Gödel algebras free over finite distributive lattices

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Gödel -Dummett logic

Gödel-Dummett (propositional) logic is the schematic extension of the intuitionistic propositional calculus by the prelinearity axiom

$$(\alpha \to \beta) \vee (\beta \to \alpha).$$

Its Tarski-Lindenbaum algebras, *Gödel algebras*, form the locally finite variety of Heyting algebras satisfying prelinearity.

Gödel logic can be also seen as an infinite-valued logic obtained as an axiomatic extension of Hájek's propositional *Basic Fuzzy Logic* BL by means of *contraction*: $\varphi \to (\varphi \& \varphi)$.

It turns out that Gödel logic $[0,1]$-semantics (also called *standard* semantics) is given by $\langle [0,1], \wedge, \to_\wedge, 0 \rangle$ for

$$x \to_\wedge y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

Free Gödel algebras over finite distributive lattices

There is a forgetful functor $\mathcal{U}$ from Gödel algebras to distributive lattices (with top and bottom elements) that associates to each Gödel algebra its underlying lattice.

$\mathcal{U}$ has a left adjoint $\mathcal{F}$. If D is a distributive lattice, one calls $\mathcal{F}(D)$ the *free Gödel algebra over the distributive lattice D*:

A Gödel algebra G is free over the distributive lattice D if there is an injective homomorphism of lattices $D \hookrightarrow G$ such that, for every injective homomorphism of lattices $E \hookrightarrow H$, with E a distributive lattice and H a Gödel algebra, and for every lattice homomorphism $D \to E$, there exits a unique homomorphism $G \to H$ of Gödel algebras making the following diagram commute:

$$\begin{array}{ccc} D & \hookrightarrow & G \\ \downarrow & & \downarrow {\scriptstyle !} \\ E & \hookrightarrow & H \end{array}$$

Equational meaning

Let $\mathcal{G}_\kappa$ denote the free Gödel algebra on κ free generators, for κ a cardinal; similarly, let $\mathcal{D}_\kappa$ be the free distributive lattice on κ free generators.

$D \cong \mathcal{D}_\kappa/\theta$, for some congruence θ and some cardinal κ.

Suppose θ is generated by equations

$$\sigma_\iota = \tau_\iota, \qquad \iota \in I$$

for σ_ι and τ_ι terms in the language of lattices over κ variables.

Then the equations $\sigma_\iota = \tau_\iota$, $\iota \in I$, generate a congruence $\hat{\theta}$ of Gödel algebras in $\mathcal{G}_\kappa$.

$$\mathcal{G}_\kappa/\hat{\theta} \cong \mathfrak{F}(D)$$

i.e., $\mathfrak{F}(D)$ is the quotient of the free Gödel algebra with respect to the set of equations $\sigma_\iota = \tau_\iota$ written in the language of lattices.

Our results

As a first result, we shall provide a more informative construction of $\mathcal{F}(D)$ whenever D is finite.

We shall then characterize Gödel algebras free over finite distributive lattices by their intrinsic properties.

For the rest of this talk, all posets and lattices are finite, unless otherwise stated.

Some well known results

- G, the category of finite Gödel algebras and their homomorphisms.
- F, the category of finite forests and open order-preserving maps.
- D, the category of finite bounded distributive lattices and their hom.
- P, the category of finite posets and order-preserving maps.

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G and F are dually equivalent categories:

$\mathrm{Spec}\colon \mathsf{G} \to \mathsf{F}^{\mathsf{op}}$ carries finite Gödel algebra G to the forest $\mathrm{Spec}\, G$ of its prime filters.

$\mathrm{Sub}\colon \mathsf{F} \to \mathsf{G}^{\mathsf{op}}$ carries forest F to the Gödel algebra $\mathrm{Sub}\, F$ of downset of F.

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D and P are dually equivalent categories.

$\mathcal{J}\colon \mathsf{D} \to \mathsf{P}^{\mathsf{op}}$ carries a distributive lattice L to the poset $\mathcal{J}(L)$ of its prime lattice filters (equivalently, to its join irreducible elements).

$\mathcal{O}\colon \mathsf{P} \to \mathsf{D}^{\mathsf{op}}$ carries a poset P to the lattice $\mathcal{O}(P)$ of downsets of P ordered by inclusion.

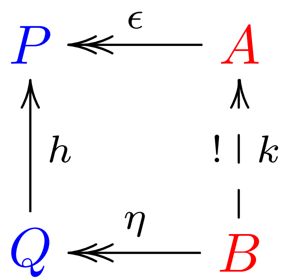
Moving to forests and posets

For any Gödel algebra G and any finite distributive lattice D with an injective homomorphism $\hat{\epsilon}\colon D \hookrightarrow G$, the following are equivalent.

(i) The Gödel algebra G is free over the distributive lattice D via $\hat{\epsilon}$.

(ii) The forest $\mathrm{Spec}\, G$ is *cofree* over the poset $\mathcal{J}(D)$ via $\epsilon = \mathrm{Spec}\,\hat{\epsilon}$.

Cofree is the dual notion of *free*: A *is cofree over* P if the diagram commutes, where A and B are forests, P nd Q are posets and ϵ, η are surjective order-preserving maps.



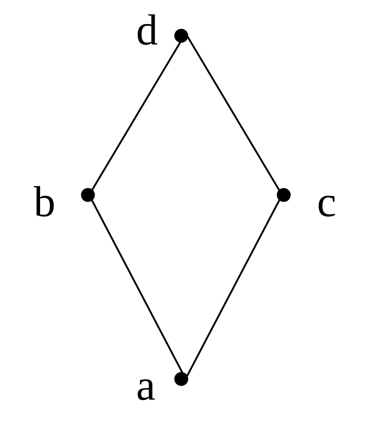
Poset of paths

A *path* in a poset P is a sequence $p_1, \ldots, p_n$ of elements of P such that $p_i < p_j$ if and only if $i < j$.

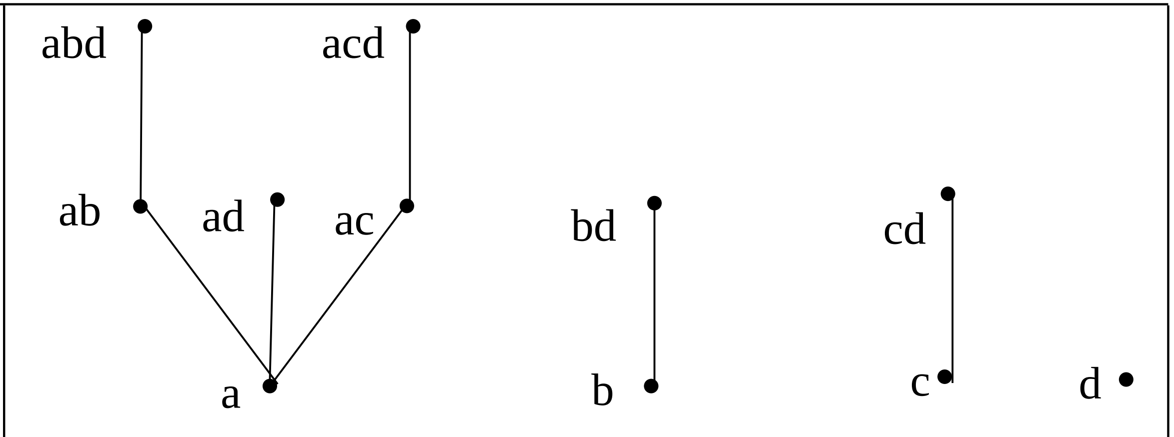
Paths in P can be partially ordered by $q_1, \ldots, q_m \sqsubseteq p_1, \ldots, p_n$ if and only if $m \leq n$ and $q_i = p_i$ for each $i = 1, \ldots, m$.

We denote by $\mathcal{P}(P)$ the poset of all paths in P.

$\mathcal{P}(P)$ is a forest for any poset P.



Poset P

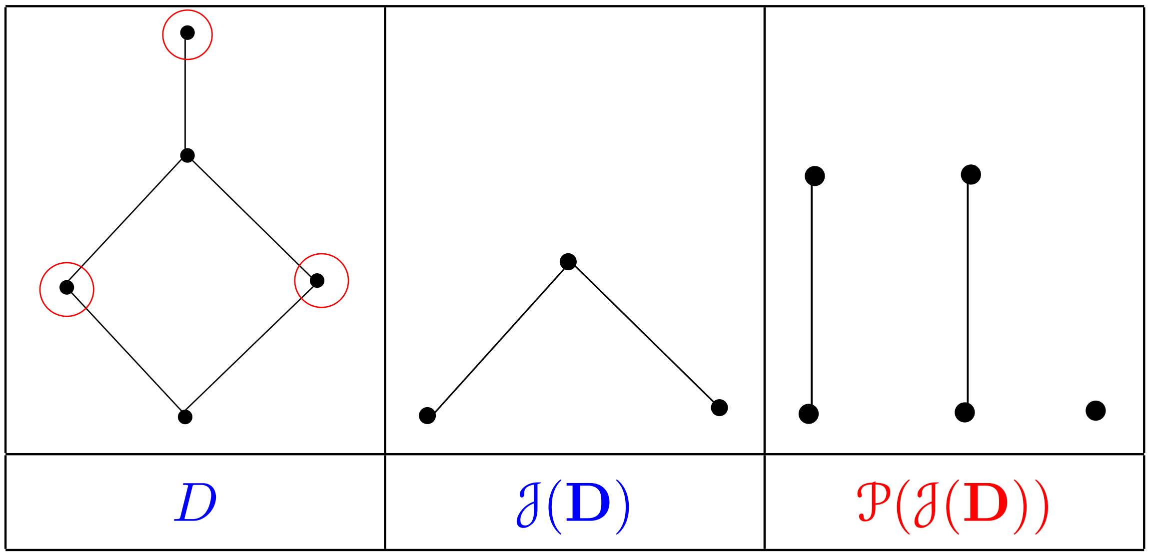


Forest of paths of P

We prove that

> A forest $\mathbf{F}$ is cofree over a poset $\mathbf{P}$ if and only if $\mathbf{F}$ is the forest of paths over $\mathbf{P}$.

So, starting from a distributive lattice D we consider the poset $\mathcal{J}(\mathbf{D})$ of its join irreducible elements and then the forest $\mathcal{P}(\mathcal{J}(\mathbf{D}))$ of paths over $\mathcal{J}(\mathbf{D})$:

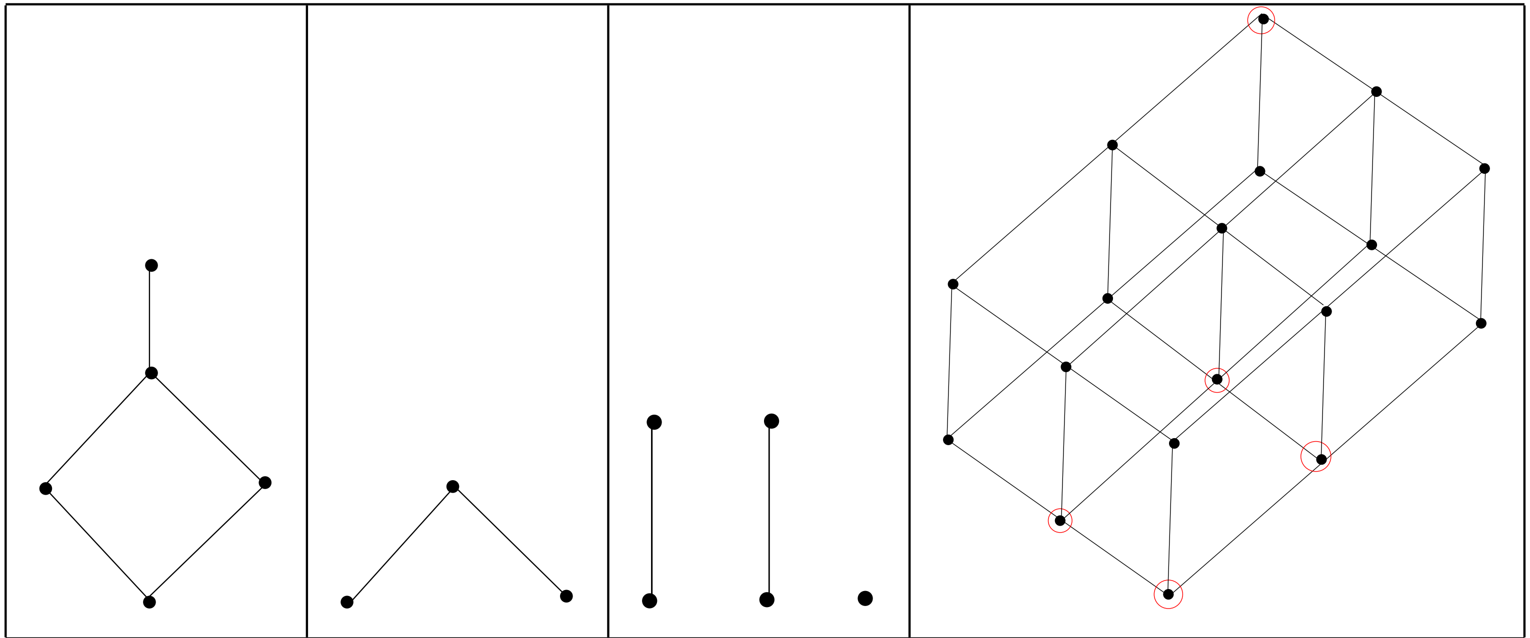


Theorem

We can now state the first theorem:

A Gödel algebra G is free over a finite distributive lattice D if and only if $G \cong \mathrm{Sub}\,\mathcal{P}(\mathcal{J}(\mathbf{D}))$, where $\mathcal{J}(\mathbf{D})$ is the poset of join-irreducible elements of D.

In the previous example we have



D	$\mathcal{J}(\mathbf{D})$	$\mathcal{P}(\mathcal{J}(\mathbf{D}))$	$\mathrm{Sub}(\mathcal{P}(\mathcal{J}(D)))$

Example

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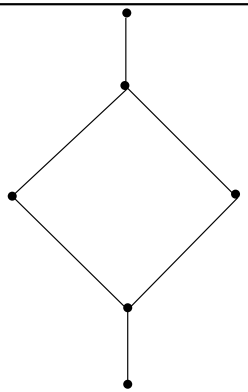
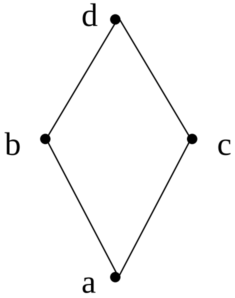
<td></td>
<td>
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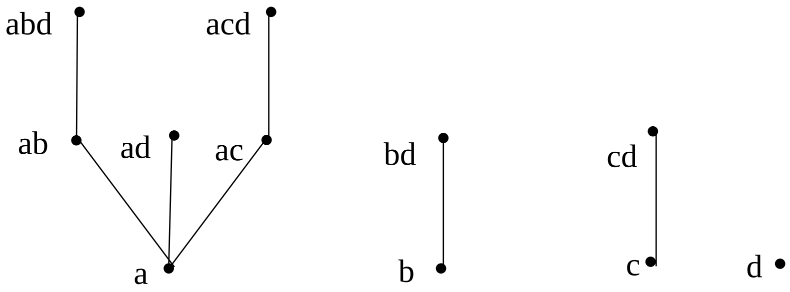
<td>
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<td>Free Gödel algebra over 2 generators.</td>
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<tr>
<td>D</td>
<td>$\mathcal{J}(\mathbf{D})$</td>
<td>$\mathcal{P}(\mathcal{J}(\mathbf{D}))$</td>
<td>$\text{Sub}(\mathcal{P}(\mathcal{J}(D)))$</td>
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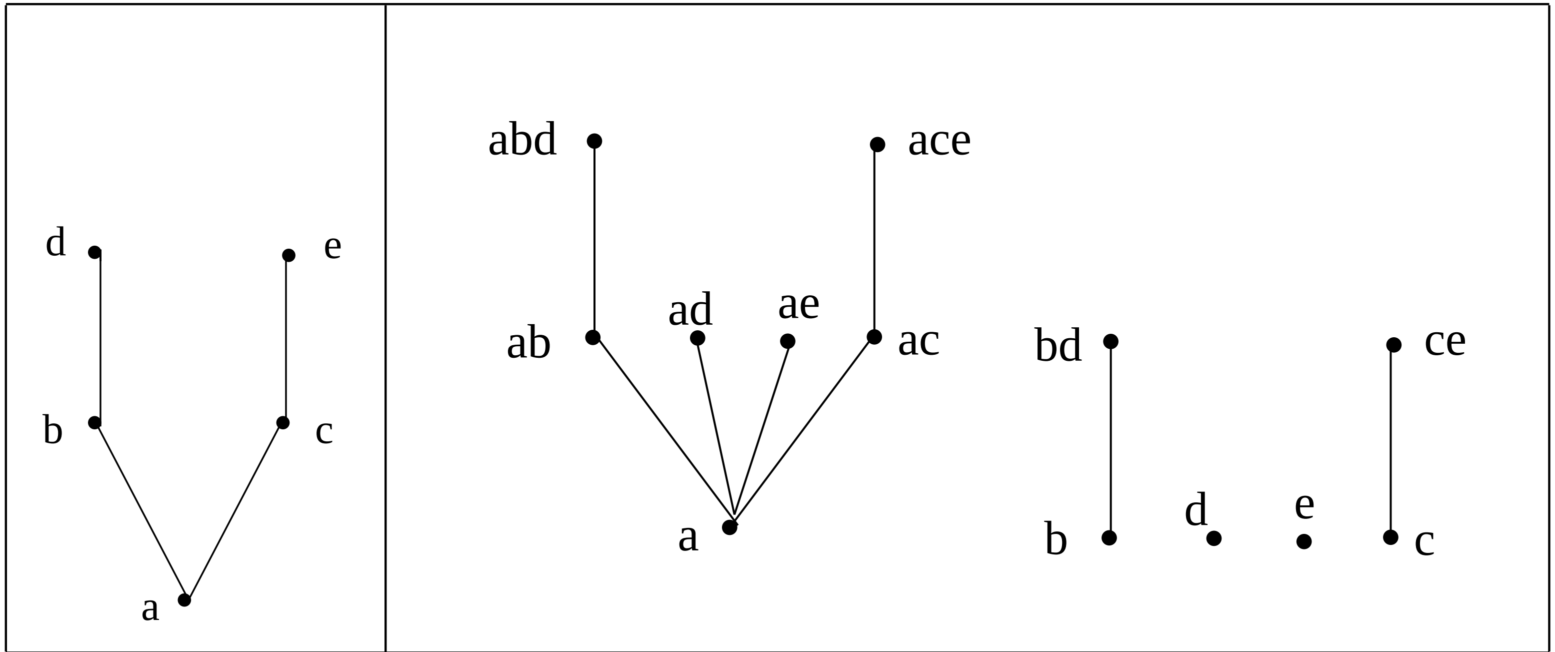
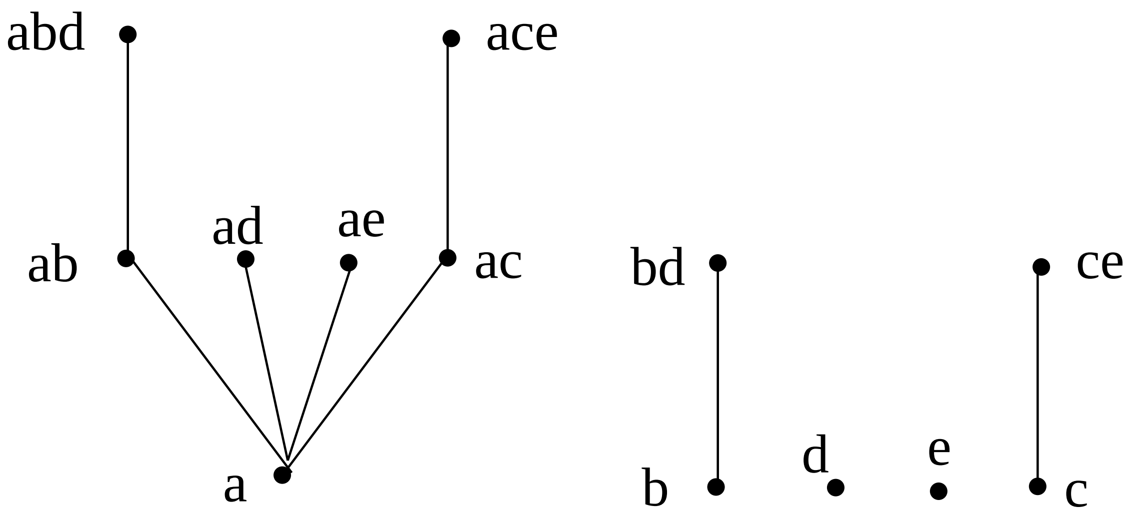
The question arises, can Gödel algebras free over some finite distributive lattice be recognized by some intrinsic property of their poset of prime filters.

Let us start from the study of *forests cofree over forests* (i.e. we consider distributive lattices whose join irreducible elements form a forest) .

Example:



$\mathcal{J}(\mathbf{D})$ $\quad$ $\mathcal{P}(\mathcal{J}(\mathbf{D}))$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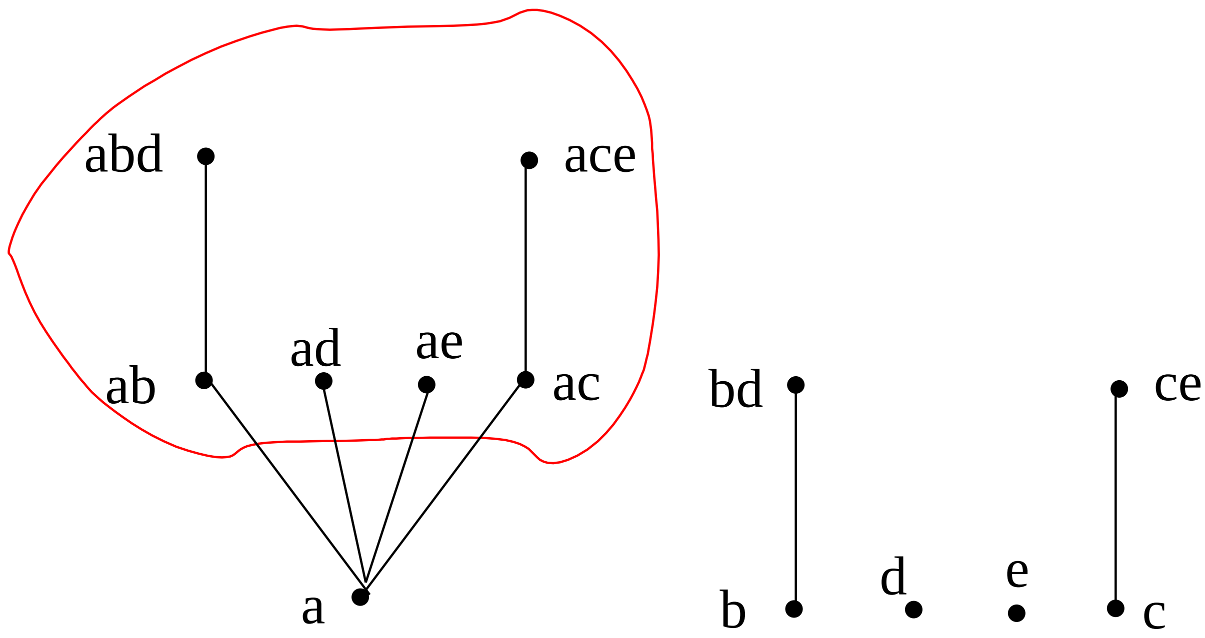
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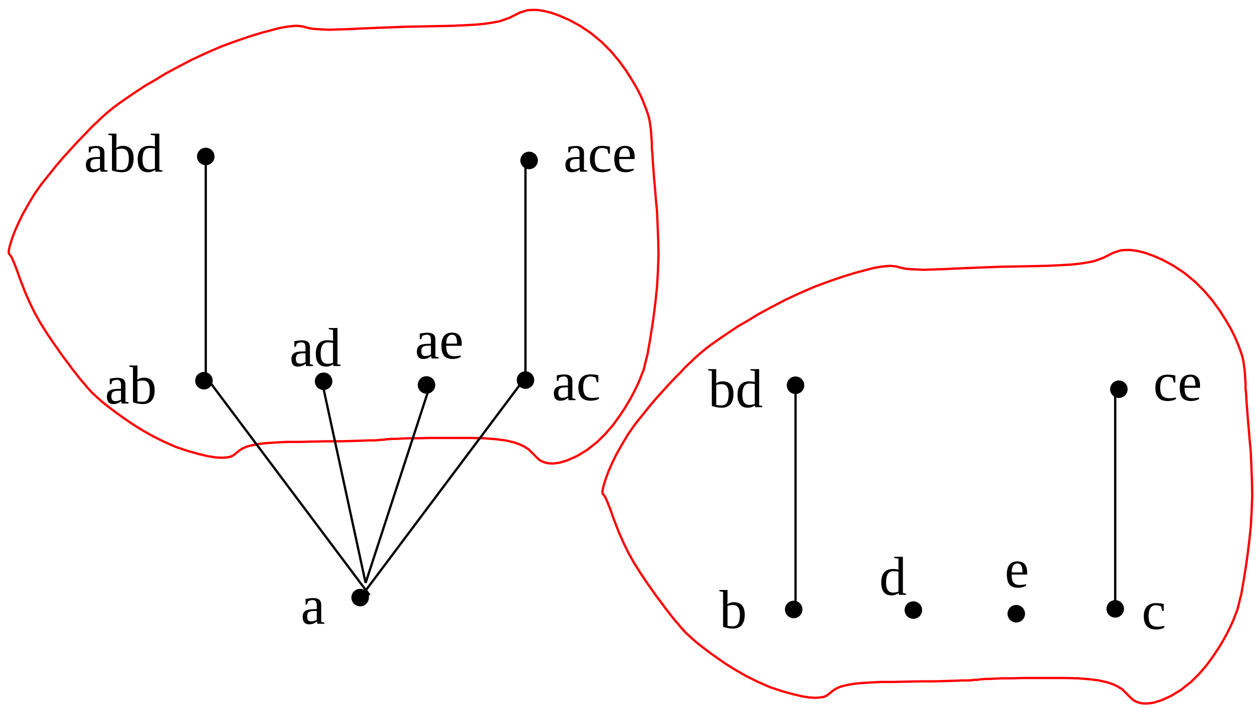
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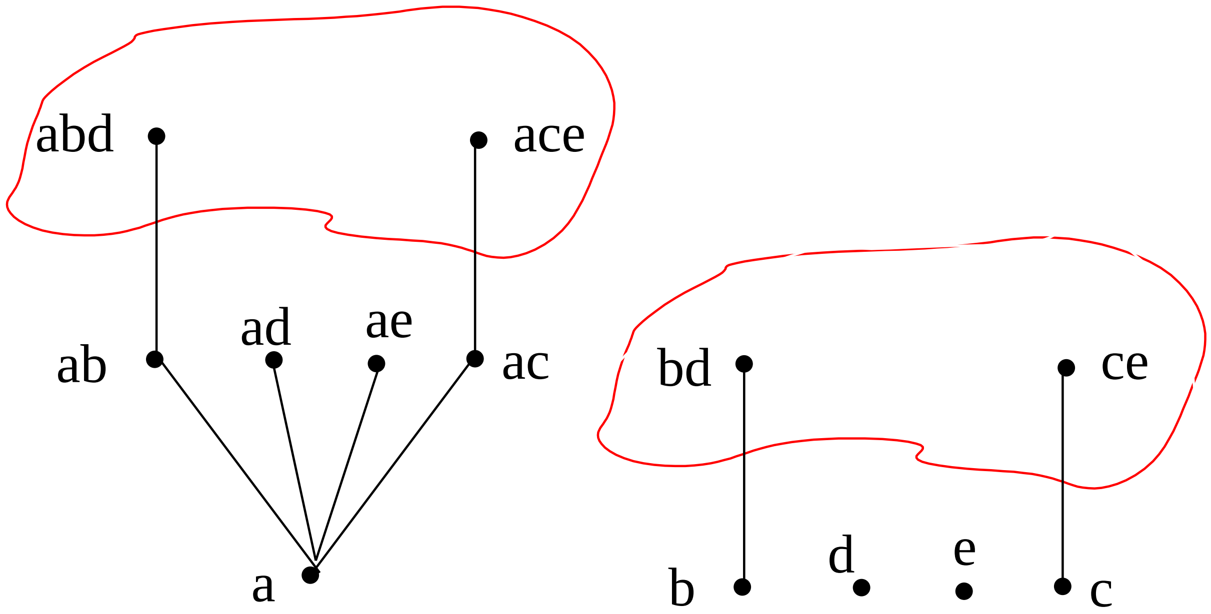
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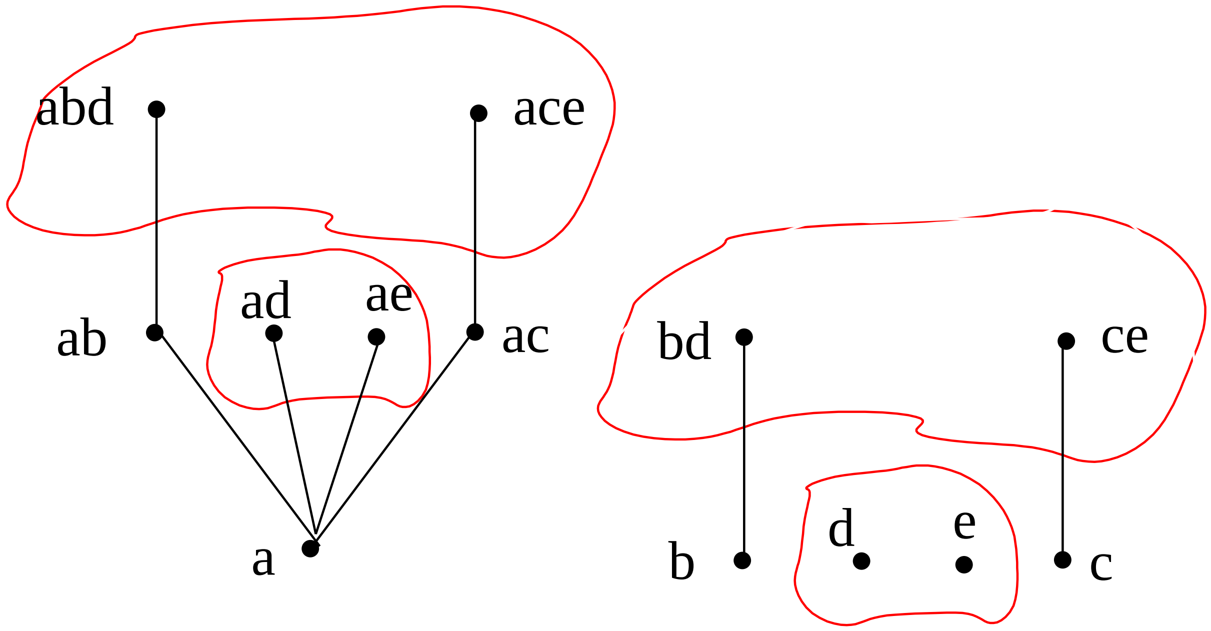
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Self similar forests

We say $p \in F$ is *inner* if it is neither a leaf nor a root.

$p^{\lhd}$ is the predecessor of the node p. $S(p)$ is the set of successors of p.

$B(p) = S(p^{\lhd}) \setminus \{p\}$ is the set of *siblings* of p.

Definition.

(i) A tree T is *self-similar* if for every inner $p \in T$ there exists an injection

$$E_p \colon S(p) \hookrightarrow B(p)$$

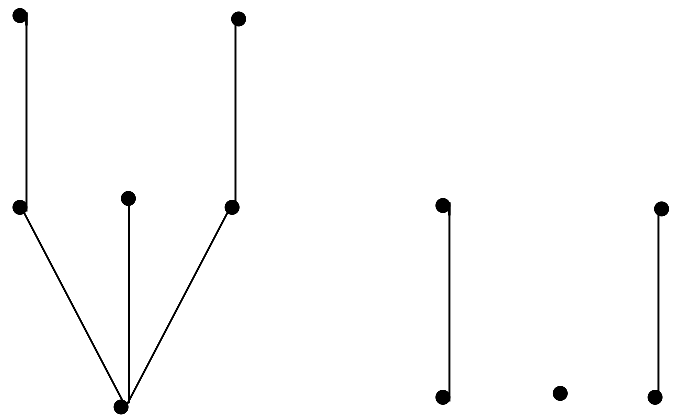
such that

$$\uparrow x \cong \uparrow E_p(x)$$

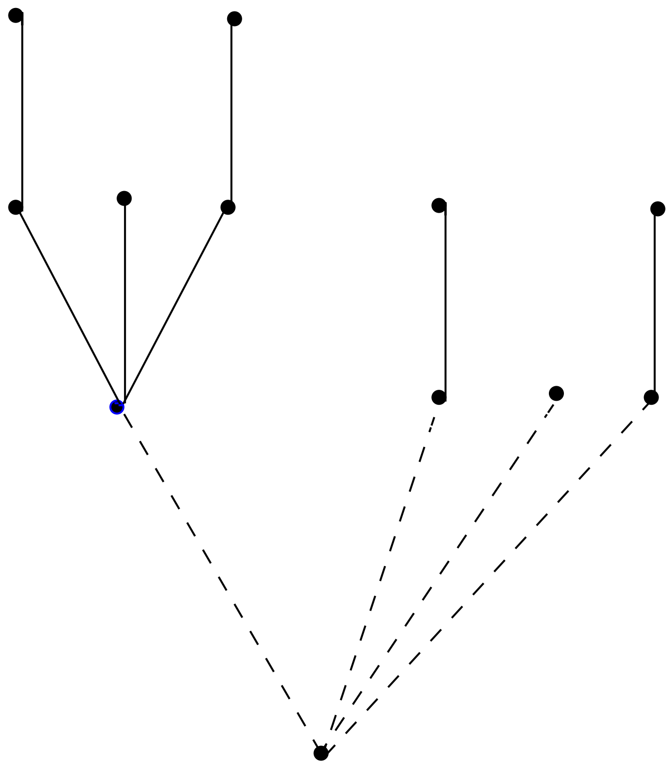
for each $x \in S(p)$.

(ii) A forest F is *self-similar* if $F_{\perp}$ is a self-similar tree.

Self similar forest: an example



Self similar forest: an example



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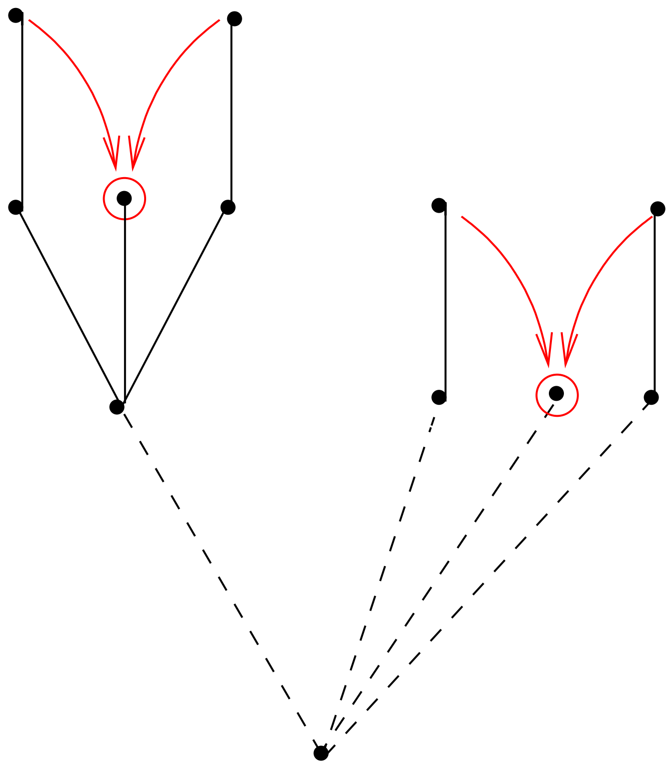
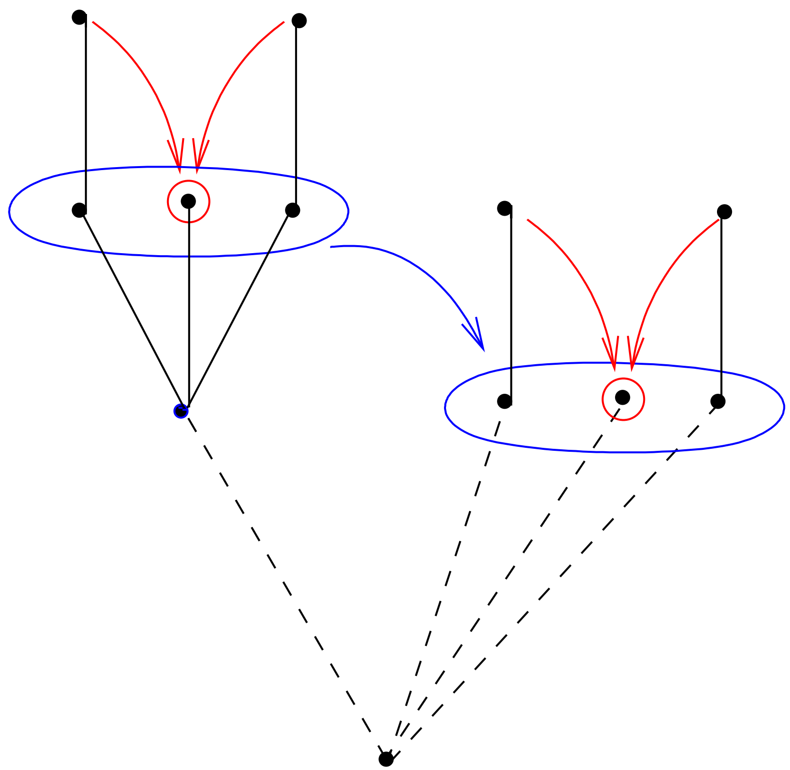


Self similar forest: an example



Strongly self similar forests

Definition.

A tree T is *strongly self-similar* if it is self similar and

if $S(p^{\lhd}) = \{q_1, \ldots, q_k\}$ then for each q_i inner node there exists

$$E_{q_i} : S(q_i) \to B(q_i)$$

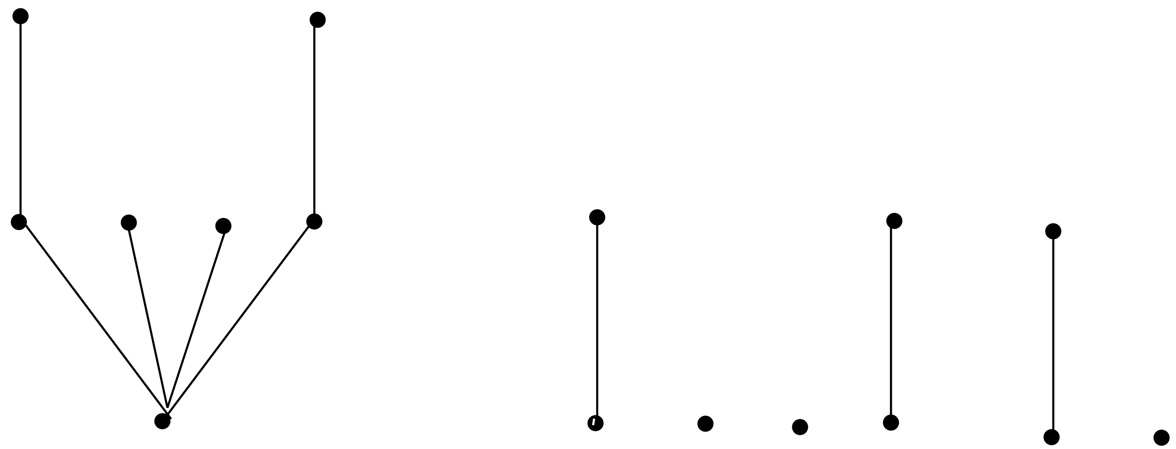
such that

if $q_j \neq q_i$ and $E_{q_i}(S(q_i)) \cap E_{q_j}(S(q_j)) \neq \emptyset$ then

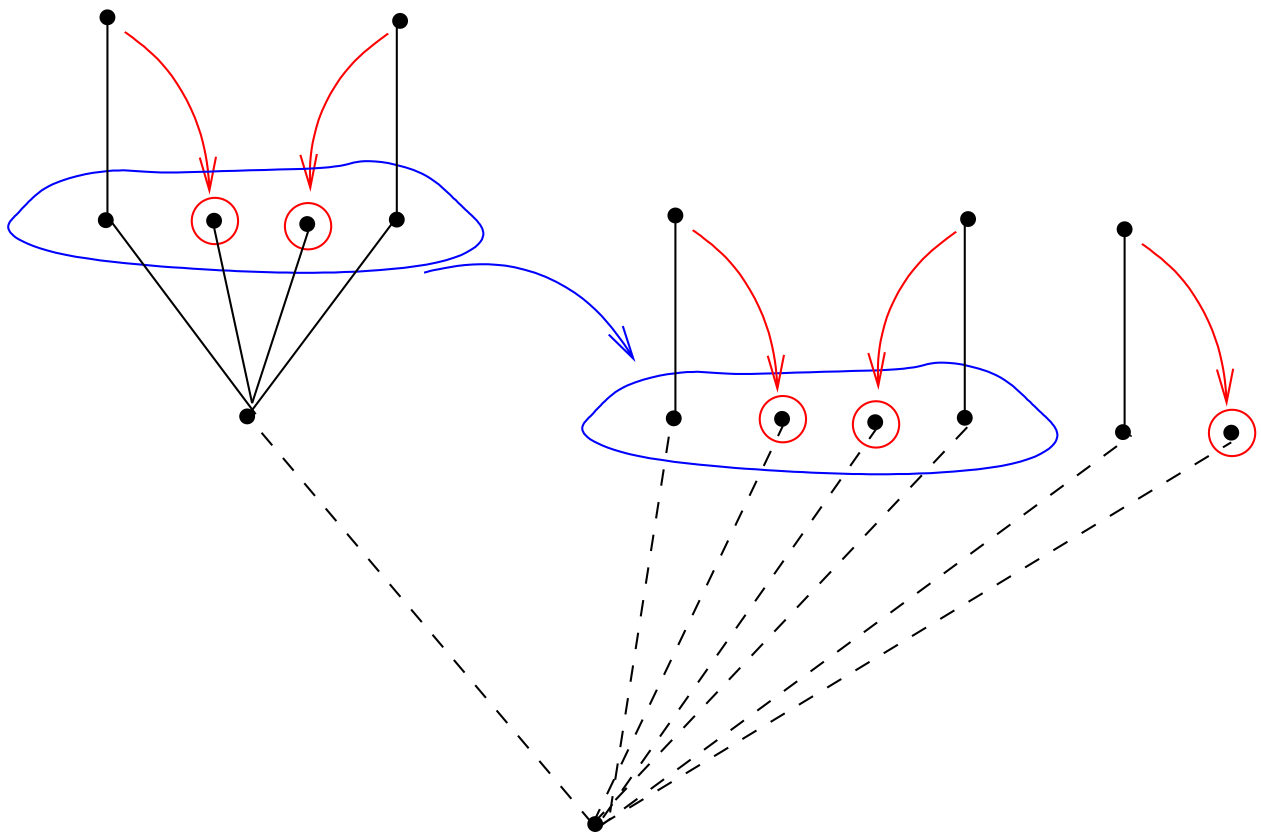
$$\begin{aligned} \text{either} \quad & \{q_i\} \cup E_{q_i}(S(q_i)) \subseteq E_{q_j}(S(q_j)) \\ \text{or} \quad & \{q_j\} \cup E_{q_j}(S(q_j)) \subseteq E_{q_i}(S(q_i)). \end{aligned}$$

A forest F is *strongly self-similar* if $F_{\perp}$ is a strongly self-similar tree.

Strongly self similar forest: an example



Strongly self similar forest: an example

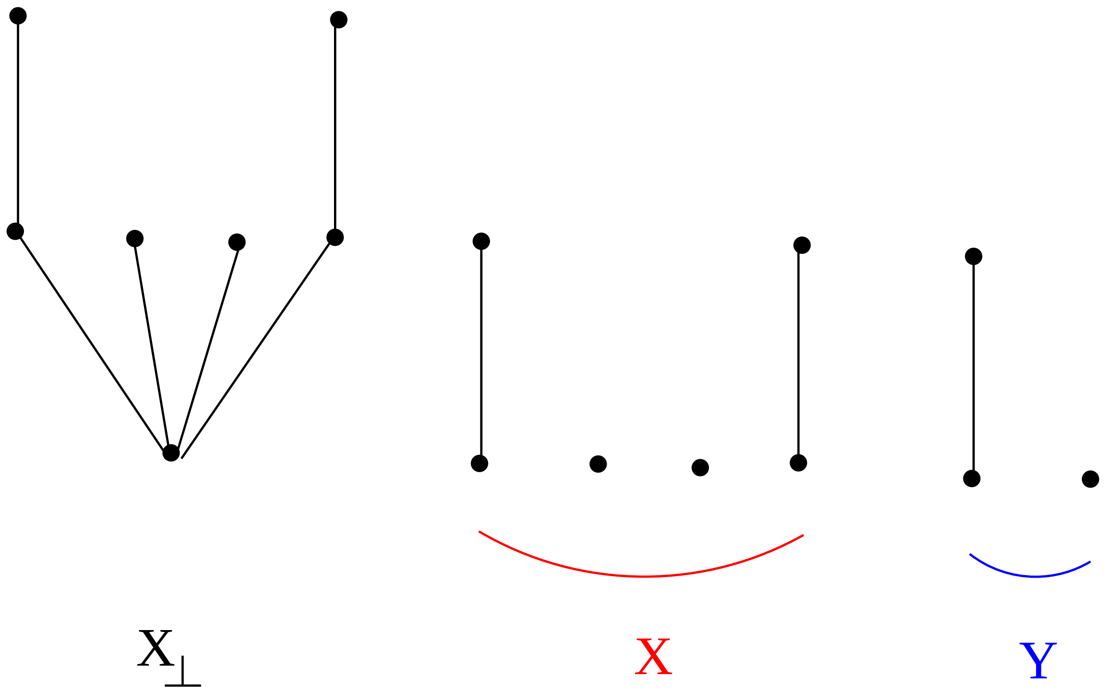


Recursive definition

A forest F is strongly self similar if and only if either is empty or

$$F = X_\perp \sqcup X \sqcup Y$$

where $X_\perp$ is a maximal tree of F and X and Y are strongly self similar forests.



Strongly self similar forests

This recursive definition allows us to prove the following:

For any Gödel algebra G, the following are equivalent.

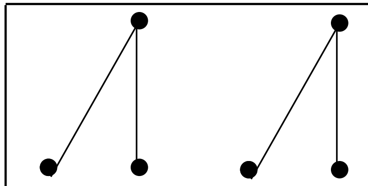
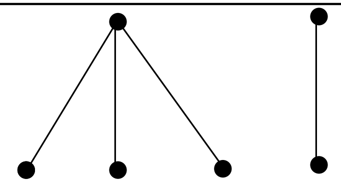
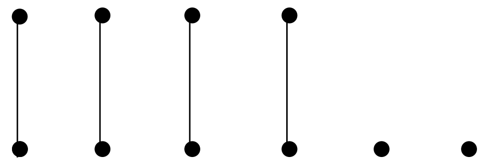
(i) G is free over some finite distributive lattice D such that $\mathcal{J}(D)$ is a forest.

(ii) $\operatorname{Spec} G$ is a strongly self similar forest.

Indeed the proof proceeds by induction on the height of the forest.

Moreover, when these conditions hold, the lattice D in (i) is unique up to an isomorphism.

Situation for self similar forest is different:

		
A	**B**	$\mathcal{P}(\mathbf{A}) = \mathcal{P}(\mathbf{B})$

A and B are different posets that have the same forest of paths. Nevertheless the following holds:

If F is a self similar forest then

$$F = X_\perp \sqcup \underbrace{X \sqcup Y}_{}$$

where:

- X is a self similar forest (hence $X_\perp \sqcup X$ is a self similar forest)
- and $X \sqcup Y$ *is self similar.*

Theorem

For any Gödel algebra G, the following are equivalent.

(i) G is free over some finite distributive lattice.

(ii) $\mathrm{Spec}\, G$ is a self-similar forest.

The proof is based on the following lemma:

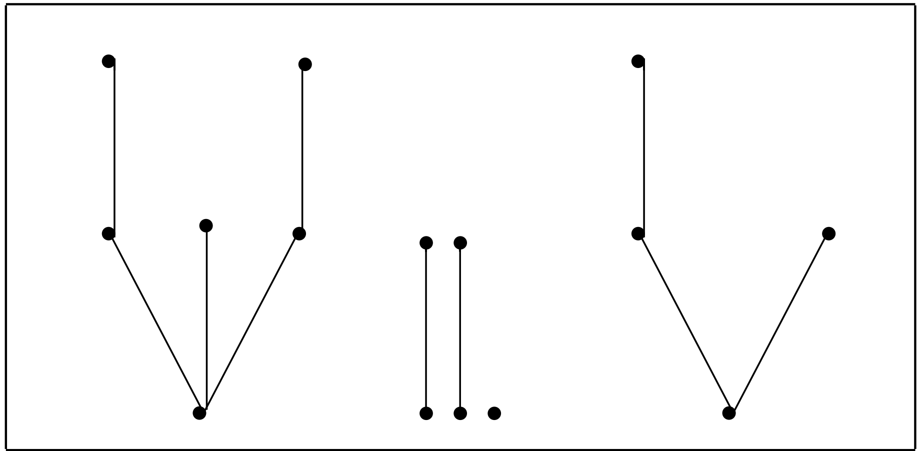
Lemma. Let $F = X_\perp \sqcup X \sqcup Y$ be a self similar forest and let B a poset such that $\mathcal{P}(B) = X \sqcup Y$. Then there exists an upward closed subposet A of B such that $\mathcal{P}(A) = X$ and

$$F = \mathcal{P}(C) \qquad \text{where } C = B(A|A_\perp)$$

where $B(A|A_\perp)$ is the poset obtained by substituting $A_\perp$ to A in B.

Let see it with an example:

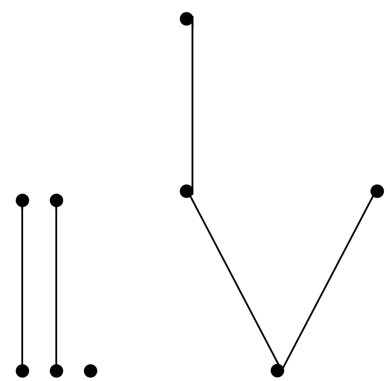
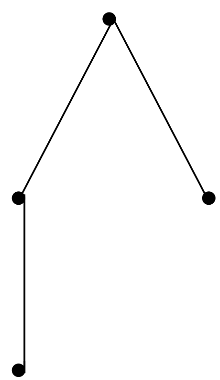
Example



$X_\perp$	X	Y

We proceed by induction on $X \sqcup Y$.

By induction we find B such that $X \sqcup Y = \mathcal{P}(B)$:

	
$X \sqcup Y = \mathcal{P}(B)$	B

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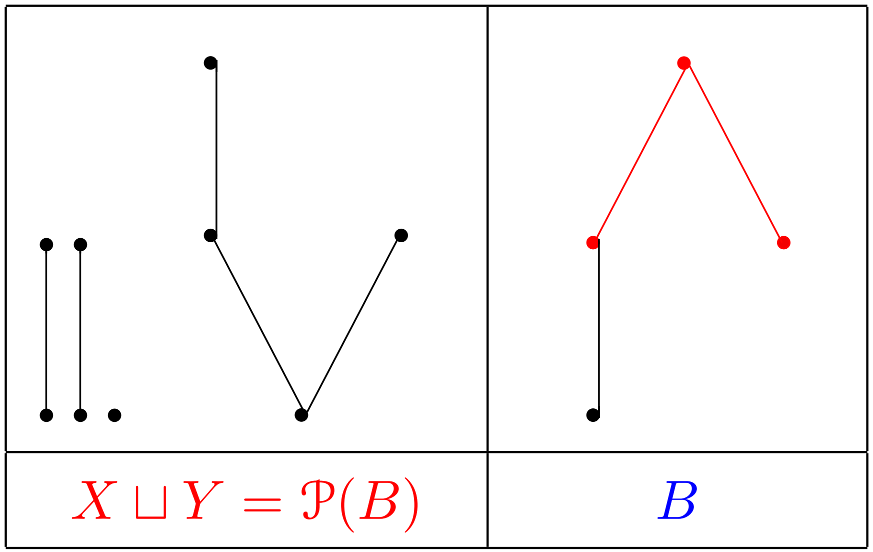
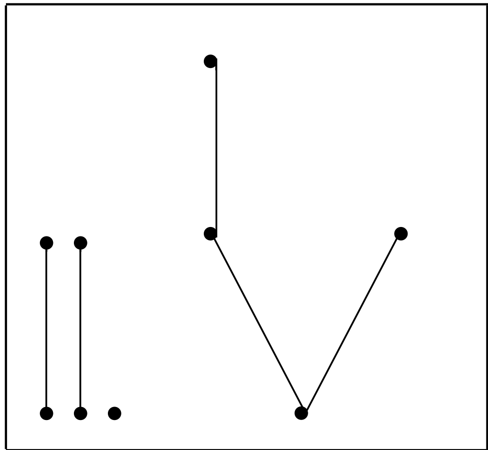
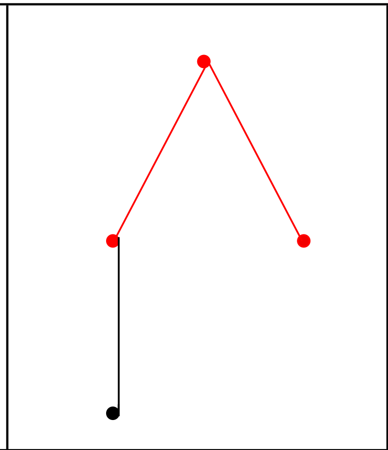
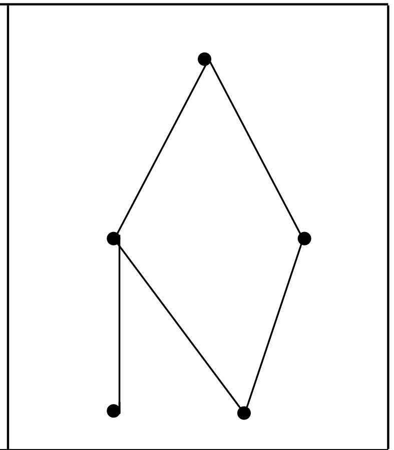
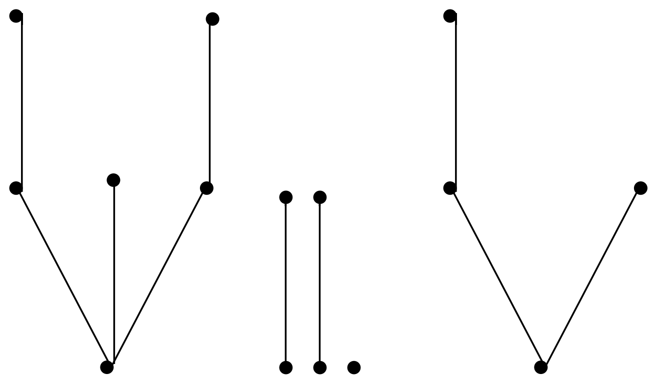
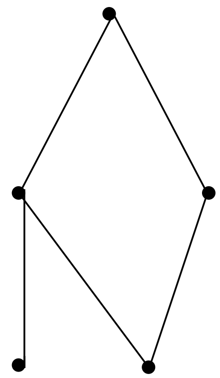


In B we may find the *contribution* of X.

By induction we find B such that $X \sqcup Y = \mathcal{P}(B)$:

		
$X \sqcup Y = \mathcal{P}(B)$	B	C

We add a bottom to it.

	
$X_{\perp} \sqcup X \sqcup Y = \mathcal{P}(C)$	C

Gödel algebras free over chains

Fix an integer $n \geq 1$, and let G_n be a Gödel algebra free over a chain of cardinality n.

(i) G_n has precisely $\binom{n}{k}$ prime filters of depth k, for each $k = 1, \ldots, n$, and thus $2^n - 1$ prime filters in all.

(ii) If g_n denotes the cardinality of G_n, then

$$g_1 = 2$$

and

$$g_n = g_{n-1}^2 + g_{n-1} \ .$$