

Localization in Abelian ℓ -groups with strong unit, via MV-algebras

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Abstract

In this paper we explore the role of Perfect MV-algebras in localization issues. Further, via suitable functors we describe a localization process for abelian lattice ordered groups with strong unit from localization of MV-algebras.

Keywords: MV-algebras, ℓ -groups.

1 MV-algebras and Perfect MV-algebras

The class of MV-algebras arises as algebraic counterpart of the infinite valued Lukasiewicz sentential calculus, as Boolean algebras did with respect to the classical propositional logic. Due to the non-idempotency of the MV-algebraic conjunction, unlike Boolean algebras, MV-algebras can be non-archimedean and can contain elements x such that $x \odot \dots \odot x$ (n times) is always greater than zero, for any $n > 0$, here \odot denotes the conjunction in the Lukasiewicz sentential calculus. In general, there are MV-algebras which are not semisimple. Non-zero elements from the radical of A are called *infinitesimals*. Perfect MV-algebras are those MV-algebras generated by their infinitesimal elements. Hence perfect MV-algebras can be seen as an extreme example of non-archimedean MV-algebras.

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In a perfect MV-algebra all infinitesimals are localized in the unique maximal ideal. Herein we explore the role of these algebras in localization issues. An important example of a perfect MV-algebra is the subalgebra S of the Lindenbaum algebra L of first order Lukasiewicz logic generated by the classes of formulas which are valid but non-provable. Hence perfect MV-algebras are directly connected with the very important phenomenon of incompleteness in Lukasiewicz first order logic (see [4], [1]). As it is well known, MV-algebras form a category which is equivalent to the category of abelian lattice ordered groups with strong unit (ℓ_u -groups for short) [3]. This makes the interest in MV-algebras relevant outside the realm of logic. Let us denote by Γ the functor implementing this equivalence. So, via Γ^{-1} each perfect MV-algebra is associated with an abelian ℓ_u -group. Also it has been proved that the category of perfect MV-algebras is equivalent to the category of abelian ℓ -groups, see ([2], Theorem 3.5, p.420). Let us denote by Δ this functor. Hence Δ maps functorially each perfect MV-algebra to an abelian ℓ -group and vice versa, without the help of a strong unit.

2 Antiarchimedean ℓ_u -groups

Let (G, u) be an abelian ℓ_u -group. Consider the subset of G defined as follows: $R(G) = \{g \in G^+ : \forall n \in \mathbb{N} \quad ng < u\}$. Elements from $R(G)$ are called *infinitesimals*. An ℓ_u group generated by infinitesimal elements is called *antiarchimedean*. An example of antiarchimedean group is the abelian ℓ -group $\mathbb{Z} * G$, the lexicographic product of \mathbb{Z} by G with strong unit $(1, 0)$ and G an arbitrary abelian ℓ -group. Let \mathbf{Ab}_u denote the category of abelian lattice ordered groups with strong unit, then easily we can define the full subcategory of \mathbf{Ab}_u having as objects the antiarchimedean groups. Denote the latter category by \mathbf{AntArc}_u . We easily get that the categories \mathbf{Perf} of perfect MV-algebras, \mathbf{AntArc}_u and the category \mathbf{Ab} of abelian ℓ -groups are mutually equivalent. Let \mathfrak{U} denote the universal class of elements from \mathbf{Ab}_u defined by the formula:

$$(\forall x)((x \geq 0 \quad \text{and} \quad x \leq u) \Rightarrow [(u - (2(u - 2x) \vee 0) = (2((2x - u) \vee 0) \wedge u)) \quad \text{and} \quad (((2x - u) \vee 0 = x) \Rightarrow (x = 0 \quad \text{OR} \quad x = u))])$$

The group $(\mathbb{Z} * G, (1, 0))$ with G an abelian ℓ -group is an example of element in \mathfrak{U} .

Proposition 1. *Let $(G, u) \in \mathbf{Ab}_u$. Then $(G, u) \in \mathfrak{U}$ iff $\Gamma(G, u) \in \mathbf{Perf}$ iff $(G, u) \cong (\mathbb{Z} * H, (1, 0))$, with H an abelian ℓ -group.*

3 Localization in MV-algebras and ℓ_u -groups

In some cases all the elements of infinite order can be "localized" into one maximal ideal. Evidently, in such an MV-algebra there can only be one maximal ideal. Perfect MV-algebras satisfy this condition. In this work we wish to study how the localized algebras relate to the original algebra. We also wish our notion of localization to parallel as closely as possible the situation in commutative rings. In our case the localizations will be homomorphic images of subalgebras.

Let A be an MV-algebra, P a prime ideal of A , (G, u) an l_u -group and H an l -ideal of (G, u) . Let us denote by $Spec(A)$ and $Spec(G)$ the set space of prime ideals of A and the space of prime l -ideals of (G, u) , respectively. Let $0_P(A) = \bigcap \{Q \mid Q \in \{Q \in Spec(A) \mid Q \subseteq P\}\}$, so $0_P(A)$ is an ideal in A . Also, let $0_H(G) = \bigcap \{K \mid K \in \{K \in Spec(G) \mid K \subseteq H\}\}$, so $0_H(G)$ is an l -ideal in (G, u) . Furthermore, set:

$\mathcal{L}(P) = \{A' \mid A' \text{ is a subalgebra of } A \text{ and } P \text{ is maximal in } A'\}$, and $\mathcal{M}(H) = \{(G', u) \mid (G', u) \text{ is a subgroup of } (G, u) \text{ and } H \text{ is maximal } l\text{-ideal in } (G', u)\}$, hence we have:

Theorem 2. (Localization Theorem for MV – algebras) *Let A be an MV-algebra and P a prime ideal of A . Then for any $A' \in \mathcal{L}(P)$ there is a natural bijection between $\{Q \in Spec(A) \mid Q \subseteq P\}$ and $Spec(A'/0_P(A))$.*

Theorem 3. (Localization Theorem for l_u – groups) *Let (G, u) be an abelian l -group with strong unit u and H a prime l -ideal of (G, u) . Then for any $G' \in \mathcal{M}(H)$ there is a natural bijection between $\{K \in Spec(G) \mid K \subseteq H\}$ and $Spec(G'/0_H(G))$.*

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