-}g]$, if* $\mathcal{R}_{\beta f} \equiv \mathcal{R}_{\alpha g}$.
- *a type II (global) connection between X and Y, denoted by $]f\text{-}g[$, if* $\mathcal{R}_{f\beta} \equiv \mathcal{R}_{g\alpha}$.
- *a type III (global) connection between X and Y, denoted by (f, g), if* $\mathcal{R}_{f\beta} \equiv \mathcal{R}_{\alpha g}$.
- *a type IV (global) connection between X and Y, denoted by $)f, g($, if* $\mathcal{R}_{\beta f} \equiv \mathcal{R}_{g\alpha}$.

Definition

Let $E \subseteq L$ be a subset of the (pre-)ordered set $(L, \leq)$. The pair of maps f and g is called

- *a type I E-connection between X and Y, denoted by $[f\text{-}g]_E$, if $\mathcal{R}_{\beta f} \equiv_E \mathcal{R}_{\alpha g}$.*
- *a type II E-connection between X and Y, denoted by $]f\text{-}g[_E$, if $\mathcal{R}_{f\beta} \equiv_E \mathcal{R}_{g\alpha}$.*
- *a type III E-connection between X and Y, denoted by $(f, g)_E$, if $\mathcal{R}_{f\beta} \equiv_E \mathcal{R}_{\alpha g}$.*
- *a type IV E-connection between X and Y, denoted by $)f, g(_E$, if $\mathcal{R}_{\beta f} \equiv_E \mathcal{R}_{g\alpha}$.*

Definition

Let $E \subseteq L$ be a subset of the (pre-)ordered set $(L, \leq)$. The pair of maps f and g is called

- *a type I Galois E-connection between X and Y, denoted by $[f\text{-}g]_{\uparrow E}$, if $\mathcal{R}_{\beta f} \equiv_{\uparrow E} \mathcal{R}_{\alpha g}$.*
- *a type II Galois E-connection between X and Y, denoted by $]f\text{-}g[_{\uparrow E}$, if $\mathcal{R}_{f\beta} \equiv_{\uparrow E} \mathcal{R}_{g\alpha}$.*
- *a type III Galois E-connection between X and Y, denoted by $(f, g)_{\uparrow E}$, if $\mathcal{R}_{f\beta} \equiv_{\uparrow E} \mathcal{R}_{\alpha g}$.*
- *a type IV Galois E-connection between X and Y, denoted by $)f, g(_{\uparrow E}$, if $\mathcal{R}_{\beta f} \equiv_{\uparrow E} \mathcal{R}_{g\alpha}$.*

Definition

Let $E \subseteq L$ *be a subset of the (pre-)ordered set* $(L, \leq)$. *The pair of maps* f *and* g *is called*

- *a type I Tarski* E*-connection between* X *and* Y, *denoted by* $< f, g >_{\downarrow E}$, *if* $\mathcal{R}_{\beta f} \equiv_{\downarrow E} \mathcal{R}_{\alpha g}$.
- *a type II Tarski* E*-connection between* X *and* Y, *denoted by* $> f, g <_{\downarrow E}$, *if* $\mathcal{R}_{f\beta} \equiv_{\downarrow E} \mathcal{R}_{g\alpha}$.
- *a type III Tarski* E*-connection between* X *and* Y, *denoted by* $\{f, g\}_{\downarrow E}$, *if* $\mathcal{R}_{f\beta} \equiv_{\downarrow E} \mathcal{R}_{\alpha g}$.
- *a type IV Tarski* E*-connection between* X *and* Y, *denoted by* $\}f, g\{_{\downarrow E}$, *if* $\mathcal{R}_{\beta f} \equiv_{\downarrow E} \mathcal{R}_{g\alpha}$.

In the classical approaches:

• to Galois connections, (X, α) and (Y, β) are possibly complete posets and $L = \mathbf{2}$. Classical Galois connections are, in our terminology, Galois 1-connections between posets; more precisely, antitonic Galois connections are Galois 1-connections of type I while isotonic Galois connections are Galois 1-connections of type III (those of type II and IV are closely related to those of type I and III, respectively).

•[−] to conjugated pairs, $X = Y$ are Boolean algebras and hence these are, in our terminology, Tarski 0-connections of any of the types I-IV on X.

The implicative structure of logic and, in particular, the order-theoretic approach to cdeo algebras, give a suitable framework to explain details of our viewpoint and allows to prove the main properties of Galois connections (most well-known) and Tarski connections (which extend results of B. Jónsson and A. Tarski).

Proposition

With the above notation, the following hold.

1. $[f\text{-}g]$ *if and only if* $[g\text{-}f]$;
2. $[f\text{-}g]$ *if and only if* $]f\text{-}g[$.
3. $[f\text{-}g]_{\uparrow e}$, *if and only if* $\forall x \in X, y \in Y$: $(x,y)\mathcal{R}_{\beta f} \geq e \Leftrightarrow (x,y)\mathcal{R}_{\alpha g} \geq e$;

Proposition

Let $(L, \rightarrow, \top)$ *be a cdeo algebra and let* $e \in L$ *that satisfy*
(n) $x \leq y \Leftrightarrow e \leq x \rightarrow y$, $\forall x, y \in L$ (*i.e.* $x \otimes e = x$, $\forall x \in L$).

1. *If* $f, g : (L, \rightarrow) \rightarrow (L, \rightarrow)$ *are two functions such that* $[f, g]_{\uparrow e}$, *then*
   1. $(y)g = \bigvee \{x |\ y \leq (x)f\}$;
   2. $(x)f = \bigvee \{y |\ x \leq (y)g\}$.
2. *Let* $f : (L, \rightarrow) \rightarrow (L, \rightarrow)$ *be a function. There exist a map* $g : (L, \rightarrow) \rightarrow (L, \rightarrow)$ *such that* $[f\text{-}g]_{\uparrow e}$ *if and only if* $(\bigvee(S))f = \bigwedge_{s \in S}(s)f$, *for every* $S \subseteq L$.

Proposition

With above notation, the following hold.

1. $< f, g >$ *if and only if* $< g, f >$;
2. $< f, g >$ *if and only if* $> f, g <$;
3. $< f, g >_{\downarrow d}$, *if and only if* $\forall x \in X, y \in Y \ (x, y)\mathcal{R}_{f\beta} \leq d \Leftrightarrow (x, y)\mathcal{R}_{g\alpha} \leq d$;

Let $(L, \to, \top)$ a cdeo algebra. If $d \in L$ the relative d-negation of $x \in L$ is defined by setting $x^{-_d} = x \to d$. If L is symmetrical, we can define a dual relative d-negation of $x \in L$ by setting $x^{\sim_d} = x \rightsquigarrow d$. An element $d \in L$ is called dualizing if it satisfies the condition $x^{\sim_d -_d} = x^{-_d \sim_d} = x, \forall x \in L$.

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Proposition

Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebras, with adjoint product $\otimes$ and d be a dualizing element of L.

1. *If $f, g : (L, \otimes) \rightarrow (L, \otimes)$ are two functions such that $< f, g >_{\downarrow d}$, then*
    1. $(y)g = \bigwedge \{x^{\sim_d} | (x)f \otimes y \leq d\} = (\bigvee \{x | (x)f \leq y^{\sim_d}\})^{\sim_d}$;
    2. $(x)f = \bigwedge \{y^{\sim_d} | (y)g \otimes x \leq d\} = (\bigvee \{y | (y)g \leq x^{\sim_d}\})^{\sim_d}$.
2. *Let $f : (L, \otimes) \rightarrow (L, \otimes)$ be a function. There exist a map $g : (L, \otimes) \rightarrow (L, \otimes)$ such that $< f, g >_{\downarrow d}$ if and only if $(\bigvee(S))f = \bigvee_{s \in S} (s)f$, for every $S \subseteq L$.*

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Remark

Although the Tarski connections are order dual of Galois connections, the contexts in which their classical notions are located are not mutually dual. The given framework explains, as already noted by Georgescu and Popescu, why the Galois connections and the conjugate pairs both fall in the scheme of so-called fuzzy Galois connections.

The relational triangles are simple tools involving L-relations and their compositions and they are useful to describe the structure of class L-relations and also its underlying algebra. In fact, in addition to those that we will see, by using the E-connections, we can define their weak (local) version that allow to describe completely the structure of the (symmetrical) cdeo algebra.

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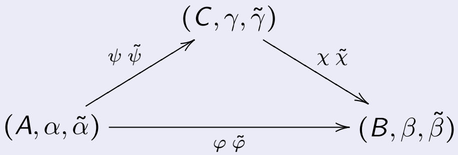
Definition

Let L be a possibly structured set and $(A, \alpha, \tilde{\alpha})$, $(B, \beta, \tilde{\beta})$ and $(C, \gamma, \tilde{\gamma})$ be sets, each equipped with two fixed binary L-relations on it. The diagram



with $\varphi, \tilde{\varphi} : A \times B \to C$, $\psi, \tilde{\psi} : A \times C \to B$ and $\chi, \tilde{\chi} : B \times C \to A$, is called double relational triangle if for all $a \in A, b \in B, c \in C$ the following equalities hold:

$$\mathcal{R}_{\alpha\chi} = \mathcal{R}_{\tilde{\varphi}\beta};\ \mathcal{R}_{\tilde{\alpha}\tilde{\chi}} = \mathcal{R}_{\psi\tilde{\beta}};\ \mathcal{R}_{\alpha\tilde{\chi}} = \mathcal{R}_{\tilde{\gamma}\varphi};\ \mathcal{R}_{\tilde{\alpha}\chi} = \mathcal{R}_{\gamma\tilde{\varphi}}.$$

Explicitly, for all $a \in A, b \in B, c \in C$ the above equalities become:

$$[1]\ (a, (c, b)\chi)\alpha = ((a, c)\tilde{\psi}, b)\beta;$$

$$[2]\ (a, (c, b)\tilde{\chi})\tilde{\alpha} = ((a, c)\psi, b)\tilde{\beta};$$

$$[3]\ (a, (c, b)\tilde{\chi})\alpha = (c, (a, b)\varphi)\tilde{\gamma};$$

$$[4]\ (a, (c, b)\chi)\tilde{\alpha} = (c, (a, b)\tilde{\varphi})\gamma.$$

- Fixing $c \in C$, we can consider the applications:

$$\tilde{\psi}_c : A \to B,\ a \mapsto (a)\tilde{\psi}_c = (a, c)\tilde{\psi}$$
$$\chi_c : B \to A,\ b \mapsto (b)\chi_c = (c, b)\chi.$$

  Hence, the equality [1] becomes $(a, (b)\chi_c)\alpha = ((a)\tilde{\psi}_c, b)\beta$; it establishes that $\tilde{\psi}_c$ and χ_c form a type III (global) connection between A and B.

- Fixing $c \in C$, we can consider the applications:

$$\psi_c : A \to B,\ a \mapsto (a)\psi_c = (a, c)\psi$$
$$\tilde{\chi}_c : B \to A,\ b \mapsto (b)\tilde{\chi}_c = (c, b)\tilde{\chi}.$$

  Hence, the equality [2] becomes $(a, (b)\tilde{\chi}_c)\tilde{\alpha} = ((a)\psi_c, b)\tilde{\beta}$; it establishes that ψ_c and $\tilde{\chi}_c$ form a type IV type (global) connection between A and B.

- Fixing $b \in B$, we can consider the applications:

$$\varphi_b : A \to C,\ a \mapsto (a)\varphi_b = (a,b)\varphi$$
$$\tilde{\chi}_b : C \to A,\ c \mapsto (c)\tilde{\chi}_b = (c,b)\tilde{\chi}.$$

  Hence, the equality [3] becomes $(a,(c)\tilde{\chi}_b)\alpha = (c,(a)\varphi_b)\tilde{\gamma}$; it establishes that φ_b and $\tilde{\chi}_b$ form a type I (global) connection between A and C.

- Fixing $b \in B$, we can consider the applications:

$$\tilde{\varphi}_b : A \to C,\ a \mapsto (a)\tilde{\varphi}_b = (a,b)\varphi$$
$$\chi_b : C \to A,\ c \mapsto (c)\tilde{\chi}_b = (c,b)\tilde{\chi}.$$

  Hence, the equality [4] becomes $(a,(c)\chi_b)\tilde{\alpha} = (c,(a)\tilde{\varphi}_b)\gamma$; it establishes that $\tilde{\varphi}_b$ and χ_b form a type I (global) connection between A and C.

- Since $(a,(c)\tilde{\chi}_b)\alpha = (c,(a)\varphi_b)\tilde{\gamma}$ is equivalent to $((c)\tilde{\chi}_b, a)\alpha_- = ((a)\varphi_b, c)\tilde{\gamma}_-$ the equalities [3] establishes, moreover, that φ_b and $\tilde{\chi}_b$ form a type II (global) connection between A and C.

The study of binary many-valued relations taking as a set of truth values some kind of cdeo algebras is developed in

M.E.D.S., C. Guido: *The structure of many-valued relations*, (preprint),

where an algebraic abstract model of L-relation, called relation pseudo-category, has been introduced and studied, as a generalization of Dedekind category and MV-relation algebras considered, respectively, by M. Winter and A. Popescu. We can give the following pointwise definitions.

Definition

1. $\perp\!\!\!\perp_{XY}: X \times Y \to L \,:\, (x,y)\perp\!\!\!\perp_{XY} = \perp;$
2. $\top\!\!\!\top_{XY}: X \times Y \to L \,:\, (x,y)\top\!\!\!\top_{XY} = \top;$
3. $\mathcal{I}_X : X \times X \to L \,:\, (x,x')\mathcal{I}_X = \begin{cases} \top & \textit{if } x = x' \\ \perp & \textit{otherwise} \end{cases};$
4. $\mathcal{R} \to \mathcal{R}' : X \times Y \to L$: $(x,y)(\mathcal{R} \to \mathcal{R}') = (x,y)\mathcal{R} \to (x,y)\mathcal{R}'.$

It is easy to show that the triple

$$(L^{X\times Y}, \to, \top\!\!\!\top_{XY})$$

is a cdeo algebra. Indeed, this algebra of L-relations inherits all the properties assumed on L.

If L is a symmetrical cdeo algebra, one can consider the following partial compositions of L-relations $\mathcal{R} : X \times Y \to L$ and $\mathcal{S} : Y \times Z \to L$ in a similar way as has been done in

R.Bělohlávek: *Fuzzy Relational Systems: Foundations and Principles*, IFSR International Series on Systems Science and Engineering, Vol. 20, Kluwer Academic, Plenum Press, Dordrecht, NewYork, 2002.

in the context of residuated lattices.

- $(x,z)(\mathcal{R} \oslash \mathcal{S}) = \bigwedge_{y \in Y} (x,y)\mathcal{R} \to (y,z)\mathcal{S}$;
- $(x,z)(\mathcal{R} \tilde{\oslash} \mathcal{S}) = \bigwedge_{y \in Y} (x,y)\mathcal{R} \rightsquigarrow (y,z)\mathcal{S}$;
- $(x,z)(\mathcal{R} \obslash \mathcal{S}) = \bigwedge_{y \in Y} (y,z)\mathcal{S} \to (x,y)\mathcal{R}$;
- $(x,z)(\mathcal{R} \tilde{\obslash} \mathcal{S}) = \bigwedge_{y \in Y} (y,z)\mathcal{S} \rightsquigarrow (x,y)\mathcal{R}$;
- $(x,z)(\mathcal{R} \odot \mathcal{S}) = \bigvee_{y \in Y} (x,y)\mathcal{R} \otimes (y,z)\mathcal{S}$;
- $(x,z)(\mathcal{R} \tilde{\odot} \mathcal{S}) = \bigvee_{y \in Y} (x,y)\mathcal{R} \tilde{\otimes} (y,z)\mathcal{S}$.

Proposition

Let $L = (L, \to, \top)$ be a symmetrical cdeo algebra. Then for any triple of sets (X, Y, Z) the diagram

$$
\begin{array}{ccccc}
 & & (L^{X\times Y}, \mathcal{S}_{X\times Y}, \tilde{\mathcal{S}}_{X\times Y}) & & \\
 & \overset{\psi\ \tilde{\psi}}{\nearrow} & & \overset{\chi\ \tilde{\chi}}{\searrow} & \\
(L^{Z\times X}), \mathcal{S}_{Z\times X}, \tilde{\mathcal{S}}_{Z\times X}) & & \xrightarrow[\varphi\ \tilde{\varphi}]{\qquad\qquad} & & (L^{Y\times Z}, \mathcal{S}_{Y\times Z}, \tilde{\mathcal{S}}_{Y\times Z})
\end{array}
$$

where

- $(\rho, \sigma)\varphi = (\sigma \owedge \rho)_-$; $(\rho, \tau)\psi = (\rho \odot \tau)_-$; $(\tau, \sigma)\chi = (\tau \tilde{\oslash} \sigma)_-$;
- $(\rho, \sigma)\tilde{\varphi} = (\sigma \tilde{\owedge} \rho)_-$; $(\rho, \tau)\tilde{\psi} = (\rho \tilde{\odot} \tau)_-$; $(\tau, \sigma)\tilde{\chi} = (\tau \oslash \sigma)_-$

is a double r-triangle if and only if L is associative.

Proof.

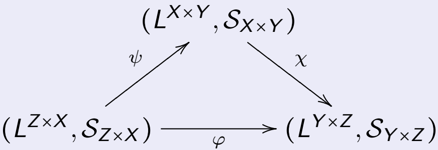
In fact, under associativity assumption the following equalities hold:

- $a \to (c \to b) = (a \tilde{\otimes} c) \to b$; $a \rightsquigarrow (c \rightsquigarrow b) = (a \otimes c) \rightsquigarrow b$;
- $a \to (c \rightsquigarrow b) = c \rightsquigarrow (a \to b)$; $a \rightsquigarrow (c \to b) = c \to (a \rightsquigarrow b)$.

□

Proposition

Let $L = (L, \rightarrow, \top)$ *be a symmetrical cdeo algebra. Then for any triple of sets* (X, Y, Z) *the diagram*



where

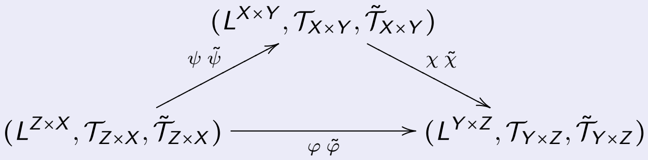
$$(\rho, \sigma)\varphi = (\sigma \obslash \rho)_-; \ (\rho, \tau)\psi = (\rho \odot \tau)_-; \ (\tau, \sigma)\chi = (\tau \oslash \sigma)_-$$

is a (double) relational triangle if and only if L is associative and commutative.

Further examples of the above defined triangles can be obtained replacing the subsethood degree with the overlap degree defined as follows.
Let X be a set and let $(L, \rightarrow, \top)$ be a cdeo algebra.
The overlap degree is the L-relation $\mathcal{T}_X : L^X \times L^X \rightarrow L$ defined by $(A, B)\mathcal{T}_X = \bigvee_{x \in X}(x)A \otimes (x)B$, for all $A, B \in L^X$.
If $(L, \rightarrow, \top)$ is a symmetrical *cdeo* algebra, we can define a further L-relation $\tilde{\mathcal{T}}_X : L^X \times L^X \rightarrow L$ such that $(A, B)\tilde{\mathcal{T}}_X = \bigvee_{x \in X}(x)A \tilde{\otimes} (x)B$, for all $A, B \in L^X$.
We notice that the equality $(A, B)\mathcal{T}_X = (B, A)\tilde{\mathcal{T}}_X$ holds for all $A, B \in L^X$.

Proposition

Let $L = (L, \to, \top)$ *be a symmetrical cdeo algebra. Then for any triple of sets* (X, Y, Z) *the diagram*



where

- $(\rho, \sigma)\varphi = \sigma \tilde{\odot} \rho$; $(\rho, \tau)\psi = \rho \tilde{\odot} \tau$; $(\tau, \sigma)\chi = \tau \odot \sigma$;
- $(\rho, \sigma)\tilde{\varphi} = \sigma \odot \rho$; $(\rho, \tau)\tilde{\psi} = \rho \odot \tau$; $(\tau, \sigma)\tilde{\chi} = \tau \tilde{\odot} \sigma$,

is a double r-triangle if and only if L *is associative.*

Proof.

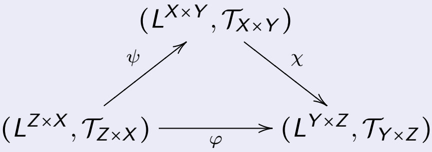
In fact, under associativity assumption:

1. $a \otimes (b \otimes c) = (a \otimes b) \otimes c$; $a \tilde{\otimes} (b \tilde{\otimes} c) = (a \tilde{\otimes} b) \tilde{\otimes} c$;
2. $a \otimes (b \tilde{\otimes} c) = (a \otimes c) \otimes b$; $a \tilde{\otimes} (b \otimes c) = b \otimes (c \otimes a)$.

□

Proposition

Let $L = (L, \to, \top)$ be a symmetrical cdeo algebra. Then for any triple of sets (X, Y, Z) the diagram



where

$$(\rho, \sigma)\varphi = \sigma \odot \rho;\ (\rho, \tau)\psi = \rho \odot \tau;\ (\tau, \sigma)\chi = \tau \odot \sigma$$

is a (double) relational triangle if and only if L is associative and commutative.